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SPECIAL PROPERTIES OF MODULES OF GENERALIZED POWER SERIES

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Abstract. Let R be a ring, M a right R-module and $(S, \leq)$ a strictly ordered monoid. In this paper, a necessary and sufficient condition is given for modules under which $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$, the module of generalized power series with coefficients in M and exponents in S is a reduced, Baer, PP. quasi-Baer module, respectively.

1. Introduction

Throughout this paper all rings R are associative with identity and all modules M are unitary right R-modules. The notation $N \leq M$ means that N is a submodule of M, and $M[x]_{R[x]}$ (resp. $M[[x]]_{R[[x]]}$ or $M[[x, x^{-1}]]_{R[[x,x^{-1}]]}$) denotes polynomial (resp. power series or Laurent power series) extension of M_R. For any nonempty subset X of R, $r_R(X)$ (resp. $l_R(x)$) denotes the right (resp. left) annihilator of X in R. Any concept and notation not defined here can be found in [10-13, 15, 16].

A ring R is called reduced if R does not have nonzero nilpotent elements. The notion of reduced rings has been studied by many authors. Some of the known results on reduced rings can be recalled as follows: R is reduced if and only if $R[x]$ is reduced if and only if $R[[x]]$ is reduced; if S is a torsion-free and cancellative monoid and $\leq$ is a strict order on S, then it is shown in [6, Lemma 2.1] that R is reduced if and only if $[[R^{S,\leq}]]$, the ring of generalized power series with coefficients in R and exponents in S, is reduced; if R is a reduced ring, then it is shown in [1, Lemma 1] that R is an Armendariz ring where an Armendariz ring is any ring R such that if $(\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} b_j x^j) = 0$ in $R[x]$ then $a_i b_j = 0$ for all i and j; if S is a torsion-free and cancellative monoid, $\leq$ is a strict order on S and R is a reduced ring, then it is shown in [6, Lemma 3.1] that R is an S-Armendariz

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ring where an S-Armendariz ring is any ring R such that if f, g in $[[R^{S,\leq}]]$ satisfy $fg = 0$ then $f(u)g(v) = 0$ for all $u, v \in S$.

The concept of a reduced ring is very useful in the investigation of certain annihilator conditions of polynomial extensions of a ring R. A ring R is called Baer (resp. right PP) if the right annihilator of every nonempty subset (resp. every element) is generated by an idempotent. A well-known result of Armendariz [1] states that, for a reduced ring R, R is Baer (resp. right PP) if and only if so is $R[x]$, and there exist non-reduced Baer rings whose polynomial ring is not Baer. In the sequel, this result has been extended in several directions by many authors, [2-9].

Recently, the notions of reduced, Armendariz, Baer, PP and quasi-Baer modules were introduced in [10]. A module M_R is called reduced if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. A module M_R is called Armendariz if, whenever $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x]$, then $m_i a_j = 0$ for all i and j. A module M_R is called Armendariz of power series type if, whenever $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x]]$, then we have $m_i a_j = 0$ for all i and j. A module M_R is called Baer if, for any nonempty subset X of M, $r_R(X) = eR$ where $e^2 = e \in R$. A module M_R is called PP if, for any $m \in M$, $r_R(m) = eR$ where $e^2 = e \in R$. A module M_R is called quasi-Baer if, for any right R-submodule X of M, $r_R(X) = eR$ where $e^2 = e \in R$. Clearly, R is reduced (resp. Armendariz, Baer, right PP, quasi-Baer) if and only if R_R is a reduced (resp. Armendariz, Baer, PP, quasi-Baer) module. And various results on reduced (resp. Baer, right PP, quasi-Baer) rings were extended to modules in [10]. It was proved that every reduced module is an Armendariz module of power series type [Lemma 1.5]; and that M_R is reduced if and only if $M[x]_{R[x]}$ is reduced if and only if $M[[x]]_{R[[x]]}$ is reduced if and only if $M[[x, x^{-1}]]_{R[[x,x^{-1}]]}$ is reduced [Theorem 1.6]. If M_R is an Armendariz module, then it was proved that M_R is Baer if and only if $M[x]_{R[x]}$ is Baer [Corollary 2.7 (1)]; and that M_R is PP if and only if $M[x]_{R[x]}$ is PP [Corollary 2.12 (1)]. If M_R is an Armendariz module of power series type, then it was proved that M_R is Baer if and only if $M[[x]]_{R[[x]]}$ is Baer if and only if $M[[x, x^{-1}]]_{R[[x,x^{-1}]]}$ is Baer [Corollary 2.7 (2)]; and that $M[[x]]_{R[[x]]}$ is PP if and only if $M[[x, x^{-1}]]_{R[[x,x^{-1}]]}$ is PP if and only if for any countable subset X of M, $r_R(X) = eR$ where $e^2 = e \in R$ [Corollary 2.12 (2)]. For quasi-Baerness, it was proved that M_R is quasi-Baer if and only if $M[x]_{R[x]}$ is quasi-Baer if and only if $M[[x]]_{R[[x]]}$ is quasi-Baer if and only if $M[[x, x^{-1}]]_{R[[x,x^{-1}]]}$ is quasi-Baer [Corollary 2.14].

As a generalization of generalized power series rings, Varadarajan introduced the notion of modules of generalized power series in [15]. Thus a natural question of characterization of reduced (Baer, PP, quasi-Baer, respectively) property of generalized power series modules is raised. In this paper, a necessary and sufficient condition is given for modules under which $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$, the module of generalized power series with coefficients in M_R and exponents in S, is a reduced (Baer, PP, quasi-Baer, respectively) module. If S is a torsion-free and cancellative monoid and $\leq$ a strict order on S, we will show that: if M_R is a reduced module, then M_R is an S-Armendariz module; M_R is reduced if and only if $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is reduced; M_R is a quasi-Baer module if and only if $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is a quasi-Baer module. If $(S,\leq)$ is a strictly ordered monoid and M_R an S-Armendariz module, we will show that: M_R is a Baer module if and only if $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is a Baer module; $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is a PP-module if and only if for any S-indexed subset X of M_R, there exists an idempotent $e \in R$ such that $r_R(X) = eR$. And many other results are obtained, which unify and extend non-trivially many of the previously known results.

2. Preliminaries

Let $(S,\leq)$ be an ordered set. Recalled that $(S,\leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and that $(S,\leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S shall be denoted additively, and the neutral element by 0. The following definition is due to [11-13].

Let $(S,\leq)$ be a strictly ordered monoid (that is, $(S,\leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s+t < s'+t$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \to R$ such that $supp(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow.

With pointwise addition, $[[R^{S,\leq}]]$ is an abelian group.

For every $s \in S$ and $f, g \in [[R^{S,\leq}]]$, let $X_s(f,g) = \{(u,v) \in S \times S \mid u+v = s, f(u) \neq 0, g(v) \neq 0\}$. It follows from [11, 4.1] that $X_s(f,g)$ is finite. This allows to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v).$$

With these operations, $[[R^{S,\leq}]]$ becomes an associative ring, with unit element e, namely $e(0) = 1, e(s) = 0$ for every $s \in S, s \neq 0$, which is called the ring of generalized power series with coefficients in R and exponents in S.

In [15, 16], Varadarajan introduced the concept of modules of generalized power series. Let M be a right R-module, $(S,\leq)$ a strictly ordered monoid. Let $[[M^{S,\leq}]]$ denotes the set of all mapping $\phi : S \to M$ with $supp(\phi)$ artinian and narrow, where $supp(\phi) = \{s \in S \mid \phi(s) \neq 0\}$.

With pointwise addition, $[[M^{S,\leq}]]$ is an abelian group.

For each $s \in S$, $f \in [[R^{S,\leq}]]$ and $\phi \in [[M^{S,\leq}]]$, let $X_s(\phi, f) = \{(u, v) \in S \times S \mid u + v = s, \phi(u) \neq 0, f(v) \neq 0\}$. Then by analogy with [11, 4.1], $X_s(\phi, f)$ is finite. This allows to define the operation of convolution:

$$(\phi f)(s) = \sum_{(u,v)\in X_s(\phi,f)} \phi(u)f(v).$$

With these operations, $[[M^{S,\leq}]]$ becomes a right $[[R^{S,\leq}]]$ -module, which is called the modules of generalized power series with coefficients in M and exponents in S.

For example, if $S = \mathbb{N}$, and $\leq$ is the usual order, then $[[M^{\mathbb{N},\leq}]]_{[[R^{\mathbb{N},\leq}]]} \cong M[[x]]_{R[[x]]}$, the power series extension of M. If $S = \mathbb{Z}$, and $\leq$ is the usual order, then $[[M^{\mathbb{Z},\leq}]]_{[[R^{\mathbb{Z},\leq}]]} \cong M[[x, x^{-1}]]_{R[[x,x^{-1}]]}$, the Laurent power series extension of M.

3. Reduced Modules

Following from [10], a module M_R is called reduced if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. It is easy to see that R is a reduced ring if and only if R_R is a reduced module. The following result appeared in [10, Lemma 1.2].

Lemma 3.1. *The following conditions are equivalent:*

(1) M_R *is reduced.*

(2) *For any* $m \in M$ *and any* $a \in R$, *the following conditions hold:*

(a) $ma = 0$ *implies* $mRa = 0$.

(b) $ma^2 = 0$ *implies* $ma = 0$.

Rege and Chhawchharia in [14] introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. Let $(S, \leq)$ be a strictly ordered monoid. Recall from [6] that R is an S-Armendariz ring if whenever f, g in $[[R^{S,\leq}]]$ satisfy $fg = 0$, then $f(u)g(v) = 0$ for all $u, v \in S$. We call a module M_R is S-Armendariz if whenever $f \in [[R^{S,\leq}]]$ and $\phi \in [[M^{S,\leq}]]$ satisfy $\phi f = 0$, then $\phi(u)f(v) = 0$ for each $u, v \in S$. Clearly, R is S-Armendariz if and only if R_R is S-Armendariz. It was proved in [6, Lemma 3.1] that if S is a torsion-free and cancellative monoid, $\leq$ a strict order on S and R is a reduced ring then R is S-Armendariz. The following proposition extends this result to modules.

Proposition 3.2. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S and M_R a reduced module. Then M_R is an S-Armendariz module.*

Proof. Let $0 \neq f \in [[R^{S,\leq}]]$ and $0 \neq \phi \in [[M^{S,\leq}]]$ satisfy $\phi f = 0$. By [11], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$ (that is, for all $s, t \in S, s \leq t$ implies $s \leq' t$). We will use transfinite induction on the strictly totally ordered set $(S, \leq')$ to show that $\phi(u)f(v) = 0$ for any $u \in supp(\phi)$ and $v \in supp(f)$. Let s and t denote the minimum elements of $supp(\phi)$ and $supp(f)$ in the $\leq'$ order, respectively. If $u \in supp(\phi)$ and $v \in supp(f)$ are such that $u + v = s + t$, then $s \leq' u$ and $t \leq' v$. If $s <' u$ then $s + t <' u + v = s + t$, a contradiction. Thus $u = s$. Similarly, $v = t$. Hence $0 = (\phi f)(s + t) = \sum_{(u,v)\in X_{s+t}(\phi,f)} \phi(u)f(v) = \phi(s)f(t)$.

Now suppose that $w \in S$ is such that for any $u \in supp(\phi)$ and $v \in supp(f)$ with $u + v <' w$, $\phi(u)f(v) = 0$. We will show that $\phi(u)f(v) = 0$ for any $u \in supp(\phi)$ and $v \in supp(f)$ with $u + v = w$. We write $X_w(\phi, f) = \{(u, v) \in S \times S \mid u + v = w, \phi(u) \neq 0, f(v) \neq 0\}$ as $\{(u_i, v_i) \mid i = 1, 2, ..., n\}$ such that $u_1 <' u_2 <' \cdots <' u_n$. Since S is cancellative, $u_1 = u_2$ and $u_1+v_1 = u_2+v_2 = w$ imply $v_1 = v_2$. Since $\leq'$ is a strict order, $u_1 <' u_2$ and $u_1 + v_1 = u_2 + v_2 = w$ imply $v_2 <' v_1$. Thus we have $v_n <' \cdots <' v_2 <' v_1$. Now,

$$0 = (\phi f)(w) = \sum_{(u,v)\in X_w(\phi,f)} \phi(u)f(v) = \sum_{i=1}^{n} \phi(u_i)f(v_i). \tag{1}$$

For any $1 \leq i \leq n - 1, u_i + v_n <' u_i + v_i = w$, and thus, by induction hypothesis, we have $\phi(u_i)f(v_n) = 0$. Since M is reduced, then $\phi(u_i)Rf(v_n) = 0$ by Lemma 3.1. Hence, multiplying (1) on the right by $f(v_n)$, we obtain

$$\sum_{i=1}^{n} \phi(u_i)f(v_i)f(v_n) = \phi(u_n)f(v_n)f(v_n) = 0.$$

Since M is reduced, then by Lemma 3.1 we have $\phi(u_n)f(v_n) = 0$. Now (1) becomes

$$\sum_{i=1}^{n-1} \phi(u_i)f(v_i) = 0. \tag{2}$$

Multiplying $f(v_{n-1})$ on (2) from the right-hand side, we obtain $\phi(u_{n-1})f(v_{n-1}) = 0$ by the same way as the above. Continuing this process, we can prove $\phi(u_i)f(v_i) = 0$ for $i = 1, 2, ..., n$. Thus $\phi(u)f(v) = 0$ for any $u \in supp(\phi)$ and $v \in supp(f)$ with $u + v = w$.

Therefore, by transfinite induction, $\phi(u)f(v) = 0$ for any $u \in supp(\phi)$ and $v \in supp(f)$.

Lee-Zhou introduced the notion of an Armendariz module of power series type in [10]. They defined a module M_R to be an Armendariz module of power series type if, whenever $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x]]$, then $m_i a_j = 0$ for all i and j. Letting $(S, \leq) = (\mathbb{N}, \leq)$, the natural number set with usual order, yields the following result.

Corollary 3.3. *Let M_R be a reduced module. Then M_R is an Armendariz module of power series type.*

In [1, Lemma 1], it was proved that if R is a reduced ring, then R is an Armendariz ring. Here we have

Corollary 3.4. *Let R be a reduced ring. Then R is an Armendariz ring of power series type.*

Let $m \in M$ and $\delta \in S$. Define a mapping $d_m^s \in [[M^{S,\leq}]]$ as follows:

$$d_m^s(s) = m, \qquad d_m^s(t) = 0, \quad s \neq t \in S.$$

Proposition 3.5. *Let $(S, \leq)$ be a strictly ordered monoid and M_R an S-Armendariz module. If $\phi \in [[M^{S,\leq}]]$ and $f_1, f_2, \cdots, f_n \in [[R^{S,\leq}]]$ are such that $\phi f_1 f_2 \cdots f_n = 0$, then $\phi(u)f_1(v_1)f_2(v_2)..., f_n(v_n) = 0$ for all $u, v_1, v_2, ..., v_n \in S$.*

Proof. Suppose $\phi f_1 f_2 \cdots f_n = 0$. Then from $\phi(f_1 f_2 \cdots f_n) = 0$ it follows that $\phi(u)(f_1 f_2 \cdots f_n)(v) = 0$ for all $u, v \in S$. Thus $(d^0_{\phi(u)} f_1 f_2 \cdots f_n)(v) = 0$ for any $v \in S$, and so $d^0_{\phi(u)} f_1 f_2 \cdots f_n = 0$. Now from $(d^0_{\phi(u)} f_1)(f_2 \cdots f_n) = 0$ it follows that $(d^0_{\phi(u)} f_1)(v_1)(f_2 \cdots f_n)\ (w) = 0$ for all $v_1, w \in S$. Since $(d^0_{\phi(u)} f_1)(v_1) = \phi(u)f_1(v_1)$ for any $u, v_1 \in S$, we have $\phi(u)f_1(v_1)(f_2 \cdots f_n)(w) = 0$ for all $u, v_1, w \in S$. Hence $d^0_{\phi(u)f_1(v_1)} f_2 \cdots f_n = 0$. Continuing in this manner, we see that $\phi(u)f_1(v_1)f_2(v_2) \cdots f_n(v_n) = 0$ for all $u, v_1, v_2, ..., v_n \in S$.

Now, combining proposition 3.2 we have

Corollary 3.6. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S and M_R a reduced module. If $\phi \in [[M^{S,\leq}]]$ and $f_1, f_2, ..., f_n \in [[R^{S,\leq}]]$ are such that $\phi f_1 f_2 \cdots f_n = 0$, then $\phi(u)f_1(v_1)f_2(v_2) \cdots f_n(v_n) = 0$ for all $u, v_1, v_2, ..., v_n \in S$.*

Let $r \in R$. Define a mapping $C_r \in [[R^{S,\leq}]]$ as follows:

$$C_r(0) = r, \qquad C_r(s) = 0, \quad 0 \neq s \in S.$$

It was proved in [10, Theorem 1.6] that M_R is reduced if and only if $M[x]_{R[x]}$ is reduced if and only if $M[[x]]_{R[[x]]}$ is reduced if and only if $M[[x, x^{-1}]]_{R[[x,x^{-1}]]}$ is reduced. Here we have

Theorem 3.7. *Let S be a torsion-free and cancellative monoid and $\leq$ a strict order on S. Then M_R is reduced if and only if $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is reduced.*

Proof. Let M_R be reduced. Suppose that $f \in [[R^{S,\leq}]]$ and $\phi \in [[M^{S,\leq}]]$ satisfy $\phi f = 0$ and $\phi g = \psi f$, where $\psi \in [[M^{S,\leq}]]$ and $g \in [[R^{S,\leq}]]$. It suffices to show that $\psi f = 0$. By Proposition 3.2, $\phi(s)f(t) = 0$ for any $s, t \in S$. Thus $\phi(s)Rf(t) = 0$ for any $s, t \in S$ by Lemma 3.1. Then

$$(\phi g f)(s) = \sum_{(u,v,w)\in X_s(\phi,g,f)} \phi(u)g(v)f(w) = 0$$

for any $s \in S$. Thus $\psi f^2 = \phi g f = 0$. Then by Corollary 3.6, $\psi(u)f(v)f(w) = 0$ for any $u, v, w \in S$. Thus $\psi(u)f(v)^2 = 0$ for any $u, v \in S$. Then $\psi(u)f(v) = 0$ for any $u, v \in S$ by Lemma 3.1, and which implies that $\psi f = 0$.

Conversely, suppose that $ma = 0$ and $mr = na \in mR \cap Ma$ where $m, n \in M$ and $r, a \in R$. Then $d_m^0 C_a = 0$. Since $[[M^{S,\leq}]]$ is reduced, we have $d_m^0[[R^{S,\leq}]] \cap [[M^{S,\leq}]]C_a = 0$. Thus $d_m^0 C_r = d_n^0 C_a = 0$, and so $mr = na = 0$. Hence M_R is reduced.

Corollary 3.8. ([8, Lemma 2.1]) *Let S be a torsion-free and cancellative monoid and $\leq$ a strict order on S. Then R is reduced if and only if $[[R^{S,\leq}]]$ is reduced.*

4. Baer Modules

Recall that R is Baer if the right annihilator of every nonempty subset is generated by an idempotent. If R is a reduced ring, then it is shown in [2, Corollary 1.10] that R is Baer if and only if $R[x]$ is Baer if and only if $R[[x]]$ is Baer. If R is commutative and $(S, \leq)$ is a strictly totally ordered monoid, then it is shown in [7, Theorem 7] that R is Baer if and only if $[[R^{S,\leq}]]$ is Baer. Recall from [10] that a right R- module M is Baer if, for any subset X of M, $r_R(X) = eR$ where $e^2 = e \in R$. It is also shown in [10, Corollary 2.7(2)] that if M_R is an Armendariz module of power series type, then M_R is Baer if and only if $M[[x]]_{R[[x]]}$ is Baer if and only if $M[[x, x^{-1}]]_{R[[x,x^{-1}]]}$ is Baer. Next, we will extend these results to generalized power series modules. First we have the following results on which our discussion is based.

Let I be a right ideal of R. Let $[[I^{S,\leq}]] = \{f \in [[R^{S,\leq}]] \mid f(s) \in I$ for any $s \in S\}$. Then it is easy to see that $[[I^{S,\leq}]]$ is a right ideal of $[[R^{S,\leq}]]$.

Lemma 4.1. *Let M be a right R-module and $(S, \leq)$ a strictly ordered monoid. Then the following conditions are equivalent:*

(1) *M_R is an S-Armendariz module.*

(2) *For any $X \subseteq [[M^{S,\leq}]]$, $[[r_R(X')^{S,\leq}]] = r_{[[R^{S,\leq}]]}(X)$, where $X' = \{\phi(s) \mid \phi \in X, s \in S\}$.*

Proof. (2)⇒(1). Let $\phi f = 0$ where $\phi \in [[M^{S,\leq}]]$ and $f \in [[R^{S,\leq}]]$. Then $f \in r_{[[R^{S,\leq}]]}(\phi)$. By (2), $f \in [[r_R(X')^{S,\leq}]]$ where $X' = \{\phi(s) \mid s \in S\}$. Take $f(s) \in r_R(X')$ for any $s \in S$. Thus $\phi(t)f(s) = 0$ for any $s, t \in S$. This means that M is an S-Armendariz module.

(1)⇒(2). Suppose that $X \subseteq [[M^{S,\leq}]]$. Take $X' = \{\phi(s) \mid \phi \in X, s \in S\}$. Let $g \in r_{[[R^{S,\leq}]]}(X)$, then $\phi g = 0$ for any $\phi \in X$. By (1), $\phi(s)g(t) = 0$ for any $s, t \in S$. Thus $g(t) \in r_R(X')$ for any $t \in S$. Thus $g \in [[r_R(X')^{S,\leq}]]$, and so $r_{[[R^{S,\leq}]]}(X) \subseteq [[r_R(X')^{S,\leq}]]$. The opposite inclusion is obviously.

Lemma 4.2. *Let M be a right R-module and $(S, \leq)$ a strictly ordered monoid. Then for any $X \subseteq M$, $[[r_R(X)^{S,\leq}]] = r_{[[R^{S,\leq}]]}(X')$, where $X' = \{d_m^0 \mid m \in X\}$.*

Proof. The proof is straightforward.

Theorem 4.3. *Let $(S, \leq)$ be a strictly ordered monoid and M_R an S-Armendariz module. Then the following conditions are equivalent:*

(1) *M_R is a Baer module.*

(2) *$[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is a Baer module.*

Proof. (1)⇒(2). Let $X \subseteq [[M^{S,\leq}]]$. Since M_R is an S-Armendariz module, by Lemma 4.1, $r_{[[R^{S,\leq}]]}(X) = [[r_R(X')^{S,\leq}]]$ where $X' = \{\phi(s) \mid \phi \in X, s \in S\}$. Since M_R is a Baer module, there exists an idempotent $e^2 = e \in R$ such that $r_R(X') = eR$. Thus $r_{[[R^{S,\leq}]]}(X) = [[r_R(X')^{S,\leq}]] = [[(eR)^{S,\leq}]] = C_e[[R^{S,\leq}]]$, and which implies $[[M^{S,\leq}]]$ is a Baer module.

(2)⇒(1). Let $X \subseteq M$. Then by Lemma 4.2, $[[r_R(X)^{S,\leq}]] = r_{[[R^{S,\leq}]]}(X')$, where $X' = \{d_m^0 \mid m \in X\}$. Since $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is a Baer module, there exists an idempotent $f^2 = f \in [[R^{S,\leq}]]$ such that $[[r_R(X)^{S,\leq}]] = r_{[[R^{S,\leq}]]}(X') = f[[R^{S,\leq}]]$. We will show that $r_R(X) = f(0)R$ and $f(0) = f(0)^2$. From $f \in [[r_R(X)^{S,\leq}]]$ it follows that $f(s) \in r_R(X)$ for any $s \in S$. Especially, $f(0) \in r_R(X)$, and so $f(0)R \subseteq r_R(X)$. Conversely, let $r \in r_R(X)$. Then $C_r \in [[r_R(X)^{S,\leq}]] = f[[R^{S,\leq}]]$. Thus $C_r = fC_r$. Then $r = C_r(0) = (fC_r)(0) = f(0)r \in f(0)R$. Thus $r_R(X) \subseteq f(0)R$. Since $f(0) \in r_R(X)$, we have $f(0) = f(0)^2$. Hence $r_R(X) = f(0)R$ and $f(0) = f(0)^2$. So M_R is a Baer module.

Applying Proposition 3.2 we can get

Corollary 4.4. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S and M_R a reduced module. Then M_R is a Baer module if and only if $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is a Baer module.*

Corollary 4.5. *Let $(S, \leq)$ be a strictly ordered monoid and R an S-Armendriz ring. Then R is Baer if and only if $[[R^{S,\leq}]]$ is Baer.*

In [5, Theorem 10], it was proved that if R is an Armendariz ring, then R is Baer if and only if $R[x]$ is Baer. Here we have

Corollary 4.6. *Let R be an Armendariz ring of power series type. Then R is Baer if and only if $R[[x]]$ is Baer.*

Applying Corollary 3.4 we can get

Corollary 4.7. ([2, Corollary 1.10.]). *Let R be a reduced ring . Then R is Baer if and only if $R[[x]]$ is Baer.*

5. PP-Modules

One of generalizations of Baer rings is PP-rings. A ring R is called right (resp. left) PP if the right (resp. left) annihilator of an element of R is generated by an idempotent. A ring is called PP if it is both right and left PP. It was proved in [1, Theorem A] that R is a reduced right PP-ring if and only if $R[x]$ is a reduced right PP-ring. It was proved in [4] that $R[[x]]$ is a reduced right PP-ring if and only if R is a reduced right PP-ring and any countable family of idempotents of R has a least upper bound in $B(R)$, the set of all central idempotents. If $(S, \leq)$ is a strictly totally ordered monoid, then it is shown in [6, Theorem 3.5] that $[[R^{S,\leq}]]$ is a reduced right PP-ring if and only if R is a reduced right PP-ring and any S-indexed family of idempotents of R has a least upper bound in $B(R)$. The notion of PP-modules was introduced in [10]. A module M_R is called PP if, for any $m \in M$, $r_R(m) = eR$ where $e^2 = e \in R$. It was also proved in [10, Corollary 2.12] that if M_R is an Armendariz module of power series type, then $M[[x]]_{R[[x]]}$ is PP if and only if for any countable subset X of M , $r_R(X) = eR$ where $e^2 = e \in R$. In this section we will consider the PP property of generalized power series modules. The following result is a corollary of Theorem 5.2. But here we give a direct and different proof.

Proposition 5.1. *Let $(S, \leq)$ be a strictly ordered monoid and M a right R-module. If $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is a PP-module, then M_R is a PP-module.*

Proof. Let $m \in M$. Then by Lemma 4.2, $[[r_R(m)^{S,\leq}]] = r_{[[R^{S,\leq}]]}(d_m^0)$. Since $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is a PP-module, there exists an idempotent $f \in [[R^{S,\leq}]]$ such that

$r_{[[R^{S,\leq}]]}(d^0_m) = f[[R^{S,\leq}]]$. We will show that $r_R(m) = f(0)R$ and $f(0) = f(0)^2$. From $f \in r_{[[R^{S,\leq}]]}(d^0_m)$ it follows that $d^0_m f = 0$. Then $mf(0) = (d^0_m f)(0) = 0$. Thus $f(0)R \subseteq r_R(m)$. Conversely, let $r \in r_R(m)$. Then $C_r \in [[(r_R(m)^{S,\leq}]] = f[[R^{S,\leq}]]$. Thus $C_r = fC_r$. Then $r = C_r(0) = (fC_r)(0) = f(0)r \in f(0)R$. Thus $r_R(m) \subseteq f(0)R$. Since $f(0) \in r_R(m)$, we have $f(0) = f(0)^2$. Hence $r_R(m) = f(0)R$ and $f(0) = f(0)^2$. So M_R is a PP-module.

Let $X \subseteq M$. We will say that X is S-indexed if there exists an artinian and narrow subset I of S such that X is indexed by I.

Theorem 5.2. *Let $(S, \leq)$ be a strictly ordered monoid and M_R an S-Armendariz module. Then the following conditions are equivalent:*

(1) $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ *is a PP-module.*
(2) *For every S-indexed subset X of M, there exists an idempotent $e \in R$ such that $r_R(X) = eR$.*

Proof. (1)⇒(2). Suppose that $[[M^{S,\leq}]]$ is a PP-module. Let $X = \{m_s \mid s \in I\}$ is an S-indexed subset of M. Define $\phi : S \to M$ via

$$\phi(s) = \begin{cases} m_s, & s \in I, \\ 0, & s \notin I. \end{cases}$$

Then $supp(\phi) = I$ is artinian and narrow, and so $\phi \in [[M^{S,\leq}]]$. Since $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is a PP-module, there exists an idempotent $f^2 = f \in [[R^{S,\leq}]]$ such that $r_{[[R^{S,\leq}]]}(\phi) = f[[R^{S,\leq}]]$. Since M_R is an S-Armendariz module, then $r_{[[R^{S,\leq}]]}(\phi) = [[r_R(X)^{S,\leq}]]$ by Lemma 4.1. Thus $[[r_R(X)^{S,\leq}]] = f[[R^{S,\leq}]]$. Then by analogy with the proof of Theorem 4.3 we can show that $r_R(X) = f(0)R$ with $f(0)^2 = f(0)$.

(2)⇒(1). Let $\phi \in [[M^{S,\leq}]]$. Then $X = \{\phi(s) \mid s \in supp(\phi)\}$ is an S- indexed subset of M. Then there exists an idempotent $e \in R$ such that $r_R(X) = eR$ by (2). Thus by Lemma 4.1, we have $r_{[[R^{S,\leq}]]}(\phi) = [[(r_R(X)^{S,\leq}]] = [[(eR)^{S,\leq}]] = C_e[[R^{S,\leq}]]$, and which implies $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is a PP-module.

Corollary 5.3. *Let $(S, \leq)$ be a torsion-free cancellative strictly ordered monoid. Then the following conditions are equivalent:*

(1) $[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ *is a reduced PP-module.*
(2) *M is a reduced PP-module, and for every S-indexed subset X of M, there exists an idempotent $e \in R$ such that $r_R(X) = eR$.*

Proof. Using Proposition 3.2, Theorem 3.7 and Theorem 5.2, we can complete the proof.

If R is reduced, then R is Abelian (that is, every idempotent of R is central). Thus, by [4], the set $B(R)$ of all idempotents is a Boolean algebra where $e \leq f$ means $ef = e$, and where the join, meet, and complement are given by $e \vee f = e + f - ef$, $e \wedge f = ef$ and $e' = 1 - e$, respectively. The following result appeared in [4] on which our following discussion is based. An element $a \in R$ will be called entire if $l_R(a) = r_R(a) = 0$.

Lemma 5.4. *The following conditions are equivalent for a ring R:*

(1) *R is a reduced right PP-ring.*

(2) *If $a \in R$ then $a = eb = be$ where $e^2 = e \in R$ and $b \in R$ is entire.*

(3) *R is an Abelian right PP-ring.*

Now, comparing with the result in [6, Theorem 3.5], we have

Corollary 5.5. *Let $(S, \leq)$ be a torsion-free cancellative strictly ordered monoid. Then the following conditions are equivalent:*

(1) *$[[R^{S,\leq}]]$ is a reduced right PP-ring.*

(2) *R is a reduced right PP-ring, and for every S-indexed subset X of R, there exists an idempotent $e \in R$ such that $r_R(X) = eR$.*

(3) *R is a reduced right PP-ring, and for every S-indexed subset X of $B(R)$, there exists an idempotent $e \in R$ such that $r_R(X) = eR$.*

(4) *R is a reduced right PP-ring and every S-indexed subset X of $B(R)$ has a least upper bound in $B(R)$.*

Proof. Letting $M = R$ in Corollary 5.3 we can get (1)⇔(2).

(2)⇒(3). It is straightforward.

(3)⇒(4). Suppose that $X = \{e_s \mid s \in I\}$ is an S-indexed subset of $B(R)$. Then by (3), $r_R(X) = eR$ where $e^2 = e \in R$. We claim that $1 - e$ is a least upper bound of X in $B(R)$. First $Xe = 0$ implies that for every $s \in I$, $e_s e = 0$, and thus $e_s(1 - e) = e_s$. Thus $e_s \leq 1 - e$. On the other hand, suppose that $e_s \leq f$ for all $s \in I$, where $f^2 = f \in R$. Then $1 - f \in r_R(X) = eR$. Thus $1 - f = e(1 - f)$. Thus $1 - e = (1 - e)f$, and which implies that $1 - e \leq f$.

(4)⇒(2). Suppose that $X = \{a_s \mid s \in I\}$ is an S-indexed subset of R. Then by Lemma 5.4, $a_s = e_s b_s$ for all $s \in I$, where $e_s^2 = e_s \in R$ and $b_s \in R$ is entire. Setting $X' = \{e_s \mid s \in I\}$. Then X' is an S-indexed subset of $B(R)$. Let e be a least upper bound of X' in $B(R)$. We will show that $r_R(X) = (1 - e)R$. First from $e_s e = e_s$ it follows that $(1 - e)e_s = 0$ for all $s \in I$. Then $1 - e \in r_R(X)$. On the other hand, let $r \in r_R(X)$. Then $a_s r = 0$ for all $s \in I$. By Lemma 5.4, there exists an idempotent $f^2 = f \in R$ and an entire element $p \in R$ such that $r = fp$.

Thus $e_s f = 0$ for all $s \in I$ since p and b_s is entire. Thus $e_s \leq 1 - f$ for all $s \in I$. Thus $e \leq 1 - f$, and so $r = (1-e)r \in (1-e)R$. Hence $r_R(X) = (1-e)R$.

In [5, Theorem 9], it was proved that if R is an Armendariz ring, then R is PP if and only if $R[x]$ is PP. Here we have

Corollary 5.6. *Let R be an Armendariz ring of power series type. Then $R[[x]]$ is right PP if and only if R is right PP and for any countable subset X of R, $r_R(X) = eR$, where $e^2 = e \in R$.*

6. Quasi-baer Modules

Another generalization of Baer rings is quasi-Baer rings. Recall that R is quasi-Baer if the right annihilator of every right ideal is generated by an idempotent. Every prime ring is quasi-Baer ring. Since Baer ring are nonsingular, the prime rings with $Z_r(R) \neq 0$ are quasi-Baer but not Baer. It was proved in [2, Theorem 1.8] that a ring R is quasi-Baer if and only if $R[x]$ is quasi-Baer if and only if $R[[x]]$ is quasi-Baer. Following from [10] a module M_R is called quasi-Baer if, for any right R-submodule X of M , $r_R(X) = eR$ where $e^2 = e \in R$. Clearly, R is quasi-Baer if and only if R_R is quasi-Baer. In [10, Corollary 2.14], it is shown that M_R is quasi-Baer if and only if $M[x]_{R[x]}$ is quasi-Baer if and only if $M[[x]]_{R[[x]]}$ is quasi-Baer. In this section we will generalize these results to generalized power series modules.

Theorem 6.1. *Let $(S \leq)$ be a torsion-free and cancellative strictly ordered monoid . Then the following conditions are equivalent:*

(1) *M_R is a quasi-Baer module.*

(2) *$[[M^{S,\leq}]]_{[[R^{S,\leq}]]}$ is a quasi-Baer module.*

Proof. (1)⇒(2). Suppose that $V \leq [[M^{S,\leq}]]$. By [11], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$ (that is, for all $s, t \in S, s \leq t$ implies $s \leq' t$). Note that $[[M^{S,\leq}]]$ (resp. $[[R^{S,\leq}]]$) is a submodule (resp. subring) of $[[M^{S,\leq'}]]$ (resp. $[[R^{S,\leq'}]]$). Thus we may assume that the order $\leq$ is total. Then for any $0 \neq f \in [[R^{S,\leq}]]$ (resp. $[[M^{S,\leq}]]$), the $supp(f)$ is a nonempty well-ordered subset of S. We denote by $\pi(f)$ the smallest element of the support of f. Let $U = \{\phi(s) \mid \phi \in V, \pi(\phi) = s\} \cup \{0\}$. Then it is easy to see that U is a right R-submodule of M. Since M_R is a quasi-Baer module, then $r_R(U) = eR$ where $e^2 = e \in R$. We will show that $r_{[[R^{S,\leq}]]}(V) = C_e[[R^{S,\leq}]]$. Let $\phi \in V$. If $\phi C_e \neq 0$. Let $\pi(\phi C_e) = s$, then $0 \neq (\phi C_e)(s) = \phi(s)e$; on the other hand, since $\phi \in V$, so $\phi C_e \in V$ and then $\phi(s)e \in U$. From $r_R(U) = eR$ it follows that $\phi(s)e =$

$(\phi(s)e)e = 0$, a contradiction. Thus $\phi C_e = 0$, and so $C_e[[R^{S,\leq}]] \subseteq r_{[[R^{S,\leq}]]}(V)$. Conversely, suppose that $0 \neq f \in r_{[[R^{S,\leq}]]}(V)$. We will show that $f(u) = ef(u)$ for all $u \in supp(f)$.

Step 1. Let $\pi(f) = s$. Then we will show that $f(s) = ef(s)$. Let $0 \neq m \in U$. Then there exists a $\phi \in V$ such that $\pi(\phi) = t$ and $\phi(t) = m$. From $f \in r_{[[R^{S,\leq}]]}(V)$ it follows that $\phi f = 0$. Thus

$$0 = (\phi f)(s+t) = \sum_{(u,v)\in X_{s+t}(\phi,f)} \phi(u)f(v).$$

If $u \in supp(\phi)$ and $v \in supp(f)$ are such that $u+v = s+t$, then $t \leq u$ and $s \leq v$. If $t < u$ then $s+t < u+v = s+t$, a contradiction. Thus $u = t$. Similarly, $v = s$. Hence $\phi(t)f(s) = 0$. Thus $Uf(s) = 0$, which implies that $f(s) \in r_R(U) = eR$. Thus $f(s) = ef(s)$.

Step 2. Assume that $f(u) = ef(u)$ for any $u < w \in supp(f)$. We will show that $f(w) = ef(w)$. Define f_w as follows:

$$f_w(x) = \begin{cases} f(x), & x < w, \\ 0, & w \leq x. \end{cases}$$

Then $f_w \in [[R^{S,\leq}]]$ and $f_w(x) = f(x) = ef(x) = ef_w(x) = (C_e f_w)(x)$ for any $x < w$ by induction hypothesis. Thus $f_w = C_e f_w \in C_e[[R^{S,\leq}]] \subseteq r_{[[R^{S,\leq}]]}(V)$. Thus $f - f_w \in r_{[[R^{S,\leq}]]}(V)$, and $\pi(f - f_w) = w$. Applying Step 1, we obtain $(f - f_w)(w) = e(f - f_w)(w)$, thus $f(w) = ef(w)$. Therefore, by transfinite induction, $f(u) = ef(u)$ for all $u \in supp(f)$. Thus $f = C_e f \in C_e[[R^{S,\leq}]]$, and which implies that $r_{[[R^{S,\leq}]]}(V) \subseteq C_e[[R^{S,\leq}]]$.

Hence $r_{[[R^{S,\leq}]]}(V) = C_e[[R^{S,\leq}]]$. This shows that $[[M^{S,\leq}]]$ is a quasi-Baer module.

(2)$\Rightarrow$(1). Suppose that $[[M^{S,\leq}]]$ is a quasi-Baer module. Let $U \leq M$, then it is easy to see that $[[U^{S,\leq}]] \leq [[M^{S,\leq}]]$. Thus there exists an idempotent $f^2 = f \in [[R^{S,\leq}]]$ such that $r_{[[R^{S,\leq}]]}([[U^{S,\leq}]]) = f[[R^{S,\leq}]]$. We claim that $r_{[[R^{S,\leq}]]}([[U^{S,\leq}]]) = f[[R^{S,\leq}]] = [[r_R(U)^{S,\leq}]]$. Let $m \in U$. Then $d_m^0 \in [[U^{S,\leq}]]$. Thus $d_m^0 f = 0$, and then $mf(s) = 0$ for all $s \in S$. Thus $Uf(s) = 0$ for all $s \in S$, and so $f \in [[r_R(U)^{S,\leq}]]$. Let $g \in [[r_R(U)^{S,\leq}]]$. Then $g(s) \in r_R(U)$ for all $s \in S$. Then $(\phi g)(t) = \sum_{(u,v)\in X_t(\phi,g)} \phi(u)g(v) = 0$ for any $\phi \in [[U^{S,\leq}]]$ and any $t \in S$. Thus $\phi g = 0$, and so $g \in r_{[[R^{S,\leq}]]}([[U^{S,\leq}]])$. Hence $r_{[[R^{S,\leq}]]}([[U^{S,\leq}]]) = f[[R^{S,\leq}]] = [[r_R(U)^{S,\leq}]]$. Then by analogy with the proof of Theorem 4.3 we can show that $r_R(U) = f(0)R$ with $f(0)^2 = f(0)$. Hence M_R is a quasi-Baer module.

It was proved in [9] that if $(S, \leq)$ is a strictly totally ordered monoid satisfying that $0 \leq s$ for all $s \in S$, then R is quasi-Baer if and only if $[[R^{S,\leq}]]$ is quasi-Baer. Here we have

Corollary 6.2. *Let $(S \leq)$ be a torsion-free and cancellative strictly ordered monoid. Then the following conditions are equivalent:*

(1) R *is quasi-Baer.*

(2) $[[R^{S,\leq}]]$ *is quasi-Baer.*

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