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α-SKEW ARMENDARIZ MODULES AND α-SEMICOMMUTATIVE MODULES

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Abstract. Let α be a ring endomorphism. We introduce α-skew Armendariz modules and α-semicommutative modules which are generalizations of Armendariz modules and semicommutative modules, respectively. And investigate their properties. Moreover, we study the relationship between a module and its polynomial module.

1. Introduction

All rings are associative and have identity, and modules are unitary right modules. $R[x]$ denotes the polynomial ring over a ring R and $M[x]$ denotes the polynomial module over a module M. Rege and Chhawchharia [9] introduced the notion of an Armendariz ring. Recently, many authors have studied Armendariz rings and given various generalizations. According to Hong, Kim and Kwak [4], for an endomorphism α of a ring R, R is called α-skew Armendariz if $p(x)q(x) = 0$ where $p(x) = \sum_{i=0}^{m} a_i x^i$ and $q(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\alpha]$ implies $a_i \alpha^i(b_j) = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. Chen [2] proved that for an endomorphism α of a ring R and $\alpha^l = 1_R$ for some positive integer l, R is α-skew Armendariz iff the polynomial ring $R[x]$ over R is α-skew Armendariz. Huh, Lee and Smoktunowicz [6] made a comparative study of Armendariz rings and semi-commutative rings. Armendariz rings need not be semicommutative rings by [6, Example 14] and semicommutative rings need not be Armendariz rings by [4, Example 3.2]. A right R-module M is an Armendariz module if $m(x)g(x) = 0$ where $m(x) = \sum_{i=0}^{t} m_i x^i \in M[x]$ and $g(x) = \sum_{j=0}^{n} a_j x^j \in R[x]$ implies $m_i a_j = 0$ for every i and j. Right R-module M is semi-commutative if $ma = 0$ implies $mRa = 0$ for $m \in M$ and $a \in R$. A

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ring R is reduced if $a^2 = 0$ implies $a = 0$ for $a \in R$. Buhphang and Rege [1] studied the basic properties of Armendariz modules and semi-commutative modules. Moreover, they proved that all flat modules over a reduced ring are both Armendariz and semi-commutative. For an endomorphism α of a ring R, a right R-module M is called α-reduced if for $m \in M$ and $a \in R$ (1) $ma = 0$ implies $mR \cap Ma = 0$ (2) $ma = 0$ iff $m\alpha(a) = 0$. If $\alpha = 1_R$, α-reduced module is called reduced module. Lee and Zhou [8] introduced those notions and proved that a right R-module M is reduced iff $M[x]/M[x](x^n)$ is an Armendariz module over $R[x]/(x^n)$ for integer $n \geq 2$.

In this paper, we introduce the notions of α-skew Armendariz module and α-semicommutative module for an endomorphism α of a ring R. Furthermore, we show that for an endomorphism α of a ring R (1) R is α-skew Armendariz if and only if every flat right R-module is α-skew Armendariz; (2) R is α-semicommutative if and only if every flat right R-module is α-semicommutative; (3) If $\alpha^l = 1_R$ for some positive integer l, then right R-module M is α-skew Armendariz if and only if $M[x]$ is α-skew Armendariz over $R[x]$; (4) If $\alpha^l = 0$ for some positive integer l, then M is α-reduced if and only if $M[x]/M[x](x^n)$ is an α-skew Armendariz module over $R[x]/(x^n)$ for integer $n \geq 2$

2. The Properties and the Equivalent Conditions

Let α be an endomorphism of a ring R and M be a right R-module. $M[x;\alpha] = \{\sum_{i=0}^{s} m_i x^i; s \geq 0, m_i \in M\}$ is an Abelian group under an obvious addition operation. Moreover, $M[x;\alpha]$ becomes a module over $R[x;\alpha]$ under the following scalar product operation: For $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x;\alpha]$, $m(x)f(x) = \sum_k (\sum_{i+j=k} m_i \alpha^i(a_j))x^k$. M is called α-Armendariz [8] if (1) $ma = 0$ iff $m\alpha(a) = 0$ for $m \in M$ and $a \in R$; (2) $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x;\alpha]$ implies $m_i \alpha^i(a_j) = 0$ for all i and j.

Definition 2.1. Let α be an endomorphism of a ring R and M be a right R-module. M is called α-skew Armendariz if $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x;\alpha]$ implies $m_i \alpha^i(a_j) = 0$ for all i and j.

We can easily prove that a ring R is α-skew Armendariz iff R_R is α-skew Armendariz, and a right R-module M is Armendariz iff it is 1_R-skew Armendariz. So α-skew Armendariz modules are not necessarily Armendariz by [4]. Moreover, α-Armendariz module is α-skew Armendariz module, but the converse may not be true. For R_4 in the following example is not α-Armendariz over R_4, however, Chen [3] proved that it is α-skew Armendariz.

Example 2.2. Let S be a domain and $R_4=\left\{\left.\begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix}\right| a, a_{ij}\in S\right\}$.

Define $\alpha : R_4 \to R_4$ by $\alpha(x) = \text{diag}(a, a, a, a)$ for any $x=\begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix}$ $\in R_4$, then R_4 is α-skew Armendariz.

Proof. Suppose that $f(x) = A_0 + A_1x + \cdots + A_nx^n$, and $g(x) = B_0 + B_1x + \cdots + B_nx^n \in R_4[x;\alpha]$ with $f(x)g(x) = 0$. We need to prove that $A_i\alpha^i(B_j) = 0$ for all i and j. Since $\alpha^2 = \alpha$, we only need $A_i\alpha(B_j) = 0$. Put

$$A_j = \begin{pmatrix} a^{(j)} & a_{12}^{(j)} & a_{13}^{(j)} & a_{14}^{(j)} \\ 0 & a^{(j)} & a_{23}^{(j)} & a_{24}^{(j)} \\ 0 & 0 & a^{(j)} & a_{34}^{(j)} \\ 0 & 0 & 0 & a^{(j)} \end{pmatrix} \text{ and } B_j = \begin{pmatrix} b^{(j)} & b_{12}^{(j)} & b_{13}^{(j)} & b_{14}^{(j)} \\ 0 & b^{(j)} & b_{23}^{(j)} & b_{24}^{(j)} \\ 0 & 0 & b^{(j)} & b_{34}^{(j)} \\ 0 & 0 & 0 & b^{(j)} \end{pmatrix}.$$

Case 1. A_0 is invertible. From $f(x)g(x) = 0$, we have $B_0 = 0$. We claim that $B_j = 0$ for all $0 \le j \le n$. If not, there exists the least k such that $B_k \ne 0$ and $B_0 = \cdots = B_{k-1} = 0$. Since $A_0B_k + A_1\alpha(B_{k-1}) + \cdots + A_k\alpha(B_0) = 0$, we have $A_0B_k = 0$ and hence $B_k = 0$, which is a contradiction.

Case 2. B_0 is invertible. Similar to the proof of case 1, we can get $A_i = 0$ for all i.

Case 3. Both A_0 and B_0 are not invertible. In the case of $A_0 \ne 0$, we claim that $\alpha(B_j) = 0$ for all j. If not, there exists the least j such that $\alpha(B_j) \ne 0$. From equation $A_0B_j + A_1\alpha(B_{j-1}) + \cdots + A_j\alpha(B_0) = 0$, we have $A_0B_j = 0$. Since $\alpha(B_j) \ne 0$, $b^{(j)} \ne 0$ and so $A_0 = 0$, a contradiction. If $A_0 = 0$, then we claim that $A_i = 0$ or $\alpha(B_j) = 0$ for all i and j. Assume to the contrary, there exist the least i and the least j such that $A_i \ne 0$ and $\alpha(B_j) \ne 0$. Now $f(x)g(x) = 0$ gives $A_0B_{i+j} + \cdots + A_{i-1}\alpha(B_{j+1}) + A_i\alpha(B_j) + A_{i+1}\alpha(B_{j-1}) + \cdots + A_{i+j}\alpha(B_0) = 0$. It follows that $A_i\alpha(B_j) = 0$. On the other hand, $\alpha(B_j) \ne 0$ implies that $b^{(j)} \ne 0$ and so $A_i = 0$, which is a contradiction. From the above discussion we have $A_i\alpha(B_j) = 0$ for all i and j. Hence R_4 is an α-skew Armendariz ring.

Definition 2.3. Let α be an endomorphism of a ring R and M be a right R-module. M is called α-semicommutative if $ma = 0$ implies $mR\alpha(a) = 0$ for $m \in M$ and $a \in R$.

A ring R is α-semicommutative if R_R is α-semicommutative. It is clear that a right R-module M is semicommutative iff it is 1_R-semicommutative. One may suspect that α-semicommutative modules are semi-commutative, however, the following example erases the possibility.

Example 2.4. R_4 in Example 2.2 is α-semicommutative.

Proof. Suppose $A = \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix}, B = \begin{pmatrix} b & b_{12} & b_{13} & b_{14} \\ 0 & b & b_{23} & b_{24} \\ 0 & 0 & b & b_{34} \\ 0 & 0 & 0 & b \end{pmatrix}$ $\in R_4$ and $AB = 0$, then we have

$$\begin{aligned} &ab = 0 \\ &ab_{12} + a_{12}b = 0 \\ &ab_{13} + a_{12}b_{23} + a_{13}b = 0 \\ &ab_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b = 0 \\ &ab_{23} + a_{23}b = 0 \\ &ab_{24} + a_{23}b_{34} + a_{24}b = 0 \\ &ab_{34} + a_{34}b = 0 \end{aligned}$$

Since S is a domain, $ab = 0$ implies $a = 0$ or $b = 0$. If $b = 0$, then $\alpha(B) = 0$, so $AR_4\alpha(B) = 0$. If $b \neq 0$, then $a = 0$. Using the equations above, we have $a_{12} = a_{13} = a_{14} = a_{23} = a_{24} = a_{34} = 0$. Thus $A = 0$, so $AR_4\alpha(B) = 0$. Therefore R_4 is α-semicommutative.

However, R_4 is not semi-commutative by [7, Example 1.3].

Remark 2.5. Let R be a subring of a ring S with $1_S \in R$ and $M_R \subseteq L_S$. Let α be an endomorphism of S such that $\alpha(R) \subseteq R$. If L_S is α-skew Armendariz (α-semicommutative), then M_R is also α-skew Armendariz (α-semicommutative).

Proposition 2.6. *Let α be an endomorphism of a ring R. The class of α-skew Armendariz (α-semicommutative) modules is closed under direct sums, direct products and submodules.*

An R-module M is torsionless if it is a submodule of a direct product of copies of R. If M is a faithful R-module, then R is a submodule of a direct product of copies of M. The following corollary is easy to be obtained by Proposition 2.6.

Corollary 2.7. ([1, Theorem 2.7]) *The following conditions are equivalent.*

(1) *R is an Armendariz (semicommutative) ring;*

(2) *Every torsionless R-module is Armendariz (semicommutative);*

(3) *Every submodule of a free R-module is Armendariz (semicommutative);*

(4) *There exists a faithful R-module which is Armendariz (semicommutative)*

Proposition 2.8. *Let α be an endomorphism of a ring R. An R-module M is α-skew Armendariz (α-semicommutative) if and only if every finitely generated (cyclic) submodule of M is α-skew Armendariz (α-semicommutative).*

The following conclusion is the generalization of Proposition 2.3 in [1].

Proposition 2.9. *Let α be an endomorphism of a commutative domain D and M be a torsion free D-module. Then M is α-skew Armendariz (α-semicommutative).*

Proposition 2.10. *Let α be a monomorphism of a commutative domain D and M be a D-module. Then M is α-skew Armendariz (α-semicommutative) if and only if its torsion submodule $T(M)$ is α-skew Armendariz (α-semicommutative).*

Proof. Let $m(x) = \sum_{i=0}^{t} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{n} a_j x^j \in D[x;\alpha]$ satisfy $m(x)f(x) = 0$, we have

$$m_0 a_0 = 0 \qquad (1)$$

$$m_0 a_1 + m_1 \alpha(a_0) = 0 \qquad (2)$$

$$m_0 a_2 + m_1 \alpha(a_1) + m_2 \alpha^2(a_0) = 0 \qquad (3)$$

$$\cdots$$

$$m_t \alpha^t(a_n) = 0 \qquad (n+t+1)$$

We can assume $a_0 \neq 0$, then $m_0 \in T(M)$ by (1). Multiplying (2) by a_0 from the right, one obtains $m_1\alpha(a_0)a_0 = 0$. Since α is monic and D is a domain, so $m_1 \in T(M)$. Multiplying (3) by $\alpha(a_0)a_0$ from the right, we obtain $m_2\alpha^2(a_0)\alpha(a_0)a_0 = 0$, so $m_2 \in T(M)$. Continuing this process, we have $m(x) \in T(M)[x]$. Since $T(M)$ is α-skew Armendariz, we conclude that $m_i\alpha^i(a_j) = 0$ for all i and j , proving that M is α-skew Armendariz. The other implication is trivial.

The proof of α-semicommutative module is similar to that above.

The following two results are the generalizations of the Theorem 2.15 and the Theorem 2.16 in [1], respectively.

Theorem 2.11. *Let α be an endomorphism of a ring R. R is α-skew Armendariz if and only if every flat right R-module is α-skew Armendariz.*

Proof. Let M be a flat right R-module. Let $0 \to K \to F \to M \to 0$ be an exact sequence with F free over R. (In what follows, for an element y of F, we denote $\overline{y} = y + K$ in M). Let $f(x) = \sum_{i=0}^{t} \overline{y}_i x^i \in M[x;\alpha]$ and $g(x) = \sum_{j=0}^{n} a_j x^j \in R[x;\alpha]$ satisfy $f(x)g(x) = 0$, then we have

$$\begin{aligned}
&\overline{y}_0 a_0 = 0\\
&\overline{y}_0 a_1 + \overline{y}_1 \alpha(a_0) = 0\\
&\overline{y}_0 a_2 + \overline{y}_1 \alpha(a_1) + \overline{y}_2 \alpha^2(a_0) = 0\\
&\quad \cdots\\
&\overline{y}_t \alpha^t(a_n) = 0
\end{aligned}$$

Therefore the elements y_0a_0, $y_0a_1 + y_1\alpha(a_0)$, $\cdots$, $y_t\alpha^t(a_n)$ all belong to K. Since M is a flat R-module, there exists an R-module homomorphism $v : F \to K$ such that $v(y_0a_0) = y_0a_0$, $v(y_0a_1 + y_1\alpha(a_0)) = y_0a_1 + y_1\alpha(a_0)$, $\cdots$, $v(y_t\alpha^t(a_n)) = y_t\alpha^t(a_n)$.

Write $w_i := v(y_i) - y_i$ for $i = 0, 1, \cdots, t$. Each w_i is an element of F, therefore the polynomial $h(x) = \sum_{i=0}^{t} w_i x^i \in F[x;\alpha]$ and $h(x)g(x) = 0$. Since R is α-skew Armendariz and F is a free R-module, F is α-skew Armendariz by Proposition 2.6. Thus, we have $w_i\alpha^i(a_j) = 0$ for all i and j. It follows that $y_i\alpha^i(a_j) \in K$ for all i and j, so $\overline{y}_i\alpha^i(a_j) = 0$ in M, proving that M is α-skew Armendariz. The other implication is obvious.

Theorem 2.12. *Let α be an endomorphism of a ring R. R is α-semicommutative if and only if every flat right R-module is α-semicommutative.*

Proof. The proof is similar to that of the Theorem 2.11.

Let α be an endomorphism of a ring R and M be a right R-module. According to Lee and Zhou [8], M is called α-reduced if, for any $m \in M$ and $a \in R$,
(1) $ma = 0$ implies $mR \cap Ma = 0$;
(2) $ma = 0$ iff $m\alpha(a) = 0$.

M is reduced if M is 1_R-reduced. It is clear that α-reduced module is reduced.

Lemma 2.13. ([8, Lemma 1.2]). *Let M be a right R-module M. The following are equivalent.*

(*1*) *M is α-reduced;*

(*2*) *The following conditions hold: For any $m \in M$ and $a \in R$,*

(a) $ma = 0$ *implies* $mRa = mR\alpha(a) = 0$*;*

(b) $ma\alpha(a) = 0$ *implies* $ma = 0$*;*

(c) $ma^2 = 0$ *implies* $ma = 0$.

R is called α-rigid [5] if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. It is easy to show that α-rigid ring is reduced.

Lemma 2.14. ([5, Lemma 4]). *Let α be an endomorphism of a ring R. R_R is α-reduced if and only if R is an α-rigid ring.*

If R is α-rigid, then R is α-skew Armendariz by [4, Corollary 4]. Therefore, if R_R is α-reduced, then R is α-skew Armendariz as well as α-semicommutative by Lemma 2.13 and 2.14.

By a regular ring we mean a von Neumann regular ring. It is well-known that all modules over a regular ring are flat, therefore the following result is immediate.

Remark 2.15. Let α be an endomorphism of a ring R. If R_R is α-reduced and R is a regular ring, then all right R-modules are α-skew Armendariz as well as α-semicommutative.

3. Polynomial Modules Over Polynomial Rings

In this section, we study the relations between right R-module M and the polynomial module $M[x]$ over M.

Proposition 3.1. *Let α be an endomorphism of a ring R and M be a right R-module. If M is α-skew Armendariz, then the following conditions are equivalent.*
(*1*) *M is α-semicommutative and semicommutative;*
(*2*) *$M[x;\alpha]$ is semicommutative over $R[x;\alpha]$.*

Proof. (1)⇒(2) Let M be an α-semicommutative and semicommutative right R-module. Let $m(x) = \sum_{i=0}^{t} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{n} a_j x^j \in R[x;\alpha]$ satisfy $m(x)f(x) = 0$. Since M is α-skew Armendariz, so $m_i\alpha^i(a_j) = 0$ for each i and j. Let $h(x) = \sum_{k=0}^{v} b_k x^k \in R[x;\alpha]$, and let c_0, c_1, c_2, $\cdots$, c_{t+v+n} be the coefficients of $m(x)h(x)f(x)$, then

$$c_0 = m_0 b_0 a_0$$
$$c_1 = m_0 b_0 a_1 + (m_0 b_1 + m_1\alpha(b_0))\alpha(a_0)$$
$$c_2 = m_0 b_0 a_2 + (m_0 b_1 + m_1\alpha(b_0))\alpha(a_1) + (m_0 b_2 + m_1\alpha(b_1) + m_2\alpha^2(b_0))\alpha^2(a_0)$$
$$\cdots$$
$$c_{t+v+n} = m_t\alpha^t(b_v)\alpha^{t+v}(a_n)$$

Since M is α-semicommutative and semicommutative, $m_0a_0 = 0$ implies $m_0R\alpha(a_0) = 0$ and $m_0Ra_0 = 0$, hence $c_0 = 0$. $m_0a_1 = 0$ and $m_1\alpha(a_0) = 0$ which imply $m_0Ra_1 = 0$ and $m_1R\alpha(a_0) = 0$, we have $c_1 = 0$. Continuing we get $c_i = 0$ for all i. Hence $m(x)h(x)f(x) = 0$, proving that $M[x; \alpha]$ is a semicommutative $R[x; \alpha]$-module.

(2) ⇒ (1) Clearly M is semicommutative. If $ma = 0$ for $m \in M$ and $a \in R$, then $mR[x; \alpha]a = 0$ since $M[x; \alpha]$ is semicommutative. So $mrxa = 0$ for any $r \in R$, and $mr\alpha(a)x = 0$, $mr\alpha(a) = 0$. Therefore M is α-semicommutative.

Corollary 3.2. *Let M be an Armendariz right R-module. The following conditions are equivalent.*
(*1*) *M is semicommutative;*
(*2*) *$M[x]$ is semicommutative over $R[x]$.*

Recall that if α is an endomorphism of a ring R, then the map $R[x] \to R[x]$ defined by $\sum_{i=0}^{m} a_ix^i \mapsto \sum_{i=0}^{m} \alpha(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α. We shall also denote the extended map $R[x] \to R[x]$ by α and the image of $f \in R[x]$ by $\alpha(f)$. By Hong, Kim and Kwak [4], R is α-skew Armendariz if and only if $R[x]$ is α-skew Armendariz provided $\alpha^l = 1_R$ for some positive integer l, but the proof had a gap. Chen [2] gave a new proof. In the following, we generalize this to modules.

Theorem 3.3. *Let α be an endomorphism of a ring R and $\alpha^l = 1_R$ for some positive integer l. Then right R-module M is α-skew Armendariz if and only if $M[x]$ is α-skew Armendariz over $R[x]$.*

Proof. Assume that M is α-skew Armendariz. Suppose that $p(y) = \sum_{i=0}^{t} m_i(x)y^i \in M[x][y; \alpha]$, $q(y) = \sum_{j=0}^{n} g_j(x)y^j \in R[x][y; \alpha]$, and $p(y)q(y) = 0$. Let $m_i(x) = m_{i0} + m_{i1}x + \cdots + m_{is_i}x^{s_i} \in M[x]$ for $0 \le i \le t$ and $g_j(x) = b_{j0} + b_{j1}x + \cdots + b_{jw_j}x^{w_j} \in R[x]$ for $0 \le j \le n$. We need to prove that $m_i(x)\alpha^i(g_j(x)) = 0$ in $M[x]$ for all i and j. Take a positive integer k such that $k > deg(m_0(x)) + deg(m_1(x)) + \cdots + deg(m_t(x)) + deg(g_0(x)) + deg(g_1(x)) + \cdots + deg(g_n(x))$, where the degree of $m_i(x)$ is as a polynomial in $M[x]$, the degree of $g_j(x)$ is as a polynomial in $R[x]$ and the degree of zero polynomial is to be 0. Since $p(y)q(y) = 0$ in $M[x][y; \alpha]$, we have the equations system

$$m_0(x)g_0(x) = 0$$

$$m_0(x)g_1(x) + m_1(x)\alpha(g_0(x)) = 0$$

$$m_0(x)g_2(x) + m_1(x)\alpha(g_1(x)) + m_2(x)\alpha^2(g_0(x)) = 0$$

$$\cdots$$

$$m_t(x)\alpha^t(g_n(x)) = 0$$

in $M[x]$. Put $m(x) = m_0(x^l) + m_1(x^l)x^{lk+1} + m_2(x^l)x^{2lk+2} + \cdots + m_t(x^l)x^{tlk+t}$ and $g(x) = g_0(x^l) + g_1(x^l)x^{lk+1} + g_2(x^l)x^{2lk+2} + \cdots + g_n(x^l)x^{nlk+n}$. Then

$$\begin{aligned} m(x) = {} & m_{00} + m_{01}x^l + m_{02}x^{2l} + \cdots + m_{0s_0}x^{ls_0} \\ & + m_{10}x^{lk+1} + m_{11}x^{lk+l+1} + m_{12}x^{lk+2l+1} + \cdots + m_{1s_1}x^{lk+ls_1+1} \\ & + \quad \cdots \\ & + m_{t0}x^{tlk+t} + m_{t1}x^{tlk+l+t} + m_{t2}x^{tlk+2l+t} + \cdots + m_{ts_t}x^{tlk+s_tl+t} \end{aligned}$$

and

$$\begin{aligned} g(x) = {} & b_{00} + b_{01}x^l + b_{02}x^{2l} + \cdots + b_{0w_0}x^{lw_0} \\ & + b_{10}x^{lk+1} + b_{11}x^{lk+l+1} + b_{12}x^{lk+2l+1} + \cdots + b_{1w_1}x^{lk+lw_1+1} \\ & + \quad \cdots \\ & + b_{n0}x^{nlk+n} + b_{n1}x^{nlk+l+n} + b_{n2}x^{nlk+2l+n} + \cdots + b_{nw_n}x^{nlk+w_nl+n} \end{aligned}$$

Using the equations system above and $\alpha^l = 1_R$, we have $m(x)g(x) = 0$ in $M[x;\alpha]$. Since M is α-skew Armendariz and $\alpha^l = 1_R$, so $m_{iu}\alpha^i(b_{jv}) = m_{iu}\alpha^{ilk+ul+i}(b_{jv}) = 0$ for all $0 \le i \le t$, $0 \le j \le n$, $u \in \{0, 1, \cdots, s_0, \cdots, s_t\}$ and $v \in \{0, 1, \cdots, w_0, \cdots, w_n\}$. So we have $m_i(x)\alpha^i(g_j(x)) = 0$ for all $0 \le i \le t$ and $0 \le j \le n$ in $M[x]$. Hence $M[x]$ is α-skew Armendariz .

Obviously, if $M[x]$ is α-skew Armendariz, then M is α-skew Armendariz.

Corollary 3.4. ([4, Theorem 6]). *Let α be an endomorphism of a ring R and $\alpha^l = 1_R$ for some positive integer l. Then R is α-skew Armendariz if and only if $R[x]$ is α-skew Armendariz.*

We write $M_n(R)$ for the $n \times n$ matrix ring over R. For a right R-module M and $A = (a_{ij}) \in M_n(R)$, let $MA = \{(ma_{ij}) : m \in M\}$. For $n \ge 2$, let $V = \sum_{i=1}^{n-1} E_{i,i+1}$ where $\{E_{i,j} : 1 \le i, j \le n\}$ are the matrix units, and set $V_n(R) = RI_n + RV + \cdots + RV^{n-1}$, $V_n(M) = MI_n + MV + \cdots + MV^{n-1}$, then $V_n(R)$ is a ring and $V_n(M)$ becomes a right module over $V_n(R)$ under usual addition and multiplication of matrices. There is a ring isomorphism $\theta: V_n(R) \to R[x]/(x^n)$ given by $\theta(r_0I_n + r_1V + \cdots + r_{n-1}V^{n-1}) = r_0 + r_1x + \cdots + r_{n-1}x^{n-1} + (x^n)$ and an Abelian group isomorphism $\varphi: V_n(M) \to M[x]/M[x](x^n)$ given by $\varphi(m_0I_n + m_1V + \cdots + m_{n-1}V^{n-1}) = m_0 + m_1x + \cdots + m_{n-1}x^{n-1} + M[x](x^n)$ such that $\varphi(WA) = \varphi(W)\theta(A)$ for all $W \in V_n(M)$ and $A \in V_n(R)$. Lee and Zhou [8] proved that M_R is reduced iff $M[x]/M[x](x^n)$ is an Armendariz right R-module over $R[x]/(x^n)$ for integer $n \ge 2$. In the following we generalize this to α-reduced module. First we prove the Lemma 3.5.

Let α be an endomorphism of a ring R, the map $V_n(R) \to V_n(R)$ defined by $a_0 I_n + a_1 V + \cdots + a_{n-1} V^{n-1} \mapsto \alpha(a_0) I_n + \alpha(a_1) V + \cdots + \alpha(a_{n-1}) V^{n-1}$ is an endomorphism of $V_n(R)$. Similarly the map $R[x]/(x^n) \to R[x]/(x^n)$ defined by $a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + (x_n) \mapsto \alpha(a_0) + \alpha(a_1) x + \cdots + \alpha(a_{n-1}) x^{n-1} + (x^n)$ is an endomorphism of $R[x]/(x^n)$. We shall also denote the two maps above by α.

Lemma 3.5. *Let α be an endomorphism of a ring R. Then $V_n(M)$ is an α-skew Armendariz module over $V_n(R)$ if and only if $M[x]/M[x](x^n)$ is an α-skew Armendariz module over $R[x]/(x^n)$.*

Proof. Let $V_n(M)$ be an α-skew Armendariz module over $V_n(R)$. $p(y) = \sum_{i=0}^{t} \overline{m}_i(x) y^i \in (M[x]/M[x](x^n))[y;\alpha]$ and $q(y) = \sum_{j=0}^{s} \overline{f}_j(x) y^j \in (R[x]/(x^n))[y;\alpha]$ where $\overline{m}_i(x) = m_{i0} + m_{i1} x + \cdots + m_{i(n-1)} x^{n-1} + M[x](x^n)$, $\overline{f}_j(x) = a_{j0} + a_{j1} x + \cdots + a_{j(n-1)} x^{n-1} + (x^n)$, $m_{iu} \in M$, $a_{jv} \in R$, $0 \le i \le t$, $0 \le j \le s$ and $0 \le u, v \le n-1$ satisfy $p(y)q(y) = 0$, we have

$$\begin{aligned}
&\overline{m}_0(x)\overline{f}_0(x) = 0\\
&\overline{m}_0(x)\overline{f}_1(x) + \overline{m}_1(x)\alpha(\overline{f}_0(x)) = 0\\
&\overline{m}_0(x)\overline{f}_2(x) + \overline{m}_1(x)\alpha(\overline{f}_1(x)) + \overline{m}_2(x)\alpha^2(\overline{f}_0(x)) = 0\\
&\quad\cdots\\
&\overline{m}_t(x)\alpha^t(\overline{f}_s(x)) = 0
\end{aligned}$$

Let $W_i = m_{i0} I_n + m_{i1} V + \cdots + m_{i(n-1)} V^{n-1} \in V_n(M)$ and $A_j = a_{j0} I_n + a_{j1} V + \cdots + a_{j(n-1)} V^{n-1} \in V_n(R)$ for $0 \le i \le t$ and $0 \le j \le s$. Let $W(y) = \sum_{i=0}^{t} W_i y^i$ and $A(y) = \sum_{j=0}^{s} A_j y^j$, we have

$$\begin{aligned}
&W_0 A_0 = 0\\
&W_0 A_1 + W_1 \alpha(A_0) = 0\\
&W_0 A_2 + W_1 \alpha(A_1) + W_2 \alpha^2(A_0) = 0\\
&\quad\cdots\\
&W_t \alpha^t(A_s) = 0
\end{aligned}$$

By the equations system above, $W(y)A(y) = 0$ in $V_n(M)[y;\alpha]$. Since $V_n(M)$ is an α-skew Armendariz module, $W_i \alpha^i(A_j) = 0$ for all i and j. Therefore we have $\overline{m}_i(x)\alpha^i(\overline{f}_j(x)) = 0$ for all i and j, proving that $M[x]/M[x](x^n)$ is a α-skew Armendariz module over $R[x]/(x^n)$.

The proof of the other implication is similar to that above.

Theorem 3.6. *Let α be an endomorphism of a ring R and $\alpha^l = 1_R$ for some positive integer l. M is α-reduced if and only if $M[x]/M[x](x^n)$ is an α-skew Armendariz module over $R[x]/(x^n)$ for integer $n \ge 2$.*

Proof. Assume that M is α-reduced. By Lemma 3.5, it suffices to show that $V_n(M)$ is an α-skew Armendariz module over $V_n(R)$.

Suppose that $W(x)A(x) = 0$ where $W(x) = \sum_{i=0}^{t} W_i x^i \in V_n(M)[x;\alpha]$ and $A(x) = \sum_{j=0}^{s} A_j x^j \in V_n(R)[x;\alpha]$. Write $W_i = m_{i0}I_n + m_{i1}V + \cdots + m_{i(n-1)}V^{n-1}$ and $A_j = a_{j0}I_n + a_{j1}V + \cdots + a_{j(n-1)}V^{n-1}$ for $0 \le i \le t$ and $0 \le j \le s$. It follows from $W(x)A(x) = 0$ that $[m_0(x)I_n + m_1(x)V + \cdots + m_{n-1}(x)V^{n-1}][a_0(x)I_n + a_1(x)V + \cdots + a_{n-1}(x)V^{n-1}] = 0$ in $V_n(M[x;\alpha])$ where $m_u(x) = m_{0u} + m_{1u}x + \cdots + m_{tu}x^t$ and $a_v(x) = a_{0v} + a_{1v}x + \cdots + a_{sv}x^s$ for $0 \le u, v \le n-1$, and hence

$$m_0(x)a_0(x) = 0 \tag{1}$$
$$m_0(x)a_1(x) + m_1(x)a_0(x) = 0 \tag{2}$$
$$m_0(x)a_2(x) + m_1(x)a_1(x) + m_2(x)a_0(x) = 0 \tag{3}$$
$$\cdots$$
$$m_0(x)a_{n-1}(x) + m_1(x)a_{n-2}(x) + \cdots + m_{n-1}(x)a_0(x) = 0 \tag{$n-1$}$$

in $M[x;\alpha]$. Since M is α-reduced, so $M[x;\alpha]$ is reduced by [8, Theorem 1.6], $m_0(x)R[x;\alpha]a_0(x) = 0$. Multiplying (2) by $a_0(x)$ from the right, one obtains $m_1(x)(a_0(x))^2 = 0$, so $m_1(x)a_0(x) = 0$, $m_0(x)a_1(x) = 0$ which imply $m_1(x)R[x;\alpha]a_0(x) = 0$, $m_0(x)R[x;\alpha]a_1(x) = 0$. Multiplying (3) by $a_0(x)$ from the right, we have $m_2(x)(a_0(x))^2 = 0$, so $m_2(x)a_0(x) = 0$, thus (3) becomes

$$m_0(x)a_2(x) + m_1(x)a_1(x) = 0 \tag{$3'$}$$

Multiplying ($3'$) by $a_1(x)$ from the right, ($3'$) becomes $m_1(x)(a_1(x))^2 = 0$, $m_1(x)a_1(x) = 0$, so $m_0(x)a_2(x) = 0$. Continuing this process , we have $m_u(x)a_v(x) = 0$ in $M[x;\alpha]$ for all u and v with $0 \le u+v \le n-1$. It follows that

$$\begin{aligned} &m_{0u}a_{0v} = 0 \\ &m_{0u}a_{1v} + m_{1u}\alpha(a_{0v}) = 0 \\ &m_{0u}a_{2v} + m_{1u}\alpha(a_{1v}) + m_{2u}\alpha^2(a_{0v}) = 0 \\ &\quad\cdots \\ &m_{tu}\alpha^t(a_{sv}) = 0 \end{aligned}$$

for all u and v with $0 \le u+v \le n-1$. Since M is α-reduced, using the similar method above, we have $m_{iu}\alpha^i(a_{jv}) = 0$ for $0 \le i \le t$, $0 \le j \le s$, $0 \le u+v \le n-1$. So $W_i\alpha^i(A_j) = 0$ for all i and j, proving that $M[x]/M[x](x^n)$ is α-skew Armendariz.

Conversely, if $l = 1$, it is true by [8, Theorem 1.9]. So we can assume $l > 1$. Suppose that $ma = 0$ for $m \in M$ and $a \in R$, then $[mI_n + (mE_{1n})x][aI_n - (\alpha(a)E_{1n})x] = 0$. By Lemma 3.5, $V_n(M)$ is an α-skew Armendariz module over $V_n(R)$, so $m\alpha(a) = 0$. Suppose that $m\alpha(a) = 0$, then $m\alpha^{l-1}(a) = 0$, so $[mI_n + (mE_{1n})x][\alpha^{l-1}(a)I_n - (aE_{1n})x] = 0$, hence $ma = 0$. If $ma = 0$, we have $m\alpha^{l-1}(a) = 0$. Let $mr = m_1a \in mR \cap Ma$, $[mI_n + (m_1E_{1n})x][\alpha^{l-1}(a)I_n - (rE_{1n})x] = 0$, so $mr = 0$. Thus M is α-reduced.

Corollary 3.7. ([8, Theorem 1.9]). *Let $n \geq 2$ be an integer. Then M_R is reduced if and only if $M[x]/M[x](x^n)$ is an Armendariz right module over $R[x]/(x^n)$*

Let α be an endomorphism of a ring R. By Lemma 3.5, we can show that R is α-rigid if and only if $R[x]/(x^n)$ is α-skew Armendariz for integer $n \geq 2$ in [3]. R_R is α-reduced iff R is α-rigid by Lemma 2.14. So we have the following open question.

Is the condition $\alpha^l = 1_R$ superfluous in Theorem 3.6?

Lemma 3.8. *Let α be an endomorphism of a ring R. Then $V_n(M)$ is an α-semicommutative module over $V_n(R)$ if and only if $M[x]/M[x](x^n)$ is an α-semicommutative module over $R[x]/(x^n)$.*

Proof. The proof is similar to that of Lemma 3.5.

Theorem 3.9. *Let α be an endomorphism of a ring R. If M is α-reduced, then $M[x]/M[x](x^n)$ is an α-semicommutative module over $R[x]/(x^n)$ for integer $n \geq 2$.*

Proof. By Lemma 3.8, it suffices to show that $V_n(M)$ is an α-semicommutative module over $V_n(R)$.

Let $W = m_0I_n + m_1V + \cdots + m_{n-1}V^{n-1}$ and $A = a_0I_n + a_1V + \cdots + a_{n-1}V^{n-1}$ satisfy WA=0 where $W \in V_n(M)$ and $A \in V_n(R)$, we have

$$m_0a_0 = 0 \qquad (1)$$

$$m_0a_1 + m_1a_0 = 0 \qquad (2)$$

$$m_0a_2 + m_1a_1 + m_2a_0 = 0 \qquad (3)$$

$$\cdots$$

$$m_0a_{n-1} + m_1a_{n-2} + \cdots + m_{n-1}a_0 = 0 \qquad (n-1)$$

Since M is α-reduced, $m_0Ra_0 = 0$. Multiplying (2) by a_0 from the right, (2) becomes $m_1{a_0}^2 = 0$, so $m_1a_0 = 0$, $m_0a_1 = 0$. Thus $m_1Ra_0 = 0$, $m_0Ra_1 = 0$. Multiplying (3) by a_0 from the right, (3) becomes $m_2{a_0}^2 = 0$, so $m_2a_0 = 0$, we have

$$m_0a_2 + m_1a_1 = 0 \qquad (3')$$

Multiplying $(3')$ by a_1 from the right, $(3')$ becomes $m_1a_1^2 = 0$, so $m_1a_1 = 0$, $m_0a_2 = 0$. Continuing this process, we have $m_ia_j = 0$ for all i and j with $0 \leq i+j \leq n-1$, so $m_iR\alpha(a_j) = 0$ for all i and j with $0 \leq i+j \leq n-1$. Thus $WV_n(R)\alpha(A) = 0$, $V_n(M)$ is α-semicommutative.

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