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A NEW SYSTEM OF GENERALIZED CO-COMPLEMENTARITY PROBLEMS IN BANACH SPACES

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Abstract. In this paper, we introduce a new system of generalized co-complementarity problems in Banach space. An iterative algorithm for finding approximate solutions of these problems is considered. Some convergence results for this iterative algorithm are derived and several existence results are also obtained.

1. Introduction

Let B be a real Banach space with dual space B^* and pairing $\langle x, f\rangle$ between $x \in B$ and $f \in B^*$. Let $CB(B)$ be the family of nonempty bounded closed subsets of B. Suppose B_1 and B_2 are two Banach spaces, $g_i, m_i : B_i \to B_i$ $(i = 1, 2)$, $F : B_1 \times B_2 \to B_1$, and $G : B_1 \times B_2 \to B_2$ are all single-valued mappings. Let $V_1 : B_1 \to CB(B_1)$ and $V_2 : B_2 \to CB(B_2)$ be two set-valued mappings. Moreover, we assume $X_1 \subset B_1$ and $X_2 \subset B_2$ are two fixed closed convex cones. Define $K_i : B_i \to 2^{B_i}$ $(i = 1, 2)$ by

$$K_i(x) = m_i(x) + X_i, \quad \forall x \in B_i.$$

In this paper, we shall study the following system of generalized co-complementarity problems $(SGCCP)$: find $(x, y) \in B_1 \times B_2$ and $(u, v) \in V_1(x) \times V_2(y)$ such that $(g_1(x), g_2(y)) \in K_1(x) \times K_2(y)$ and

$$\begin{cases} F(u, v) \in (J_1(K_1(x) - g_1(x)))^*, \\ G(u, v) \in (J_2(K_2(y) - g_2(y)))^*, \end{cases} \tag{1.1}$$

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.

where $J_i : B_i \to B_i^*$ $(i = 1, 2)$ are the normalized duality mappings, $(J_1(K_1(x) - g_1(x)))^*$ and $(J_2(K_2(y) - g_2(y)))^*$ denote the dual cones of the sets $(J_1(K_1(x) - g_1(x)))$ and $(J_2(K_2(y) - g_2(y)))$, respectively.

Recall that the normalized duality operator $J : B \to B^*$ is defined for arbitrary Banach space by the condition

$$\|Jx\|_{B^*} = \|x\| \quad \text{and} \quad \langle x, Jx\rangle = \|x\|^2, \quad \forall x \in B.$$

Some examples and properties of the mapping J can be found in [1, p. 19]. When B is a Hilbert space, $Jx = x$ reduces to the identity mapping. Note that every nonzero $x \in B$ is weak* continuous, and thus, attains its norm on the weak* compact unit ball of B^*. In this case where B^* is strictly convex, the point x attains its norm on the ball of B^* is unique, namely, $Jx/\|x\|$. In this paper, we are mainly interested in uniformly smooth Banach space B. Therefore, the construction of J is concrete to us here.

Before we proceed any further, we make a few observations. There is evidence that our results generalize many known important results obtained in the literature.

(i) If $B_1 = B_2$, $K_1 = K_2$, $g_1 = g_2$, and $F = G$, then problem (1.1) reduces to finding $x \in B_1$, $u \in V_1(x)$, and $v \in V_2(y)$ such that $g_1(x) \in K_1(x)$ and

$$F(u, v) \in (J_1(K_1(x) - g_1(x)))^*. \tag{1.2}$$

(ii) If $V_1(x) = T(x)$ is a single valued mapping and $F(u, v) = u + A(v)$, then problem (1.2) reduces to finding $x \in B_1$ and $v \in V_2(x)$ such that $g_1(x) \in K_1(x)$ and

$$T(x) + A(v) \in (J_1(K_1(x) - g_1(x)))^*, \tag{1.3}$$

which is the generalized co-complementarity problem studied by Chen, Wong and Yao [3].

(iii) If B_1 is a Hilbert space, then problem (1.3) reduces to finding $x \in B_1$ and $v \in V_2(x)$ such that $g_1(x) \in K_1(x)$ and

$$T(x) + A(v) \in (K_1(x) - g_1(x))^*, \tag{1.4}$$

which is the generalized multivalued complementarity problem studied by Jou and Yao [8].

(iv) If g_1 is an identity mapping, then problem (1.4) reduces to finding $x \in K_1(x)$ and $v \in V_2(x)$ such that

$$T(x) + A(v) \in (K_1(x) - x)^*, \tag{1.5}$$

which is known as the generalized strongly nonlinear quasi-complementarity problem studied by Chang and Huang [2].

(v) If g_1 and V_2 are identity mappings, A and m are zero mappings, then problem (1.3) equivalent to finding $x \in X_1$ such that

$$T(x) \in X_1^*, \quad \langle T(x), x\rangle = 0, \tag{1.6}$$

which is known as the generalized complementarity problem studied by Habetler and Price [5] and Karamardian [10].

The complementarity theory derives its importance from the face that it unifies problems in fields such as: mathematical programming, game theory, the theory of equilibrium in a competitive economy, equilibrium of traffic flows, mechanics, engineering, lubricant evaporation in the cavity of a cylindrical bearing, elasticity theory, maximizing oil production, computation of fixed point etc., see Isac [6, 7].

The aim of this paper is to construct the projection iterative methods of finding approximate solutions of $(SGCCP)$ in (especially uniformly smooth) Banach space. As pointed out by Chen, Wong and Yao [3], such research fields are new, interesting, and should be applicable to all those classical complementarity problems mentioned above. The present results improve and extend many know results in the literature.

2. Preliminaries

We first recall the following definitions.

Definition 2.1. Let B be a Banach space with the normalized duality mapping $J : B \to B^*$. A mapping $A : B \to B$ is said to be

(1) strongly accretive if there exists a constant $\gamma > 0$ such that

$$\langle Ax - Ay, J(x-y)\rangle \geq \gamma \|x-y\|^2, \quad \forall x, y \in B;$$

(2) Lipschitz continuous if there exists a positive constant β such that

$$\|A(x) - A(y)\| \leq \beta\|x-y\|, \quad \forall x, y \in B.$$

Definition 2.2. Let B_1 and B_2 be two Banach spaces, $F : B_1 \times B_2 \to B_1$ a single-valued mapping, and $V : B_1 \to CB(B_1)$ a set-valued mapping. For any given $y \in B_2$, $F(\cdot, y)$ is said to be ξ-strongly accretive with respect to V if there exists a constant $\xi > 0$ such that

$$\begin{aligned}&\langle F(u_1, y) - F(u_2, y), J_1(x_1 - x_2)\rangle \\ &\geq \xi\|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in B_1, \forall u_1 \in V(x_1), \forall u_2 \in V(x_2),\end{aligned}$$

where $J_1 : B_1 \to B_1^*$ is the normalized duality mapping.

Definition 2.3. The mapping $V : B \to CB(B)$ is said to be H-Lipschitz continuous if there exists a constant $\eta > 0$ such that

$$H(V(x), V(y)) \leq \eta\|x - y\|, \quad \forall x, y \in B,$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on $CB(B)$.

We remark that the *uniform convexity* of the Banach space B means that for any given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in B$, $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$ ensure the following inequality:

$$\|x + y\| \leq 2(1 - \delta).$$

The function

$$\delta_B(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| = 1, \|y\| = 1, \|x - y\| \geq \epsilon\}$$

is called *the modulus of the convexity* of the space B.

The *uniform smoothness* of the Banach space B means that for any given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in B$, $\|x\| \leq 1$, $\|y\| < \delta$ ensure the following inequality:

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 \leq \epsilon\|y\|$$

holds. The function

$$\rho_B(t) = \sup\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| \leq t\}$$

is called *the modulus of the smoothness* of the space B.

We also remark that the space B is uniformly convex if and only if $\delta_B(\epsilon) > 0$ for all $\epsilon > 0$, and it is uniformly smooth if and only if $\lim_{t\to 0} t^{-1}\rho_B(t) = 0$. Moreover, B^* is uniformly convex if and only if B is uniformly smooth. In this case, B is reflexive by the Milman theorem. A Hilbert space is uniformly convex and uniformly smooth. The proof of the following inequalities can be found, e.g., in [1, p. 24].

Proposition 2.1. *Let B be a uniformly smooth Banach space and J be the normalized duality mapping from B into B^*. Then, for all $x, y \in B$, we have* (*i*) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle$,

(*ii*) $\langle x-y, J(x)-J(y)\rangle \leq 2d^2\rho_B(4\|x-y\|/d)$, *where* $d = ((\|x\|^2+\|y\|^2)/2)^{1/2}$.

Let B be a real Banach space and Ω be a nonempty closed convex subset of B. A mapping $Q_\Omega : B \to \Omega$ is said to be a retraction on Ω if $Q_\Omega^2 = Q_\Omega$. The mapping Q_Ω is said to be a nonexpansive retraction if, in addition,

$$\|Q_\Omega(x) - Q_\Omega(y)\| \leq \|x-y\|, \quad \forall x, y \in B,$$

and Q_Ω is a sunny retraction if for all $x \in B$,

$$Q_\Omega(Q_\Omega(x) + t(x - Q_\Omega(x))) = Q_\Omega(x), \quad \forall t \in R.$$

The following characterization of a sunny nonexpansive retraction mapping can be found, e.g., in [4].

Proposition 2.2. *Q_Ω is a sunny nonexpansive retraction if and only if for all $x \in B$ and $y \in \Omega$,*

$$\langle x - Q_\Omega(x), J(Q_\Omega(x) - y)\rangle \geq 0.$$

From Proposition 2.2, we have the following retraction shift equality.

Proposition 2.3. *Let B be a Banach space and Ω be a nonempty closed convex subset of B. Let Q_Ω be a sunny nonexpansive retraction mapping and $m : B \to B$ be a single valued mapping. Then for all $x \in B$, we have*

$$Q_{\Omega+m(x)}(x) = m(x) + Q_\Omega(x - m(x)).$$

3. Iterative Algorithm and Convergence

In this section, we first derive some characterizations of solutions of the system of generalized co-complementarity problem.

Theorem 3.1. *Let B_1 and B_2 be two Banach spaces with normalized duality mapping J_1 and J_2, respectively. Suppose $X_1 \subset B_1$ and $X_2 \subset B_2$ are two closed convex cones such that the sunny nonexpansive retraction mappings Q_{X_1} and Q_{X_2} exist. Let $F : B_1 \times B_2 \to B_1$, $G : B_1 \times B_2 \to B_2$, $V_i : B_i \to CB(B_i)$ and $g_i, m_i : B_i \to B_i$ for $i = 1, 2$. Assume $K_i(x) = m_i(x) + X_i$ for all $x \in B_i$ and $i = 1, 2$. Then, for any given $(x, y) \in B_1 \times B_2$ and $(u, v) \in V_1(x) \times V_2(y)$ are solutions of SGCCP (1.1) if and only if*

$$\begin{cases} g_1(x) = m_1(x) + Q_{X_1}(g_1(x) - \tau_1 F(u, v)), \\ g_2(y) = m_2(y) + Q_{X_2}(g_2(y) - \tau_2 G(u, v)), \end{cases} \tag{3.1}$$

where $\tau_1 > 0$ *and* $\tau_2 > 0$ *are constants.*

Proof. From Proposition 2.3, we know that (3.1) holds if and only if

$$\begin{cases} g_1(x) = Q_{K_1(x)}(g_1(x) - \tau_1 F(u,v)), \\ g_2(y) = Q_{K_2(y)}(g_2(y) - \tau_2 G(u,v)), \end{cases} \tag{3.2}$$

From Proposition 2.2, it is easy to see that (3.2) holds if and only if

$$\langle g_1(x) - \tau_1 F(u,v) - g_1(x), J_1(g_1(x) - z_1)\rangle \geq 0, \quad \forall z_1 \in K_1(x)$$

and

$$\langle g_2(y) - \tau_2 G(u,v) - g_2(y), J_2(g_2(y) - z_2)\rangle \geq 0, \quad \forall z_2 \in K_2(y).$$

That is,

$$\begin{cases} \langle F(u,v), J_1(z_1 - g_1(x))\rangle \geq 0, \quad \forall z_1 \in K_1(x), \\ \langle G(u,v), J_2(z_2 - g_2(y))\rangle \geq 0, \quad \forall z_2 \in K_2(y). \end{cases} \tag{3.3}$$

We note that (3.3) holds if and only if

$$F(u,v) \in (J_1(K_1(x) - g_1(x)))^*, \quad G(u,v) \in (J_2(K_2(y) - g_2(y)))^*.$$

This is complete the proof.

Remark 3.1. In theorem 3.1, we suppose the sunny nonexpansive retraction mappings Q_{X_1} and Q_{X_2} exist. Such conditions can be satisfied under some assumptions, see, for example, Theorem 1 and Remark 2 in [9], or Theorem 5 and Remark 6 in [9].

Next we shall construct an iterative algorithm for finding approximate solutions of SGCCP (1.1) and discuss the convergence analysis of the algorithm.

Algorithm 3.1. Let B_i, X_i, g_i, m_i, V_i, K_i, F and G be the same as in Theorem 3.1 for $i = 1, 2$. Let $\tau_1 > 0$ and $\tau_2 > 0$ be fixed. For any given $(x_0, y_0) \in B_1 \times B_2$ and $(u_0, v_0) \in V_1(x_0) \times V_2(y_0)$, from Theorem 3.2, let

$$\begin{cases} x_1 = x_0 - g_1(x_0) + m_1(x_0) + Q_{X_1}(g_1(x_0) - \tau_1 F(u_0, v_0) - m_1(x_0)), \\ y_1 = y_0 - g_2(y_0) + m_2(y_0) + Q_{X_2}(g_2(y_0) - \tau_2 G(u_0, v_0) - m_2(y_0)). \end{cases}$$

Since $u_0 \in V_1(x_0)$ and $v_0 \in V_2(y_0)$, by Nadler's Theorem [11], there exist $u_1 \in V_1(x_1)$ and $v_1 \in V_2(y_1)$ such that

$$\|u_0 - u_1\| \leq (1+1)H(V_1(x_0), V_1(x_1)), \quad \|v_0 - v_1\| \leq (1+1)H(V_2(y_0), V_2(y_1)),$$

where H is the Hausdorff metric on $CB(B)$. Let

$$\begin{cases} x_2 = x_1 - g_1(x_1) + m_1(x_1) + Q_{X_1}(g_1(x_1) - \tau_1 F(u_1, v_1) - m_1(x_1)), \\ y_2 = y_1 - g_2(y_1) + m_2(y_1) + Q_{X_2}(g_2(y_1) - \tau_2 G(u_1, v_1) - m_2(y_1)). \end{cases}$$

Again by Nadler's Theorem, there exist $u_2 \in V_1(x_2)$ and $v_2 \in V_2(y_2)$ such that

$$\|u_1 - u_2\| \leq (1 + \frac{1}{2})H(V_1(x_1), V_1(x_2)), \quad \|v_1 - v_2\| \leq (1 + \frac{1}{2})H(V_2(y_1), V_2(y_2)).$$

Continuing in this way, we can obtain the following:

For any given $(x_0, y_0) \in B_1 \times B_2$ and $(u_0, v_0) \in V_1(x_0) \times V_2(y_0)$, compute the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by iterative schemes such that

$$\text{(3.4)} \quad \begin{cases} x_{n+1} = x_n - g_1(x_n) + m_1(x_n) + Q_{X_1}(g_1(x_n) - \tau_1 F(u_n, v_n) - m_1(x_n)), \\ y_{n+1} = y_n - g_2(y_n) + m_2(y_n) + Q_{X_2}(g_2(y_n) - \tau_2 G(u_n, v_n) - m_2(y_n)) \end{cases}$$

and

$$\text{(3.5)} \quad \begin{cases} u_n \in V_1(x_n), \quad \|u_n - u_{n+1}\| \leq (1 + \frac{1}{n+1})H(V_1(x_n), V_1(x_{n+1})), \\ v_n \in V_2(y_n), \quad \|v_n - v_{n+1}\| \leq (1 + \frac{1}{n+1})H(V_2(y_n), V_2(y_{n+1})) \end{cases}$$

for all $n = 0, 1, 2, \cdots$, where $\tau_1 > 0$ and $\tau_2 > 0$ are two constants.

Now we have the following convergence and existence result.

Theorem 3.3. *Let B_1 and B_2 be two uniformly smooth Banach spaces with $\rho_{B_1}(t) \leq C_1 t^2$, $\rho_{B_2}(t) \leq C_2 t^2$ for some $C_1 > 0$, $C_2 > 0$, respectively. Let $X_1 \subset B_1$, $X_2 \subset B_2$ be two closed convex cones such that the sunny nonexpansive retraction mappings Q_{X_1} and Q_{X_2} exist. Let $F : B_1 \times B_2 \to B_1$, $G : B_1 \times B_2 \to B_2$, $V_i : B_i \to CB(B_i)$, and $g_i, m_i : B_i \to B_i$ be mappings for $i = 1, 2$. Suppose $K_i : B_i \to 2^{B_i}$ is defined by $K_i(x) = m_i(x) + X_i$ for all $x \in B_i$ $(i = 1, 2)$ and*

- (i) *g_i and m_i are Lipschitz continuous with constants δ_i and θ_i, respectively, and V_i is H-Lipschitz continuous with constant η_i for $i = 1, 2$;*
- (ii) *g_i is strongly accretive with constant γ_i with $i = 1, 2$. For any given $(x, y) \in B_1 \times B_2$, $F(\cdot, y)$ is ξ_1-strongly accretive with respect to V_1 and $G(x, \cdot)$ is ξ_2-strongly accretive with respect to V_2;*
- (iii) *for any given $(x, y) \in B_1 \times B_2$, $F(\cdot, y)$, $F(x, \cdot)$, $G(\cdot, y)$, and $G(x, \cdot)$ are Lipschitz continuous with constants β_1, β_2, α_1, α_2, respectively;*
- (iv) *there exist $\tau_1 > 0$ and $\tau_2 > 0$ such that*

$$2(1 - 2\gamma_1 + 64C_1\delta_1^2)^{1/2} + (1 - 2\tau_1\xi_1 + 64C_1\tau_1^2\beta_1^2\eta_1^2)^{1/2} + 2\theta_1 + \tau_2\alpha_1\eta_1 < 1,$$

$$2(1 - 2\gamma_2 + 64C_2\delta_2^2)^{1/2} + (1 - 2\tau_2\xi_2 + 64C_2\tau_2^2\alpha_2^2\eta_2^2)^{1/2} + 2\theta_2 + \tau_1\beta_2\eta_2 < 1.$$

Then for any given $(x_0, y_0) \in B_1 \times B_2$ *and* $(u_0, v_0) \in V_1(x_0) \times V_2(y_0)$*, the sequences* $\{(x_n, y_n)\}$ *and* $\{(u_n, v_n)\}$ *generated by Algorithm 3.1 converge strongly to some* $(x, y) \in B_1 \times B_2$ *and* $(u, v) \in V_1(x) \times V_2(y)$*, respectively, which solve SGCCP* (1.1).

Proof. It follows from iterative schemes (3.4) that

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
=\ &\|x_n-g_1(x_n)+m_1(x_n)+Q_{X_1}(g_1(x_n)-\tau_1 F(u_n,v_n)-m_1(x_n)) \\
&-(x_{n-1}-g_1(x_{n-1})+m_1(x_{n-1})+Q_{X_1}(g_1(x_{n-1}) \\
&-\tau_1 F(u_{n-1},v_{n-1})-m_1(x_{n-1})))\| \\
\le\ &\|x_n-x_{n-1}-(g_1(x_n)-g_1(x_{n-1}))\|+\|m_1(x_n)-m_1(x_{n-1})\| \\
&+\|g_1(x_n)-\tau_1 F(u_n,v_n)-m_1(x_n)-(g_1(x_{n-1}) \\
&-\tau_1 F(u_{n-1},v_{n-1})-m_1(x_{n-1}))\| \\
\le\ &2\|x_n-x_{n-1}-(g_1(x_n)-g_1(x_{n-1}))\|+2\|m_1(x_n)-m_1(x_{n-1})\| \\
&+\|x_n-x_{n-1}-\tau_1(F(u_n,v_n)-F(u_{n-1},v_{n-1}))\| \\
\le\ &2\|x_n-x_{n-1}-(g_1(x_n)-g_1(x_{n-1}))\|+2\|m_1(x_n)-m_1(x_{n-1})\| \\
&+\|x_n-x_{n-1}-\tau_1(F(u_n,v_n)-F(u_{n-1},v_n))\| \\
&+\tau_1\|F(u_{n-1},v_n)-F(u_{n-1},v_{n-1})\|.
\end{aligned} \tag{3.6}
$$

By Proposition 2.3 and the assumptions,

$$
\begin{aligned}
&\|x_n-x_{n-1}-(g_1(x_n)-g_1(x_{n-1}))\|^2 \\
\le\ &\|x_n-x_{n-1}\|^2+2\langle-(g_1(x_n)-g_1(x_{n-1})), J_1(x_n-x_{n-1} \\
&-(g_1(x_n)-g_1(x_{n-1})))\rangle \\
=\ &\|x_n-x_{n-1}\|^2-2\langle g_1(x_n)-g_1(x_{n-1}), J_1(x_n-x_{n-1})\rangle \\
&+2\langle-(g_1(x_n)-g_1(x_{n-1})), J_1(x_n-x_{n-1}-(g_1(x_n)-g_1(x_{n-1}))) \\
&-J_1(x_n-x_{n-1})\rangle \\
\le\ &\|x_n-x_{n-1}\|^2-2\gamma_1\|x_n-x_{n-1}\|^2+4d^2\rho_{B_1}(4\|g_1(x_n)-g_1(x_{n-1})\|/d) \\
\le\ &\|x_n-x_{n-1}\|^2-2\gamma_1\|x_n-x_{n-1}\|^2+64C_1\|g_1(x_n)-g_1(x_{n-1})\|^2 \\
\le\ &(1-2\gamma_1+64C_1\delta_1^2)\|x_n-x_{n-1}\|^2
\end{aligned} \tag{3.7}
$$

and

$$\|x_n-x_{n-1}-\tau_1(F(u_n,v_n)-F(u_{n-1},v_n))\|^2$$

$$\begin{aligned}
&\le \|x_n - x_{n-1}\|^2 + 2\langle -\tau_1(F(u_n, v_n) - F(u_{n-1}, v_n)), J_1(x_n - x_{n-1} \\
&\quad -\tau_1(F(u_n, v_n) - F(u_{n-1}, v_n))\rangle \\
&= \|x_n - x_{n-1}\|^2 - 2\tau_1\langle F(u_n, v_n) - F(u_{n-1}, v_n), J_1(x_n - x_{n-1})\rangle \\
&\quad -2\tau_1\langle F(u_n, v_n) - F(u_{n-1}, v_n), J_1(x_n - x_{n-1} - \tau_1(F(u_n, v_n) \\
&\quad -F(u_{n-1}, v_n))) - J_1(x_n - x_{n-1})\rangle \\
&\le \|x_n - x_{n-1}\|^2 - 2\tau_1\xi_1\|x_n - x_{n-1}\|^2 + 4d^2\rho_{B_1}(4\tau_1\|F(u_n, v_n) \\
&\quad -F(u_{n-1}, v_n)\|/d) \\
&\le \|x_n - x_{n-1}\|^2 - 2\tau_1\xi_1\|x_n - x_{n-1}\|^2 \\
&\quad +64C_1\tau_1^2\|F(u_n, v_n) - F(u_{n-1}, v_n)\|^2 \\
&\le (1 - 2\tau_1\xi_1 + 64C_1\tau_1^2\beta_1^2\eta_1^2(1 + \frac{1}{n+1})^2)\|x_n - x_{n-1}\|^2,
\end{aligned} \tag{3.8}$$

where $J_1 : B_1 \to B_1^*$ is the normalized duality mapping. It follows from the Lipschitz continuity of the mappings m_1 and F that

$$\|m_1(x_n) - m_1(x_{n-1})\| \le \theta_1\|x_n - x_{n-1}\| \tag{3.9}$$

and

$$\|F(u_{n-1}, v_n) - F(u_{n-1}, v_{n-1}) \le \beta_2\eta_2(1 + \frac{1}{n})\|y_n - y_{n-1}\|. \tag{3.10}$$

From (3.6)-(3.10), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| \le\ &\{2(1 - 2\gamma_1 + 64C_1\delta_1^2)^{1/2} \\
&+(1 - 2\tau_1\xi_1 + 64(1 + \frac{1}{n+1})^2 C_1\tau_1^2\beta_1^2\eta_1^2)^{1/2} \\
&+ 2\theta_1\}\|x_n - x_{n-1}\| + \tau_1\beta_2\eta_2(1 + \frac{1}{n})\|y_n - y_{n-1}\|.
\end{aligned} \tag{3.11}$$

Similarly, we have

$$\begin{aligned}
\|y_{n+1} - y_n\| \le\ &\{2(1 - 2\gamma_2 + 64C_2\delta_2^2)^{1/2} \\
&+(1 - 2\tau_2\xi_2 + 64(1 + \frac{1}{n+1})^2 C_2\tau_2^2\alpha_2^2\eta_2^2)^{1/2} \\
&+ 2\theta_2\}\|y_n - y_{n-1}\| + \tau_2\alpha_1\eta_1(1 + \frac{1}{n})\|x_n - x_{n-1}\|.
\end{aligned} \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \le k_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|), \tag{3.13}$$

where $k_n = \max\{\varepsilon_n, \lambda_n\}$ and

$$\varepsilon_n = 2(1-2\gamma_1+64C_1\delta_1^2)^{1/2} + (1-2\tau_1\xi_1+64(1+\frac{1}{n+1})^2C_1\tau_1^2\beta_1^2\eta_1^2)^{1/2} +2\theta_1+\tau_2\alpha_1\eta_1(1+\frac{1}{n}),$$
$$\lambda_n = 2(1-2\gamma_2+64C_2\delta_2^2)^{1/2} + (1-2\tau_2\xi_2+64(1+\frac{1}{n+1})^2C_2\tau_2^2\alpha_2^2\eta_2^2)^{1/2} +2\theta_2+\tau_1\beta_2\eta_2(1+\frac{1}{n}).$$

Let

$$\varepsilon = 2(1-2\gamma_1+64C_1\delta_1^2)^{1/2} + (1-2\tau_1\xi_1+64C_1\tau_1^2\beta_1^2\eta_1^2)^{1/2} +2\theta_1+\tau_2\alpha_1\eta_1,$$
$$\lambda = 2(1-2\gamma_2+64C_2\delta_2^2)^{1/2} + (1-2\tau_2\xi_2+64C_2\tau_2^2\alpha_2^2\eta_2^2)^{1/2} +2\theta_2+\tau_1\beta_2\eta_2.$$

Then,

$$\epsilon_n \to \epsilon \quad \text{and} \quad \lambda_n \to \lambda \quad \text{as} \quad n \to \infty.$$

Let $k = \max\{\varepsilon, \lambda\}$. Then $k_n \to k$ as $n \to \infty$. It follows from condition (iv) that $0 < k < 1$. Hence, there are a positive number k_0 and an integer $n_0 \geq 1$ such that $k_n \leq k_0 < 1$ for all $n \geq n_0$.

Now we define $\|\cdot\|_1$ on $B_1 \times B_2$ by

$$\|(x,y)\|_1 = \|x\| + \|y\|, \quad \forall (x,y) \in B_1 \times B_2.$$

It is easy to see that $(B_1 \times B_2, \|\cdot\|_1)$ is a Banach space. Let $z_n = (x_n, y_n) \in B_1 \times B_2$. It follows from (3.13) that

$$\|z_{n+1} - z_n\|_1 \leq k_n \|z_n - z_{n-1}\|_1.$$

This implies that $\{z_n\}$ is a Cauchy sequence in $(B_1 \times B_2, \|\cdot\|_1)$. Suppose that $\{z_n\}$ converges to some $z = (x,y) \in B_1 \times B_2$. Since

$$\|x_n - x\| \leq \|x_n - x\| + \|y_n - y\| = \|z_n - z\|_1 \to 0 \quad (n \to +\infty),$$
$$\|y_n - y\| \leq \|x_n - x\| + \|y_n - y\| = \|z_n - z\|_1 \to 0 \quad (n \to +\infty),$$

it is easy to see that $\{x_n\}$ converges to $x \in B_1$ and $\{y_n\}$ converges to $y \in B_2$, respectively. By (3.5), we obtain

$$(3.14) \qquad \begin{cases} \|u_n - u_{n+1}\| \leq (1+\frac{1}{n+1})H(V_1(x_n), V_1(x_{n+1})) \\ \qquad \leq (1+\frac{1}{n+1})\eta_1\|x_n - x_{n+1}\|, \\ \|v_n - v_{n+1}\| \leq (1+\frac{1}{n+1})H(V_2(y_n), V_2(y_{n+1})) \\ \qquad \leq (1+\frac{1}{n+1})\eta_2\|y_n - y_{n+1}\|. \end{cases}$$

Let $w_n = (u_n, v_n) \in B_1 \times B_2$. By (3.14),

$$\|w_n - w_{n-1}\|_1 \leq s_n \|z_n - z_{n-1}\|_1,$$

where

$$s_n = \max\{(1 + \frac{1}{n+1})\eta_1,\ (1 + \frac{1}{n+1})\eta_2\}.$$

Since $\{z_n\}$ is a Cauchy sequence, we know that $\{w_n\}$ is also a Cauchy sequence in $B_1 \times B_2$. Suppose that $\{w_n\}$ converges to some $w = (u, v) \in B_1 \times B_2$. Then it is easy to see that $\{u_n\}$ converges to u and $\{v_n\}$ converges to v, respectively. Since F, G, Q_{X_i}, g_i, m_i, and V_i are all continuous $(i = 1, 2)$, we have

$$x = x - g_1(x) + m_1(x) + Q_{X_1}(g_1(x) - \tau_1 F(u, v) - m_1(x)),$$
$$y = y - g_2(y) + m_2(y) + Q_{X_2}(g_2(y) - \tau_2 G(u, v) - m_2(y)).$$

It remains to show that $(u, v) \in V_1(x) \times V_2(y)$. In fact,

$$\begin{aligned} d(u, V_1(x)) &\leq \|u - u_n\| + d(u_n, V_1(x)) \\ &\leq \|u - u_n\| + H(V_1(x_n), V_1(x)) \\ &\leq \|u - u_n\| + \eta_1 \|x - x_n\|, \end{aligned}$$

where

$$d(u, V_1(x)) = \inf\{\|u - z\| : z \in V_1(x)\}.$$

It follows that $d(u, V_1(x)) = 0$ and so $u \in V_1(x)$ since $V_1(x)$ is closed. Similarly, we have $v \in V_2(y)$. By Theorem 3.2, we know that $(x, y) \in B_1 \times B_2$ and $(u, v) \in V_1(x) \times V_2(y)$ are solutions of SGCCP (1.1). This completes the proof.

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