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APPROXIMATION OF COMMON FIXED POINTS OF FAMILIES OF NONEXPANSIVE MAPPINGS*

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Abstract. Let X be a reflexive and smooth Banach space which has a weakly sequentially continuous duality mapping. We consider in this paper the iteration scheme $x_{n+1} = \lambda_{n+1}y + (1 - \lambda_{n+1})T_{n+1}x_n$ for infinitely many nonexpansive maps $T_1, T_2, T_3, ...$ in X as well as for finitely many nonexpansive retraction. We establish several strong convergence results which generalize [10, Theorem 3.3] and [10, Theorem 4.1] from Hilbert space setting to Banach space setting.

1. Introduction

In 1967 for N nonexpansive maps $T_1, T_2, ..., T_N$, Halpern [7] first introduced the iteration scheme

$$x_{n+1} = \lambda_{n+1}y + (1 - \lambda_{n+1})T_{n+1}x_n$$

in which he considered the case when $y = 0$ and $N = 1$; i.e., one map T. He proved that the conditions $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$ were necessary conditions for the convergence of the iterates to a fixed point of T. In 1977 Lions [9] considered the above scheme with the additional assumption $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$ on the parameters and proved convergence of the iterates. In 1983 Reich [13] posed the following problem:

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In a Banach space, what conditions on the sequence $\{\lambda_n\}$ of parameters will ensure convergence of the iterates?

In 1992 Wittmann [19] proved convergence of the iterates in a Hilbert space under the assumption that the parameters satisfy $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$ in addition to the above two necessary conditions. In 1994 under the assumption that the parameters satisfy the two necessary conditions and are decreasing, Reich [12] proved strong convergence of the iterates for the case of a single map (i.e., $N = 1$) in a uniformly smooth Banach space which has a weakly continuous duality map. In 1996 Bauschke [1] generalized Wittmann's result to finitely many maps where $T_n := T_{n \bmod N}$. The additional condition imposed by him on the parameters was $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$. He also provided an algorithmic proof which has been used successfully with modifications by many authors [4, 15, 23]. In 1997 Shioji and Takahashi [17] extended Wittmann's result to a Banach space. This paper provides some answers to the problem posed by Reich [13] by introducing a new condition on the parameters $\lim_{n\to\infty} \lambda_n/\lambda_{n+N} = 1$ in the framework of a Hilbert space. Shimizu and Takahashi (see [15, Theorem 1]) in 1997 considered the above iteration scheme with the necessary conditions on the parameters and some additional conditions imposed on the mappings. In 2003 O'Hara, Pillay and Xu [10] established the following strong convergence result in a Hilbert space which generalizes Theorem 1 of Shimizu and Takahashi [15].

Theorem 1.1. [10, Theorem 3.3]. *Let* $\{\lambda_n\} \subset (0,1)$ *satisfy* $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$. *Let* C *be a nonempty, closed and convex subset of a Hilbert space* H *and let* $T_n : C \to C (n = 1, 2, 3, ...)$ *be nonexpansive mappings such that*

$$F := \bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$$

where $Fix(T_n) = \{x \in C : x = T_n x\},\ n = 1, 2, 3, ...$ *Assume that* $V_1, ..., V_N : C \to C$ *are nonexpansive mappings with the property: for all* $k = 1, 2, ..., N$ *and for any bounded subset* $\tilde{C}$ *of* C, *there holds*

$$\lim_{n\to\infty} \sup_{x\in\tilde{C}} \|T_n x - V_k(T_n x)\| = 0.$$

For $x_0 \in C$ *and* $y \in C$ *define*

$$(1) \qquad x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n \quad n \geq 0.$$

Then $\{x_n\}$ *converges strongly to* Py *where* P *is the projection from* H *onto* $\bigcap_{k=1}^{N} Fix(V_k)$.

Furthermore for the same iteration scheme (1) with finite many maps $T_1, T_2, ..., T_N$, O'Hara, Pillay and Xu [10] established the following complementary result to Theorem 3.1 of Bauschke [1] with condition $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$ replaced by condition $\lim_{n\to\infty} \lambda_n/\lambda_{n+N} = 1$.

Theorem 1.2. [10, Theorem 4.1]. *Let Cbe a nonempty, closed and convex subset of a Hilbert space H and let $T_1, T_2, ..., T_N$ be nonexpansive self-mappings of C with $F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. Assume that*

$$F = Fix(T_N...T_1) = Fix(T_1T_N...T_2) = \cdots = Fix(T_{N-1}T_{N-2}...T_1T_N).$$

Let $\{\lambda_n\} \subset (0, 1)$ satisfy the following conditions:

(1) $\lim_{n\to\infty} \lambda_n = 0$;

(2) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(3) $\lim_{n\to\infty} \lambda_n/\lambda_{n+N} = 1$.

Given points $x_0, y \in C$, the sequence $\{x_n\} \subset C$ is defined by

$$x_{n+1} = \lambda_{n+1}y + (1 - \lambda_{n+1})T_{n+1}x_n \quad n \geq 0.$$

Then $\{x_n\}$ converges strongly to $P_F y$ where P_F is the projection of C onto F.

Let X be a reflexive Banach space which has a weakly sequentially continuous duality map. For example, every l^p $(1 < p < \infty)$ space has a weakly sequentially continuous duality map with gauge function $\varphi(t) = t^{p-1}$. In this paper the iteration scheme (1) is considered for infinitely many nonexpansive maps $T_1, T_2, T_3, ...$ in X. Theorem 3.3 of O'Hara, Pillay and Xu [10] is extended to the setting of Banach space X and it is shown that the sequence of iterates converges strongly to Py where P is some sunny and nonexpansive retraction. For this same iteration scheme (1) with finitely many nonexpansive maps $T_1, T_2, ..., T_N$ in X, Theorem 4.1 of O'Hara, Pillay and Xu [10] is also extended to the setting of Banach space X under the same conditions imposed by them on the sequence $\{\lambda_n\}$ of parameters. The iterates converge strongly to Py where P is the sunny and nonexpansive retraction onto the intersection of the fixed point sets of the T_i, $i = 1, 2, ..., N$.

2. Preliminaries

Throughout this paper let X be a real Banach space and X^* be its dual space. Let C be a nonempty subset of X and $T : C \to C$ be a mapping of C into itself. T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The fixed point set of T is denoted by $Fix(T) := \{x \in C : Tx = x\}$. The notation $\rightharpoonup$ denotes weak convergence and the notation $\to$ denotes strong convergence. By a gauge function we mean a continuous strictly increasing function φ defined on $R^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. The mapping $J_\varphi : X \to 2^{X^*}$ defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x^*\| = \varphi(\|x\|)\}, \quad \forall x \in X$$

is called the duality mapping with gauge function φ. In particular the duality mapping with gauge function $\varphi(t) = t$ denoted by J is referred to as the normalized duality mapping. Browder [2] initiated the study of certain classes of nonlinear operators by means of the duality mapping J_φ. Set for every $t \geq 0$,

$$\Phi(t) = \int_0^t \varphi(r) dr.$$

Then it is known [8, p. 1350] that $J_\varphi(x)$ is the subdifferential of the convex functional $\Phi(\|\cdot\|)$ at x. Thus it is easy to see that the normalized duality mapping $J(x)$ can also be defined as the subdifferential of the convex functional $\Phi(\|x\|) = \|x\|^2/2$, that is

$$(2) \quad J(x) = \partial\Phi(\|x\|) = \{f \in X^* : \Phi(\|y\|) - \Phi(\|x\|) \geq \langle y - x, f\rangle \ \forall y \in X\} \quad \forall x \in X.$$

We will use the following properties of duality mappings.

Proposition 2.1. [22, p. 193-194].

(*i*) *$J = I$ (i.e., the identity mapping of X) if and only if X is a Hilbert space.*

(*ii*) *J is surjective if and only if X is reflexive.*

(*iii*) *$J_\varphi(\lambda x) = \operatorname{sign}(\lambda)(\varphi(|\lambda| \cdot \|x\|)/\|x\|)J(x)$ $\forall x \in X \setminus \{0\}, \lambda \in R$ where R is the set of all real numbers; in particular $J(-x) = -J(x), \forall x \in X$.*

Recall that a Banach space X is said to satisfy Opial's condition [11] if for any sequence $\{x_n\}$ in X the condition that $\{x_n\}$ converges weakly to $x \in X$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X, y \neq x$. It is known [18] that any separable Banach space can be equivalently renormed so that it satisfies Opial's condition. Recall also that X is said to have a weakly sequentially continuous duality mapping if there exists a gauge function φ such that the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology; i.e., for any sequence $\{x_n\}$ in X, if $x_n \to x$ in X, then $J_\varphi(x_n) \to J_\varphi(x)$ in the weak* topology of X. A space with a weakly sequentially continuous duality mapping is easily seen to satisfy Opial's condition; see [2] for more details. Every l^p space $(1 < p < \infty)$ has a weakly sequentially continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$.

Let $U = \{x \in X : \|x\| = 1\}$, the unit sphere of X. The norm of X is said to be Gâteaux differentiable if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. In this case X is said to be smooth. It is known [24] that X is smooth if and only if the normalized duality mapping J is single-valued. In this case, the normalized duality mapping J is continuous from the strong topology to the weak* topology. Moreover, if X admits a weakly sequentially continuous duality mapping, then X satisfies Opial's condition and X is smooth, see Lemma 1 in [25].

In the sequel we will use the following concepts and lemmas.

Lemma 2.1. (see [6, Lemma 4]). *Let X be a Banach space satisfying Opial's condition and let C be a nonempty, closed and convex subset of X. Let $T : C \to C$ be a nonexpansive mapping. Then $(I - T)$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C which converges weakly to x and if the sequence $\{x_n - Tx_n\}$ converges strongly to zero, then $x - Tx = 0$.*

Lemma 2.2. *Let φ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$, and let*

$$\Phi(t) = \int_0^t \varphi(r)dr.$$

Then there holds the following inequality

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x + y)\rangle, \ \forall x, y \in X,$$

where $j_\varphi(x + y) \in J_\varphi(x + y)$.

Proof. The proof of this lemma is essentially due to Lim and Xu [8]. For the completeness, we give its proof. Indeed it is known that $J_\varphi(x)$ is the subdifferential of the convex function $\Phi(\|\cdot\|)$ at x, that is,

$$J_\varphi(x) = \partial\Phi(\|x\|) = \{f \in X^* : \Phi(\|y\|) - \Phi(\|x\|) \geq \langle y - x, f\rangle \ \forall y \in X\}.$$

Consequently, it follows that for each $x, y \in X$

$$\Phi(\|x\|) - \Phi(\|x + y\|) \geq \langle x - (x + y), j_\varphi(x + y)\rangle \quad \forall j_\varphi(x + y) \in J_\varphi(x + y).$$

The conclusion follows from the above inequality. ∎

Let C be a convex subset of X, K be a nonempty subset of C and let P be a retraction from C onto K, i.e., $Px = x$ for each $x \in K$. P is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. If there is a sunny and nonexpansive retraction from C onto K, K is said to be a sunny and nonexpansive retract of C. For a sunny and nonexpansive retraction, there exists the following useful characterization.

Lemma 2.3 [16, Proposition 4, p. 59]. *Let C be a convex subset of a smooth Banach space X, K be a nonempty subset of C and let P be a retraction from C onto K. Then P is sunny and nonexpansive if and only if for all $x \in C$ and $y \in K$,*

$$\langle x - Px, J(y - Px)\rangle \leq 0.$$

Hence there is at most one sunny and nonexpansive retraction from C onto K. More information regarding sunny and nonexpansive retractions can be found in [5, 14].

Remark 2.1. If $X = H$ is a real Hilbert space and C is a nonempty, closed and convex subset of H, then every nearest point projection of H onto C is a sunny and nonexpansive retraction of H onto C where the mapping $P_C : H \to C$ is defined as follows: for each $x \in H$, $P_C x$ is the unique element of C that satisfies $\|x - P_C x\| = d(x, C) := \inf_{y\in C} \|x - y\|$. Indeed it is easy to see that P_C is a retraction of H onto C. Moreover it follows from Lemma 2.3 in [10] that for all $x \in H$ and $y \in C$,

$$\langle x - P_C x, P_C x - y\rangle \geq 0.$$

According to Lemma 2.3, we know that P_C is a sunny and nonexpansive retraction of H onto C.

Lemma 2.4. (see [1]). *Let $\{\lambda_n\}$ be a sequence in $[0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 0$. Then*

$$\sum_{n=1}^{\infty} \lambda_n = \infty \Leftrightarrow \prod_{n=1}^{\infty}(1 - \lambda_n) = 0.$$

Lemma 2.5. [10, Lemma 2.2]. *Let $\{\lambda_n\}$ be a sequence in $[0, 1]$ that satisfies $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $\{a_n\}$ be a sequence of nonnegative real numbers that satisfies any one of the following conditions:*

(a) *For all $\varepsilon > 0$, there exists an integer $N \geq 1$ such that for all $n \geq N$,*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n\varepsilon.$$

(b) $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n c_n$ *where* $\limsup_{n\to\infty} c_n \leq 0$.
Then $\lim_{n\to\infty} a_n = 0$.

The proof of Lemma 2.5 can be found in [20].

3. Strong Convergence by Imposing Conditions on the Mappings

In this section we establish the following strong convergence result in a real Banach space which generalizes Theorem 3.3 of O'Hara, Pillay and Xu [10].

Theorem 3.1. *Let* $\{\lambda_n\} \subset (0,1)$ *satisfy* $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$. *Let* X *be a reflexive Banach space which has a weakly sequentially continuous duality mapping* J_φ *with gauge function* φ. *Let* C *be a nonempty, closed and convex subset of* X *and let* $T_n : C \to C (n = 1, 2, 3, ...)$ *be nonexpansive mappings such that*

$$F := \bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset.$$

Assume that $V_1, ..., V_N : C \to C$ *are nonexpansive mappings with the following property: for all* $k = 1, 2, ..., N$ *and for any bounded subsets* $\tilde{C}$ *of* C, *there holds*

$$\lim_{n\to\infty} \sup_{x\in\tilde{C}} \|T_n x - V_k(T_n x)\| = 0. \tag{3}$$

For $x_0 \in C$ *and* $y \in C$ *define*

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n \quad n \geq 0.$$

If there exists a sunny and nonexpansive retraction P *of* C *onto* $\bigcap_{k=1}^{N} Fix(V_k)$, *then*

$$\limsup_{n\to\infty} \langle y - Py, J_\varphi(x_n - Py)\rangle \leq 0.$$

Suppose additionally that Py *lies in* F. *Then* $x_n \to Py$.

Proof. The proof given below employs the same idea as in the proof of Theorem 3.3 [10]. We note that assumption (3) implies that $\bigcap_{k=1}^{N} Fix(V_k) \supset F$. We proceed with the following steps.

Step 1. We claim that for all $n \geq 0$,

$$\|x_n - f\| \leq \max\{\|x_0 - f\|, \|y - f\|\} \quad \forall f \in F.$$

Indeed we use an inductive argument. The result is clearly true for $n = 0$. Suppose the result is true for n. Let $f \in F$. Then by the nonexpansivity of T_{n+1},

$$\begin{aligned} \|x_{n+1} - f\| &= \|\lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n - f\| \\ &= \|\lambda_{n+1}(y - f) + (1 - \lambda_{n+1})(T_{n+1} x_n - f)\| \\ &\leq \lambda_{n+1}\|y - f\| + (1 - \lambda_{n+1})\|T_{n+1} x_n - f\| \\ &\leq \lambda_{n+1}\|y - f\| + (1 - \lambda_{n+1})\|x_n - f\| \end{aligned}$$

$$\begin{aligned}&\le \lambda_{n+1}\max\{\|x_0-f\|,\|y-f\|\}\\ &\quad +(1-\lambda_{n+1})\max\{\|x_0-f\|,\|y-f\|\}\\ &= \max\{\|x_0-f\|,\|y-f\|\}.\end{aligned}$$

Step 2. We claim that $\{x_n\}$ is bounded. Indeed for all $n \ge 0$ and for any $f \in F$,

$$\begin{aligned}\|x_n\| &\le \|x_n-f\|+\|f\|\\ &\le \max\{\|x_0-f\|,\|y-f\|\}+\|f\|.\end{aligned}$$

Step 3. Step 3: We claim that $\{T_{n+1}x_n\}$ is bounded. Indeed for all $n \ge 0$ and for any $f \in F$,

$$\begin{aligned}\|T_{n+1}x_n\| &\le \|T_{n+1}x_n-f\|+\|f\|\\ &\le \|x_n-f\|+\|f\|\\ &\le \max\{\|x_0-f\|,\|y-f\|\}+\|f\|.\end{aligned}$$

Step 4. We claim that $x_{n+1}-T_{n+1}x_n \to 0$. Indeed we have

$$\begin{aligned}\|x_{n+1}-T_{n+1}x_n\| &= \lambda_{n+1}\|y-T_{n+1}x_n\|\\ &\le \lambda_{n+1}(\|y\|+\|T_{n+1}x_n\|)\\ &\le \lambda_{n+1}(\|y\|+M) \quad \text{for some } M.\end{aligned}$$

Since $\lambda_{n+1} \to 0$, we obtain $x_{n+1}-T_{n+1}x_n \to 0$.

Step 5. We claim that $\limsup_{n\to\infty}\langle y-Py, J_\varphi(x_{n+1}-Py)\rangle \le 0$. Indeed, since X is reflexive and $\{x_n\}$ is bounded by Step 2, there exists a subsequence $\{x_{n_j+1}\}$ of $\{x_n\}$ such that

$$x_{n_j+1} \rightharpoonup p$$

for some $p \in C$ and

$$\limsup_{n\to\infty}\langle y-Py, J_\varphi(x_{n+1}-Py)\rangle = \lim_{j\to\infty}\langle y-Py, J_\varphi(x_{n_j+1}-Py)\rangle.$$

By our assumption we have for any $k = 1, 2, ..., N$ and for $\tilde{C} = \{x_n\}$,

$$\begin{aligned}0 = \lim_{n\to\infty}\sup_{x\in\tilde{C}}\|T_{n+1}x-V_k(T_{n+1}x)\| &\ge \limsup_{n\to\infty}\|T_{n+1}x_n-V_k(T_{n+1}x_n)\|\\ &\ge \limsup_{j\to\infty}\|T_{n_j+1}x_{n_j}-V_k(T_{n_j+1}x_{n_j})\|.\end{aligned}$$

Thus

$$\lim_{j\to\infty} \|T_{n_j+1}x_{n_j} - V_k(T_{n_j+1}x_{n_j})\| = 0 \quad \text{for all } k = 1, 2, ..., N.$$

Therefore $p \in Fix(V_k)$ for $k = 1, 2, ..., N$ by Lemma 2.1; i.e., $p \in \bigcap_{k=1}^{N} Fix(V_k)$. Thus we deduce from Lemma 2.3 that

$$\begin{aligned}\limsup_{n\to\infty}\langle y - Py, J_\varphi(x_{n+1} - Py)\rangle &= \lim_{j\to\infty}\langle y - Py, J_\varphi(x_{n_j+1} - Py)\rangle \\ &= \langle y - Py, J_\varphi(p - Py)\rangle \le 0,\end{aligned}$$

since $p \in \bigcap_{k=1}^{N} Fix(V_k)$.

Step 6. Suppose additionally that Py lies in F. Then we claim that $x_n \to Py$. Indeed using Lemma 2.2, we obtain

$$\begin{aligned}&\Phi(\|x_{n+1} - Py\|)\\ =\ &\Phi(\|(1-\lambda_{n+1})(T_{n+1}x_n - Py) + \lambda_{n+1}(y - Py)\|)\\ \le\ &\Phi(\|(1-\lambda_{n+1})(T_{n+1}x_n - Py)\|) + \lambda_{n+1}\langle y - Py, J_\varphi(x_{n+1} - Py)\rangle\\ \le\ &(1-\lambda_{n+1})\Phi(\|x_n - Py\|) + \lambda_{n+1}\langle y - Py, J_\varphi(x_{n+1} - Py)\rangle.\end{aligned}$$

Applying Lemma 2.5, we conclude that $\Phi(\|x_n - Py\|) \to 0$; that is, $\|x_n - Py\| \to 0$. Consequently, $x_n \to Py$. The proof is now complete. ■

4. Strong Convergence by Imposing Conditions on the Parameters

In 1996 Bauschke [1] defined the following control conditions on the parameters $\{\lambda_n\}$:

[B1] $\lim_{n\to\infty} \lambda_n = 0$.

[B2] $\sum_{n=1}^{\infty} \lambda_n = \infty$.

[B3] $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$.

In 2003 O'Hara, Pillay and Xu [10] replaced [B3] by the condition

[N3] $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$.

This condition also improves Lions' condition [9] as follows

[L3] $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0$.

Note that both [N3] and [B3] cover the natural candidate of $\lambda_n = (n+1)^{-1}$ but [L3] does not. However [B3] and [N3] are independent of each other (even coupled with conditions [B1] and [B2]); see [20]. Theorem 4.1 given in [10] is a complementary result to Theorem 3.1 of Bauschke [1] with condition [B3] replaced by condition [N3]. Its proof employs the same idea as in the proof of Theorem 3.1 [1]. We will now extend Theorem 4.1 [10] to the setting of Banach space X under the same conditions as those imposed on the parameters $\{\lambda_n\}$ in [10, Theorem 4.1].

We consider N maps $T_1, T_2, ..., T_N$. For $n > N$, set

$$T_n := T_{n\mathrm{mod}N},$$

where $n\mathrm{mod}N$ is defined as follows: if $n = kN + l$ $0 \le l < N$, then

$$n\mathrm{mod}N := \begin{cases} l, \text{ if } l \neq 0, \\ N, \text{ if } l = 0. \end{cases}$$

Theorem 4.1. *Let X be a reflexive Banach space which has a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty, closed and convex subset of X and let $T_1, T_2, ..., T_N$ be nonexpansive self-mappings of C with $F := \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. Assume that*

$$F = Fix(T_N ... T_1) = Fix(T_1 T_N ... T_2) = \cdots = Fix(T_{N-1} T_{N-2} \cdots T_1 T_N).$$

Let $\{\lambda_n\} \subset (0, 1)$ satisfy the following conditions:

$[N1]$ $\lim_{n\to\infty} \lambda_n = 0$.

$[N2]$ $\sum_{n=1}^\infty \lambda_n = \infty$.

$[N3]$ $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$.

Given points $x_0, y \in C$, the sequence $\{x_n\} \subset C$ is defined by

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n \quad n \ge 0.$$

If there exists a sunny and nonexpansive retraction P_F of C onto F, then $x_n \to P_F y$.

Proof. Following the idea of the proof in [10, Theorem 4.1], we divide the proof into several steps.

Step 1. $\|x_n - f\| \le \max\{\|x_0 - f\|, \|y - f\|\}$ for all $n \ge 0$ and for all $f \in F$.

Step 2. $\{x_n\}$ is bounded.

Step 3. $\{T_{n+1} x_n\}$ is bounded.

Step 4. $x_{n+1} - T_{n+1} x_n \to 0$.

Step 5. $x_{n+N} - x_n \to 0$.

Step 6. $x_n - T_{n+N}...T_{n+1}x_n \to 0.$

Step 7. $\limsup_{n\to\infty}\langle y - P_F y, J_\varphi(x_n - P_F y)\rangle \le 0.$

Step 8. $x_n \to P_F y.$

At first it is easy to see that Steps 1-4 are the same as those in Theorem 3.1 and the proofs are thus omitted. Nex we give the proofs of Steps 5-8, respectively.
Step 5: By Step 3, there exists a constant $L > 0$ such that for all $n \ge 1$,

$$\|y - T_{n+1}x_n\| \le L.$$

Since for all $n \ge 1,\ T_{n+N} = T_n$, we have

$$\begin{aligned}\|x_{n+N} - x_n\| &= \|(\lambda_{n+N} - \lambda_n)(y - T_{n+N}x_{n+N-1}) \\ &\quad +(1-\lambda_{n+N})(T_n x_{n+N-1}) - T_n x_{n-1}\| \\ &\le L|\lambda_{n+N} - \lambda_n| + (1-\lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| \\ &= (1-\lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| + \lambda_{n+N}L\left|1 - \frac{\lambda_n}{\lambda_{n+N}}\right|.\end{aligned}$$

By [N3] we have $\lim_{n\to\infty} L|1 - \frac{\lambda_n}{\lambda_{n+N}}| = 0$ and so by Lemma 2.5,

$$x_{n+N} - x_n \to 0.$$

Step 6: The proof of this step is taking from Step 4 in the proof of Theorem 3.2 [21]; see [21, p. 195]. Noting that each T_i is nonexpansive and using Step 4, we get the finite table

$$\begin{aligned}&x_{n+N} - T_{n+N}x_{n+N-1} \to 0, \\ &T_{n+N}x_{n+N-1} - T_{n+N}T_{n+N-1}x_{n+N-2} \to 0, \\ &\qquad\qquad\vdots \\ &T_{n+N}...T_{n+2}x_{n+1} - T_{n+N}...T_{n+1}x_n \to 0.\end{aligned}$$

Adding up this table yields

$$x_n - T_{n+N}...T_{n+1}x_n \to 0.$$

Step 7: By Step 2, $\{\langle y - P_F y, J_\varphi(x_n - P_F y)\rangle\}$ is bounded and hence

$$\limsup_{n\to\infty}\langle y - P_F y, J_\varphi(x_n - P_F y)\rangle$$

exists. Thus we can pick a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty}\langle y - P_F y, J_\varphi(x_n - P_F y)\rangle = \lim_{i\to\infty}\langle y - P_F y, J_\varphi(x_{n_i} - P_F y)\rangle$$

and $x_{n_i} \rightharpoonup p$ for some $p \in C$.

The proof of $p \in F$ given below is taking from Step 5 in the proof of Theorem 3.2 [21]; see [21, p. 195]. Since the pool of mappings $\{T_i : 1 \le i \le N\}$ is finite, we may further assume (passing to a further subsequence if necessary) that for some integer $k \in \{1, 2, ..., N\}$,

$$n_i \mathrm{mod} N \equiv k, \quad \forall i \ge 1.$$

Then it follows from Step 6 that

$$x_{n_i} - T_{k+N}...T_{k+1}x_{n_i} \to 0.$$

Hence by Lemma 2.1, we conclude that $p \in Fix(T_{k+N}...T_{k+1})$ which implies that $p \in F$ from our assumption. Now by similar argument of Step 5 in the proof of Theorem 3.1, we can show that

$$\limsup_{n\to\infty}\langle y - P_F y, J_\varphi(x_n - P_F y)\rangle \le 0.$$

Finally Step 8 can be shown by the same argument of Step 6 in the proof of Theorem 3.1. The proof is now complete. ■

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