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A-STATISTICAL CONVERGENCE OF SEQUENCES OF CONVOLUTION OPERATORS

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Abstract. In this paper, using the concept of $A-$statistical convergence we are concerned with the Korovkin type approximation theory for a sequence of positive convolution operators defined on $C[a, b]$, the space of all real valued continuous functions on $[a, b]$. We also study rates of $A-$statistical convergence of these operators.

1. INTRODUCTION

In this paper, we are concerned with the Korovkin type approximation theory for positive convolution operators via statistical convergence. The study of the Korovkin type approximation theory is a well-established area of research, which deals with the problem of approximating a function f by means of a sequence $\{L_n(f)\}$ of positive linear operators. Statistical convergence, while introduced over nearly fifty years ago, has only recently become an area of active research. Especially it has made an appearance in approximation theory [11] (see also [5, 6). Recall that approximation theory has important applications in various areas of functional analysis, and in numerical solutions of differential and integral equations [1, 4, 15].

The first section of this paper introduces some basic ideas related to statistical convergence while the second section describes some Korovkin type approximation theorems for a sequence of positive convolution operators defined on the space of all real valued continuous functions on an interval $[a, b]$. The third section addresses some problems concerning rates of statistical convergence of the sequence of convolution operators. In the last section, we consider positive convolution operators on C^*, the space of all $2\pi-$periodic and continuous functions on the whole real axis, and give an $A-$statistical approximation result.

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We now turn to introducing some notation and basic definitions used in this paper.

Let $A = (a_{jn})$ be an infinite summability matrix. For a given sequence $x := (x_n)$, the $A-$transform of x, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn}x_n$ provided the series converges for each j. We say that A is regular if $\lim Ax = L$ whenever $\lim x = L$ [12]. Assume now that A is a non-negative regular summability matrix and K is a subset of $\mathbb{N}$, the set of all natural numbers. The $A-$density of K, denoted by $\delta_A(K)$, is defined by $\delta_A(K) := \lim_j \sum_{n=1}^{\infty} a_{jn}\chi_K(n)$ provided the limit exists, where χ_K is the characteristic function of K. If $x = (x_n)$ is a sequence such that x_n satisfies a property P for all n except a set of $A-$density zero, then we say that x_n satisfies P for "almost all n", and we abbreviate this by "$a.\,a.\ n$". A sequence $x = (x_n)$ is said to be $A-$statistically convergent to a number L if, for every $\varepsilon > 0$, $\delta_A\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$; or equivalently

$$\lim_j \sum_{n:\,|x_n - L| \geq \varepsilon} a_{jn} = 0.$$

We denote this limit by $st_A - \lim x = L$ [8] (see also [2, 3, 13, 14 17). For $A = C_1$, the Cesáro matrix, $A-$statistical convergence reduces to statistical convergence [7, 9, 10]. Taking $A = I$, the identity matrix, $A-$statistical convergence coincides with the ordinary convergence. We note that if $A = (a_{jn})$ is a regular summability matrix for which $\lim_j \max_n |a_{jn}| = 0$, then $A-$statistical convergence is stronger than convergence [14]. It should be also noted that the concept of $A-$statistical convergence may also be given in normed spaces (see [13] for details).

2. A-Statistical Approximation by Convolution Operators

As usual, $C[a, b]$ denotes the space of all real valued continuous functions defined on $[a, b]$. Then $C[a, b]$ is a Banach space with the usual norm $\|\cdot\|_{C[a,b]}$ defined by

$$\|f\| := \|f\|_{C[a,b]} = \sup_{x \in [a,b]} |f(x)|, \quad f \in C[a, b].$$

Let L be a linear operator from $C[a, b]$ into $C[a, b]$. Then we say that L is positive linear operator provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of $L(f)$ at a point $x \in [a, b]$ by $L(f; x)$.

We now consider the following convolution operators defined on $C[a, b]$:

$$L_n(f; x) = \int_a^b f(y)K_n(y - x)dy, \quad n \in \mathbb{N},\ x \in [a, b] \text{ and } f \in C[a, b], \tag{2.1}$$

where a and b are two real numbers such that $a < b$.

Throughout the paper we assume that K_n is a continuous function on $[a-b, b-a]$ and also that $K_n(u) \geq 0$ for all $n \in \mathbb{N}$ and for every $u \in [a-b, b-a]$. Note that if $x,\ y \in [a, b]$ then $u := y - x \in [a-b, b-a]$. In this case our convolution operators L_n given by (2.1) are positive and linear.

Recently, Srivastava and Gupta [18] have studied on approximation properties of a certain family of summation-integral type operators in the classical sense. However, in this section we obtain a Korovkin type approximation theorem for positive convolution operators via the concept of $A-$statistical convergence which is a more general and stronger method than the ordinary convergence.

We first recall that Gadjiev and Orhan [11] proved the following Korovkin type result for any sequence of positive linear operators defined on $C[a, b]$ by using the concept of statistical convergence.

Theorem A. [11]. *Let $\{L_n\}$ be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$. If*

$$st - \lim_n \|L_n(f_i) - f_i\| = 0 \text{ with } f_i(y) = y^i,\ \ i = 0, 1, 2,$$

then, for all $f \in C[a, b]$, we have

$$st - \lim_n \|L_n(f) - f\| = 0.$$

Assume now that $A = (a_{jn})$ is a non-negative regular summability matrix. Then Theorem A works for $A-$statistical convergence. Furthermore, using the function φ on $[a, b]$ defined by $\varphi(y) := (y - x)^2$ for each $x \in [a, b]$ we have the following result that we need in proving the main result of this section. Note that if L_n is a positive and linear, then $L_n(\varphi; x) = L_n(f_2; x) - 2xL_n(f_1; x) + x^2 L_n(f_0; x)$ with $f_i(y) = y^i$ $(i = 0, 1, 2)$ since $\varphi \in C[a, b]$.

Theorem 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}$ be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$. If*

$$st_A - \lim_n \|L_n(f_0) - f_0\| = 0 \textit{ with } f_0(y) = 1$$

and

$$st_A - \lim_n \|L_n(\varphi)\| = 0,$$

then, for all $f \in C[a, b]$, we have

$$st_A - \lim_n \|L_n(f) - f\| = 0.$$

Proof. Let $f \in C[a,b]$ and $x \in [a,b]$. Since f is continuous on $[a,b]$, for every $\varepsilon > 0$ there exists a real number $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for y satisfying $|y - x| \leq \delta$. Letting $I_\delta := [x-\delta, x+\delta] \cap [a,b]$, we can write that

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f(x)| \, \chi_{I_\delta}(y) + |f(y) - f(x)| \, \chi_{[a,b] \setminus I_\delta}(y) \\ &\leq \varepsilon + 2M \, \delta^{-2} (y-x)^2 \end{aligned} \tag{2.2}$$

where $M := \|f\|$. Using (2.2), positivity and linearity of the operators L_n, we have

$$\begin{aligned} |L_n(f;x) - f(x)| \leq\ & L_n(|f(y) - f(x)|\,; x) \\ & + |f(x)| \, |L_n(f_0;x) - f_0(x)| \\ \leq\ & \varepsilon L_n(f_0;x) + \frac{2M}{\delta^2} L_n(\varphi;x) \\ & + M \, |L_n(f_0;x) - f_0(x)| \\ \leq\ & \varepsilon + \alpha \left\{ |L_n(f_0;x) - f_0(x)| + L_n(\varphi;x) \right\} \end{aligned}$$

where $\alpha := \max \left\{ \varepsilon + M, \ \dfrac{2M}{\delta^2} \right\}$. This implies that

$$\|L_n(f) - f\| \leq \varepsilon + \alpha \left\{ \|L_n(f_0) - f_0\| + \|L_n(\varphi)\| \right\}. \tag{2.3}$$

Given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. Define

$$\begin{aligned} D &: = \{ n : \|L_n(f) - f\| \geq r \}, \\ D_1 &: = \left\{ n : \|L_n(f_0) - f_0\| \geq \frac{r - \varepsilon}{2\alpha} \right\}, \\ D_2 &: = \left\{ n : \|L_n(\varphi)\| \geq \frac{r - \varepsilon}{2\alpha} \right\}. \end{aligned}$$

Then it follows from (2.3) that $D \subseteq D_1 \cup D_2$. So we get, for all $j \in \mathbb{N}$, that

$$\sum_{n \in D} a_{jn} \leq \sum_{n \in D_1} a_{jn} + \sum_{n \in D_2} a_{jn}. \tag{2.4}$$

Letting $j \to \infty$ in (2.4) and using the hypotheses we have

$$\lim_j \sum_{n \in D} a_{jn} = 0,$$

which yields the proof. ∎

Let δ be a positive real number so that $\delta < \frac{b-a}{2}$, and let

$$\|f\|_\delta := \sup_{a+\delta \le x \le b-\delta} |f(x)|\,, \quad f \in C[a,b].$$

In order to give our main result we need the following lemmas.

Theorem 2.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that δ is a fixed positive number such that $\delta < \frac{b-a}{2}$. If the conditions*

$$st_A - \lim_n \int_{-\delta}^{\delta} K_n(y)dy = 1 \tag{2.5}$$

and

$$st_A - \lim_n \left(\sup_{|y|\ge\delta} K_n(y) \right) = 0 \tag{2.6}$$

hold, then for the operators L_n given by (2.1), *we have*

$$st_A - \lim_n \|L_n(f_0) - f_0\|_\delta = 0 \text{ with } f_0(y) = 1.$$

Proof. Fix $0 < \delta < \frac{b-a}{2}$ and let $x \in [a+\delta, b-\delta]$. Then it is easy to see that

$$-(b-a) \le a - x \le -\delta \tag{2.7}$$

and

$$\delta \le b - x \le b - a. \tag{2.8}$$

It follows from (2.1) that, for all $n \in \mathbb{N}$,

$$L_n(f_0;x) = \int_a^b K_n(y-x)dy = \int_{a-x}^{b-x} K_n(y)dy. \tag{2.9}$$

Taking into consideration (2.7), (2.8) and (2.9) we have

$$\int_{-\delta}^{\delta} K_n(y)dy \le L_n(f_0;x) \le \int_{-(b-a)}^{b-a} K_n(y)dy. \tag{2.10}$$

Hence (2.10) and (2.7) imply that

$$\|L_n(f_0) - f_0\|_\delta \le u_n \tag{2.11}$$

where
$$u_n := \max\left\{\left|\int_{-\delta}^{\delta} K_n(y)dy - 1\right|, \left|\int_{-(b-a)}^{b-a} K_n(y)dy - 1\right|\right\}.$$

Note that since condition (2.5) holds for all $\delta > 0$ such that $\delta < (b-a)/2$, it is clear that

$$st_A - \lim_n u_n = 0. \tag{2.12}$$

Now, for a given $\varepsilon > 0$, we get from (2.11) that

$$D := \{n : \|L_n(f_0) - f_0\|_\delta \geq \varepsilon\} \subseteq \{n : u_n \geq \varepsilon\} =: D'.$$

Then, for all $j \in \mathbb{N}$, we have

$$\sum_{n \in D} a_{jn} \leq \sum_{n \in D'} a_{jn}. \tag{2.13}$$

Taking limit as $j \to \infty$ in (2.13) and using (2.12) we immediately conclude the result. ■

Lemma 2.3. *Let $A = (a_{jn})$ be a non-negative regular summability matrix. If* (2.5) *and* (2.6) *hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for the operators L_n given by* (2.1), *we have*

$$st_A - \lim_n \|L_n(\varphi)\|_\delta = 0 \text{ with } \varphi(y) := (y-x)^2.$$

Proof. For a fixed $0 < \delta < \frac{b-a}{2}$, let $x \in [a+\delta, b-\delta]$. Note that, for $x \in [a+\delta, b-\delta]$, since $\varphi(y) = y^2 - 2xy + x^2$, it is obvious that $\varphi \in C[a,b]$ for each $x \in [a+\delta, b-\delta]$. So we can compute $L_n(\varphi; x)$. Actually, $L_n(\varphi; x) = L_n(f_2; x) - 2xL_n(f_1; x) + x^2 L_n(f_0; x)$ with $f_i(y) = y^i$ $(i = 0, 1, 2)$. Then using (2.1), (2.7) and (2.8) we get, for all $n \in \mathbb{N}$, that

$$L_n(\varphi; x) = \int_{a-x}^{b-x} y^2 K_n(y)dy \leq \int_{-(b-a)}^{b-a} y^2 K_n(y)dy. \tag{2.14}$$

Since the function f_2 is continuous at $y = 0$, given $\varepsilon > 0$ there exists $\eta > 0$ such that $y^2 < \varepsilon$ for all y satisfying $|y| \leq \eta$. Here we have two cases such that $\eta \geq b - a$ or $\eta < b - a$.

Case 1. Let $\eta \geq b-a$. Then it follows from (2.14) that

$$0 \leq L_n(\varphi; x) \leq \varepsilon^2 \int\limits_{-(b-a)}^{b-a} K_n(y)dy,$$

and hence, by (2.5) the proof is completed.

Case 2. Now let $\eta < b-a$. Then we can write from (2.14) that

$$L_n(\varphi; x) \leq \int\limits_{|y|\geq\eta} y^2 K_n(y)dy + \int\limits_{|y|\leq\eta} y^2 K_n(y)dy$$

and hence we obtain, for all $n \in \mathbb{N}$, that

$$\|L_n(\varphi)\|_\delta \leq a_n \left(\frac{(b-a)^3 - \eta^3}{3} \right) + \varepsilon^2 b_n, \tag{2.15}$$

where

$$a_n := \sup_{|y|\geq\eta} K_n(y) \quad \text{and} \quad b_n := \int\limits_{|y|\leq\eta} K_n(y)dy.$$

Observe that conditions (2.5) and (2.6) yield $st_A-\lim_n a_n = 0$ and $st_A-\lim_n b_n = 1$, respectively. Taking $M := \max\left\{ \frac{(b-a)^3 - \eta^3}{3},\ \varepsilon^2 \right\}$ in (2.15), we conclude, for all $n \in \mathbb{N}$, that

$$\|L_n(\varphi)\|_\delta \leq \varepsilon^2 + M\left(a_n + |b_n - 1|\right). \tag{2.16}$$

Given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon^2 < r$. Define the following sets:

$$\begin{aligned} D &:= \left\{ n : \|L_n(\varphi)\|_\delta \geq r \right\}, \\ D_1 &:= \left\{ n : a_n \geq \frac{r-\varepsilon^2}{2M} \right\}, \\ D_2 &:= \left\{ n : |b_n - 1| \geq \frac{r-\varepsilon^2}{2M} \right\}. \end{aligned}$$

Then, by (2.16) we immediately get $D \subseteq D_1 \cup D_2$. Hence, for all $j \in \mathbb{N}$, we have

$$\sum_{n\in D} a_{jn} \leq \sum_{n\in D_1} a_{jn} + \sum_{n\in D_2} a_{jn}. \tag{2.17}$$

Since $st_A-\lim_n a_n = st_A-\lim_n |b_n - 1|$, letting $j \to \infty$ in (2.17) the proof follows. ■

Now the following main result follows from Theorem 2.1, Lemmas 2.2 and 2.3 at once.

Theorem 2.4. *Let* $A = (a_{jn})$ *be a non-negative regular summability matrix and let* $\{L_n\}$ *be a sequence of convolution operators given by* (2.1)*. If conditions* (2.5) *and* (2.6) *hold for a fixed* $\delta > 0$ *such that* $\delta < \frac{b-a}{2}$*, then for all* $f \in C[a,b]$*, we have*

$$st_A - \lim_n \|L_n(f) - f\|_\delta = 0. \tag{2.18}$$

If we take $A = I$, the identity matrix, we then get the following

Corollary 2.5. *Assume that* δ *is a fixed positive number such that* $\delta < \frac{b-a}{2}$*. If the conditions*

$$\lim_n \int_{-\delta}^{\delta} K_n(y-x)dy = 1 \quad \textit{and} \quad \lim_n \left(\sup_{|y| \geq \delta} K_n(y) \right) = 0$$

hold, then for all $f \in C[a,b]$*, we have*

$$\lim_n \|L_n(f) - f\|_\delta = 0,$$

i.e., for all $f \in C[a,b]$*, the sequence* $\{L_n(f)\}$ *is uniformly convergent to* f *on the interval* $[a+\delta, b-\delta]$.

Remark. We now exhibit a sequence of positive convolution operators for which Corollary 2.5 does not apply but our Theorem 2.4 does.

Let $A = (a_{jn})$ be a non-negative regular summability matrix for which $\lim_j \max_n\{a_{jn}\} = 0$. In this case $A-$statistical convergence is stronger than ordinary convergence [14]. So we can choose a sequence (d_n) which is $A-$statistically null but non-convergent. Without loss of generality we may assume that (d_n) is non-negative. Otherwise we would replace (d_n) by $(|d_n|)$. Now let the operators L_n on $C[a,b]$ be defined by

$$L_n(f;x) = \frac{n(1+d_n)}{\sqrt{\pi}} \int_a^b f(y) e^{-n^2(y-x)^2} dy. \tag{2.19}$$

If we choose

$$K_n(y) = \frac{n(1+d_n)}{\sqrt{\pi}} e^{-n^2 y^2}, \tag{2.20}$$

then the operators L_n given by (2.19) have form of the convolution operators as in (2.1).

Observe that the functions K_n given by (2.20) do not satisfy the hypotheses of Corollary 2.5. However, we now show that each function K_n in (2.20) satisfies conditions (2.5) and (2.6). Indeed, for every $\delta > 0$ such that $\delta < \frac{b-a}{2}$, we have

$$\begin{aligned}\int_{-\delta}^{\delta} K_n(y)dy &= \frac{n(1+d_n)}{\sqrt{\pi}}\left(\int_{-\infty}^{\infty} e^{-n^2y^2}dy - \int_{|y|\geq\delta} e^{-n^2y^2}dy\right)\\ &= \frac{2(1+d_n)}{\sqrt{\pi}}\left(\int_{0}^{\infty} e^{-y^2}dy - \int_{\delta.n}^{\infty} e^{-y^2}dy\right).\end{aligned}$$

Since $\int_0^\infty e^{-y^2}dy = \dfrac{\sqrt{\pi}}{2} < \infty$, it is clear that $\lim_n \int_{\delta.n}^{\infty} e^{-y^2}dy = 0$. Also, since $st_A - \lim_n(1+d_n) = 1$, we immediately get

$$st_A - \lim_n \int_{-\delta}^{\delta} K_n(y)dy = 1,$$

which gives (2.5).

On the other hand, we have

$$\begin{aligned}\sup_{|y|\geq\delta} K_n(y) &= \frac{n(1+d_n)}{\sqrt{\pi}} \sup_{|y|\geq\delta} e^{-n^2y^2}\\ &\leq \frac{n(1+d_n)}{e^{n^2\delta^2}}.\end{aligned}$$

Since $\lim_n \dfrac{n}{e^{n^2\delta^2}} = 0$ and $st_A - \lim_n(1+d_n) = 1$, we conclude that

$$st_A - \lim_n\left(\sup_{|y|\geq\delta} K_n(y)\right) = 0,$$

hence (2.6) holds. Therefore, by Theorem 2.4, the operators L_n given by (2.19) satisfy condition (2.18) for all $f \in C[a,b]$.

3. Rates of A-Statistical Convergence

In this section, using the modulus of continuity we study rates of $A-$ statistical convergence in Theorem 2.4.

The concepts of the rates of $A-$statistical convergence have been introduced in [5] as follows:

Let $A = (a_{jn})$ be a non-negative regular summability matrix and let (a_n) be a positive non-increasing sequence of real numbers. Then a sequence $x = (x_n)$ is $A-$statistically convergent to a number L with the rate of $o(a_n)$ if for every $\varepsilon > 0$, $\lim_j \frac{1}{a_j} \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0$. In this case we write $x_n - L = st_A - o(a_n)$, (as $n \to \infty$). If for every $\varepsilon > 0$, $\sup_j \frac{1}{a_j} \sum_{n:|x_n| \geq \varepsilon} a_{jn} < \infty$, then x is $A-$statistically bounded with the rate of $O(a_n)$ and it is denoted by $x_n = st_A - O(a_n)$, (as $n \to \infty$). In the above two definitions the "rate" is more controlled by the entries of the summability method rather than the terms of the sequence $x = (x_n)$. For instance, when one takes the identity matrix I, if $a_{nn} = o(a_n)$ then $x_n - L = st_A - o(a_n)$ for any convergent sequence $(x_n - L)$ regardless of how slowly it goes to zero. To avoid such an unfortunate situation we may consider the concept of convergence in measure from measure theory to define the rate of convergence as follows: $x = (x_n)$ is $A-$statistically convergent to L with the rate of $o_m(a_n)$, denoted by $x_n - L = st_A - o_m(a_n)$, (as $n \to \infty$), if for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n - L| \geq \varepsilon a_n} a_{jn} = 0$. Finally, the sequence $x = (x_n)$ is $A-$statistically bounded with the rate of $O_m(a_n)$ provided that there is a positive number M such that $\lim_j \sum_{n:|x_n| \geq M a_n} a_{jn} = 0$. In this case we write $x_n = st_A - O_m(a_n)$, (as $n \to \infty$).

Let $f \in C[a, b]$. The modulus of continuity (see, for instance, [15]), denoted by $w(f, \alpha)$, is defined to be

$$w(f, \alpha) = \sup_{|y-x| \leq \alpha} |f(y) - f(x)| .$$

The modulus of continuity of the function f in $C[a, b]$ gives the maximum oscillation of f in any interval of length not exceeding $\alpha > 0$. It is well-known that if $f \in C[a, b]$, then

$$\lim_{\alpha \to 0} w(f, \alpha) = w(f, 0) = 0,$$

and that for any constants $c > 0$, $\alpha > 0$,

$$w(f, c\alpha) \leq (1 + [c])\, w(f, \alpha), \tag{3.1}$$

where $[c]$ is defined to be the greatest integer less than or equal to c.

Hence we get the following

Theorem 3.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}$ be a sequence of convolution operators given by* (2.1). *Assume further* *that* (a_n) *and* (b_n) *are two positive non-increasing sequences. If, for a fixed* $\delta > 0$ *such that* $\delta < \frac{b-a}{2}$,

$$\|L_n(f_0) - f_0\|_\delta = st_A - o(a_n), \text{ (as } n \to \infty), \tag{3.2}$$

and

$$w(f, \alpha_n) = st_A - o(b_n), \ \text{(as } n \to \infty), \tag{3.3}$$

where $\alpha_n := \sqrt{\|L_n(\varphi)\|_\delta}$, *then for all* $f \in C[a, b]$, *we have*

$$\|L_n(f) - f\|_\delta = st_A - o(c_n), \text{ (as } n \to \infty),$$

where $c_n := \max\{a_n, b_n\}$. *Similar results hold when little "o" is replaced by big "O".*

Proof. Let $0 < \delta < \frac{b-a}{2}$, $f \in C[a, b]$ and $x \in [a + \delta, b - \delta]$. By positivity and linearity of the operators L_n and using inequality (3.1), we get, for any $\alpha > 0$, that

$$\begin{aligned}|L_n(f; x) - f(x)| &\le L_n\left(|f(y) - f(x)|; x\right) + |f(x)|\,|L_n(f_0; x) - f_0(x)| \\ &\le L_n\left(w\left(f, \alpha\frac{|y-x|}{\alpha}\right); x\right) + |f(x)|\,|L_n(f_0; x) - f_0(x)| \\ &\le w(f, \alpha) L_n\left(1 + \left[\frac{|y-x|}{\alpha}\right]; x\right) + |f(x)|\,|L_n(f_0; x) - f_0(x)| \\ &\le w(f, \alpha)\left\{L_n(f_0; x) + \frac{1}{\alpha^2} L_n(\varphi; x)\right\} + |f(x)|\,|L_n(f_0; x) - f_0(x)|.\end{aligned}$$

This yields that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|L_n(f) - f\|_\delta \le w(f, \alpha)&\left\{\|L_n(f_0)\|_\delta + \frac{1}{\alpha^2}\|L_n(\varphi)\|_\delta\right\} \\ &+ M_1\|L_n(f_0) - f_0\|_\delta\end{aligned} \tag{3.4}$$

where $M_1 := \|f\|_\delta$. Now letting $\alpha := \alpha_n = \sqrt{\|L_n(\varphi)\|_\delta}$ in (3.4), we have

$$\begin{aligned}\|L_n(f) - f\|_\delta &\le w(f, \alpha_n)\{\|L_n(f_0)\|_\delta + 1\} + M_1\|L_n(f_0) - f_0\|_\delta \\ &\le 2w(f, \alpha_n) + w(f, \alpha_n)\|L_n(f_0) - f_0\|_\delta + M_1\|L_n(f_0) - f_0\|_\delta.\end{aligned}$$

Let $M := \max\{2, M_1\}$. Then we can write, for all $n \in \mathbb{N}$, that

$$\begin{aligned}\|L_n(f) - f\|_\delta \le M\{&w(f, \alpha_n) + w(f, \alpha_n)\|L_n(f_0) - f_0\|_\delta \\ &+ \|L_n(f_0) - f_0\|_\delta\}\end{aligned} \tag{3.5}$$

Given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} D &: = \{n : \|L_n(f) - f\|_\delta \geq \varepsilon\}, \\ D_1 &: = \left\{n : w(f, \alpha_n) \geq \frac{\varepsilon}{3M}\right\}, \\ D_2 &: = \left\{n : w(f, \alpha_n) \|L_n(f_0) - f_0\|_\delta \geq \frac{\varepsilon}{3M}\right\}, \\ D_3 &: = \left\{n : \|L_n(f_0) - f_0\|_\delta \geq \frac{\varepsilon}{3M}\right\}. \end{aligned}$$

Then we easily see from (3.5) that $D \subseteq D_1 \cup D_2 \cup D_3$. Also, defining

$$D_2' := \left\{n : w(f, \alpha_n) \geq \sqrt{\frac{\varepsilon}{3M}}\right\}$$

and

$$D_2'' := \left\{n : \|L_n(f_0) - f_0\|_\delta \geq \sqrt{\frac{\varepsilon}{3M}}\right\}$$

one can deduce that $D_2 \subseteq D_2' \cup D_2''$. Hence we get $D \subseteq D_1 \cup D_2' \cup D_2'' \cup D_3$. Since $c_n = \max\{a_n, b_n\}$, we obtain, for all $j \in \mathbb{N}$, that

$$\begin{aligned} \frac{1}{c_j} \sum_{n \in D} a_{jn} \leq \frac{1}{b_j} \sum_{n \in D_1} a_{jn} + \frac{1}{b_j} \sum_{n \in D_2'} a_{jn} + \frac{1}{a_j} \sum_{n \in D_2''} a_{jn} \\ + \frac{1}{a_j} \sum_{n \in D_3} a_{jn}. \end{aligned} \tag{3.6}$$

Letting $j \to \infty$ in (3.6) and using (3.2) and (3.3) we have

$$\lim_j \frac{1}{c_j} \sum_{n \in D} a_{jn} = 0,$$

whence the result. ∎

Finally, the above proof can easily be modified to prove the following analog.

Theorem 3.2. *Let $A = (a_{jn})$, $\{L_n\}$, (α_n), (a_n) and (b_n) be as in Theorem 3.1. If, for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$,*

$$\|L_n(f_0) - f_0\|_\delta = st_A - o_m(a_n), \ (as\ n \to \infty),$$

and

$$w(f, \alpha_n) = st_A - o_m(b_n), \ (as\ n \to \infty),$$

then, for all $f \in C[a, b]$, we have

$$\|L_n(f) - f\|_\delta = st_A - o_m(c_n), \ (as\ n \to \infty),$$

where $c_n := \max\{a_n, b_n, a_n b_n\}$. Similar conclusions hold when little "o_m" is replaced by big "O_m".

4. Some Further Results

In this section using the $A-$statistical convergence, we deal with an approximation result by positive convolution operators defined on C^*, the space of all $2\pi-$periodic and continuous functions on the whole real axis with the usual norm

$$\|f\|_{C^*} = \sup_{x\in\mathbb{R}} |f(x)|, \quad f \in C^*.$$

We now consider the convolution operators L_n defined on C^* by

$$L_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)K_n(y-x)dy, \quad n \in \mathbb{N} \text{ and } f \in C^*, \tag{4.1}$$

where $K_n \in C^*$ for all $n \in \mathbb{N}$ and $K_n(y) \geq 0$ for every $y \in [-\pi, \pi]$. So K_n is non-negative on the whole real axis. Then using the similar technique as in the proof of Theorem 2.4 one can also get the following result.

Theorem 4.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}$ be a sequence of convolution operators given by* (4.1)*. If*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)dy = 1 \quad a.\ a.\ n$$

and, for any $\delta > 0$

$$st_A - \lim_n \left(\sup_{|y|\geq\delta} K_n(y) \right) = 0,$$

then for all $f \in C^*$, we have

$$st_A - \lim_n \|L_n(f) - f\|_{C^*} = 0.$$

Of course, if the matrix A in Theorem 4.1 is replaced by the identity matrix I, then we immediately get the classical approximation result (see, e.g., [16, p. 9]).

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