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AN ENGEL CONDITION WITH GENERALIZED DERIVATIONS ON LIE IDEALS

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Abstract. Let R be a prime ring, with extended centroid C, g a non-zero generalized derivation of R, L a non-central Lie ideal of R, $k \geq 1$ a fixed integer. If $[g(u), u]_k = 0$, for all u, then either $g(x) = ax$, with $a \in C$ or R satisfies the standard identity s_4. Moreover in the latter case either $char(R) = 2$ or $char(R) \neq 2$ and $g(x) = ax + xb$, with $a, b \in Q$ and $a - b \in C$.

We also prove a more generalized version by replacing L with the set $[I, I]$, where I is a right ideal of R.

1. INTRODUCTION

Let R be a prime ring with center $Z(R)$ and extended centroid C, Q the Martindale quotients ring, U the Utumi quotients ring. We denote by $[a, b] = ab - ba$ the simple commutator of the elements $a, b \in R$ and by $[a, b]_k = [[a, b]_{k-1}, b]$, for $k > 1$, the k-th commutator of a, b. A well known result of Posner [16] says that if d is a derivation of R such that $[d(x), x] \in Z(R)$, for all $x \in R$, then R is commutative. In [7] Lanski generalizes the result of Posner, by replacing the element $x \in R$ with an element of a non-central Lie ideal L of R. More precisely he proves that if $[d(x), x]_k = 0$ for all $x \in L$ and $k \geq 1$ a fixed integer, then $char(R) = 2$ and R satisfies s_4, the standard identity of degree 4. Later in [8] Lee and Lee consider a similar Engel-condition, $[d(x), x]_k = 0$, in case $x \in \{f(x_1, .., x_n), x_1, .., x_n \in I\}$, where I is a two-sided ideal of R and $f(x_1, .., x_n)$ a multilinear polynomial in R. They show that either $f(x_1, .., x_n)$ is central valued in R or $char(R) = 2$ and R satisfies s_4. More recently in [9] Lee extendes this last result to the case when the valutations of $f(x_1, .., x_n)$ are in a right ideal I of R. In particular the author studies what happens when $f(x_1, ..x_n)$ is multilinear. In this case, the conclusion

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is that $I = eRC$ for a suitable idempotent element $e \in I$ and either $f(x_1, .., x_n)$ is central valued in $eRCe$ or $char(R) = 2$ and $eRCe$ satisfies s_4.

In this paper we will continue the line of investigation concerning the Engel-conditions $[g(x), x]_k = 0$ for all $x \in S$ a suitable subset of R, with g additive mapping in R. More precisely, in what follows $S = L$ denotes a non-central Lie ideal of R and g is a generalized derivation on R, i.e. an additive mapping on R such that $g(xy) = g(x)y + xd(y)$, for all $x, y \in R$ and d a derivation of R. In the first section we will prove the following:

Theorem. *Let R be a prime ring, with extended centroid C, g a non-zero generalized derivation of R, L a non-central Lie ideal of R, $k \geq 1$ a fixed integer. If $[g(u), u]_k = 0$, for all u, then either $g(x) = ax$, with $a \in C$ or R satisfies the standard identity s_4. Moreover in the latter case either $char(R) = 2$ or $char(R) \neq 2$ and $g(x) = ax + xb$, with $a, b \in Q$ and $a - b \in C$.*

Then we will extend the above result to the one-sided case, more precisely we will prove:

Theorem. *Let R be a prime ring, g a non-zero generalized derivation of R, I a non-zero right ideal of R such that $[I, I]I \neq 0$, $k \geq 1$.*

If $[g([r_1, r_2]), [r_1, r_2]]_k = 0$, for any $r_1, r_2 \in I$, then either $g(x) = cx$, for suitable $c \in R$, such that $(c - \gamma)I = 0$ for a suitable $\gamma \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and $eRCe$ satisfies s_4. In the latter case either $char(R) = 2$ or $char(R) \neq 2$ and $g(x) = cx + xb$, for suitable $c, b \in R$ and there exists $\gamma \in C$ such that $(c - b + \gamma)I = 0$.

We would like to point out that in [10] Lee proves that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U. In particular Lee proves the following result:

Theorem 3 in [10]. *Every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$, for some $a \in U$ and a derivation d on U.*

For more details on generalized derivations we refer the reader to [5, 10, 14].

1. Engel Condition on Lie Ideals

Here we begin with the following:

Theorem 1. *Let R be a non-commutative prime ring, $a, b \in R$, I a two-sided ideal of R, $k \geq 1$ a fixed integer such that $[a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]]_k = 0$, for any $r_1, r_2 \in I$. Then either $a, b \in Z(R)$ or R satisfies the standard identity s_4. In the latter case either $char(R) = 2$ or $char(R) \neq 2$ and $a - b \in Z(R)$.*

Proof. Suppose that either $a \notin Z(R)$ or $b \notin Z(R)$. In both cases

$$[a[x_1, x_2] + [x_1, x_2]b, [x_1, x_2]]_k$$

is a non-trivial generalized polynomial identity for I ando so also for R. By Theorem 2 in [1], $[a[x_1, x_2]+[x_1, x_2]b, [x_1, x_2]]_k$ is also an identity for RC. By Martindale's result in [15] RC is a primitive ring with non-zero socle. There exists a vectorial space V over a division ring D such that RC is dense of D-linear transformations over V.

Suppose that $dim_D V \geq 3$ and $\{v, va\}$ are linearly D-independent for some $v \in V$. By the density of RC, there exists $w \in V$ such that $\{w, v, va\}$ are linearly D-independent and $x_0, y_0 \in RC$ such that $vx_0 = 0$, $vy_0 = 0$, $(va)x_0 = w$, $(va)y_0 = 0$ $wy_0 = va$. This leads to the contradiction $0 = v[a[x_0, y_0] + [x_0, y_0]b, [x_0, y_0]]_k = va \neq 0$. Thus $\{v, va\}$ are linearly D-dependent, for all $v \in V$, which implies that $a \in C$. From this, RC satisfies $[[x_1, x_2]b, [x_1, x_2]]_k$. As above suppose that there exists $v \in V$ such that $\{v, vb\}$ are linearly D-independent. Then there exists $w \in V$ such that $\{v, vb, w\}$ are linearly D-independent and there exist $x_0, y_0 \in RC$ such that $vx_0 = w$, $vy_0 = 0$, $wy_0 = v$, $(vb)x_0 = v$, $(vb)y_0 = 0$. This implies that $0 = v[[x_0, y_0]b, [x_0, y_0]]_k = (-1)^k vb \neq 0$, a contradiction. Also in this case we conclude that $\{v, vb\}$ are linearly D-dependent, for all $v \in V$, and so $b \in C$.

Consider now the case when $dim_D V \leq 2$. In this condition RC is a simple ring which satisfies a non-trivial generalized polynomial identity. By [17, Theorem 2.3.29] $RC \subseteq M_t(F)$, for a suitable field F, moreover $M_t(F)$ satisfies the same generalized identity of RC, hence $[a[r_1, r_2]+[r_1, r_2]b, [r_1, r_2]]_k = 0$, for any $r_1, r_2 \in M_t(F)$. If $t \geq 3$, by the above argument, we get $a, b \in F$. If $t = 1$ there is nothing to prove. Let $t = 2$.

Suppose that $char(R) \neq 2$, if not we are done. Denote e_{ij} the usual matrix unit and $a = \sum a_{ij}e_{ij}$, $b = \sum b_{ij}e_{ij}$, for $a_{ij}, b_{ij} \in F$.

Notice that, if k is even:

$$[a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]]_k = 2^{k-1}\left((a-b)[r_1, r_2]^{k+1} - [r_1, r_2]^{k+1}(a-b)\right) \tag{1}$$

and if k is odd:

$$[a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]]_k = 2^{k-1}\left((a-b)[r_1, r_2]^{k+1} - [r_1, r_2]^{k}(a-b)[r_1, r_2]\right). \tag{2}$$

Choose $[r_1, r_2] = e_{ii} - e_{jj}$ for any $i \neq j$.

In case k is even, from (1) and since $char(R) \neq 2$, we get

$$0 = (a-b)(e_{ii} - e_{jj}) - (e_{ii} - e_{jj})(a-b)$$

and right multiplying by e_{ii} and left multiplying by e_{jj}:

$$0 = e_{jj}(a-b)e_{ii} + e_{jj}(a-b)e_{ii}$$

that is $2(a_{ji} - b_{ji}) = 0$, which means that $a-b$ is a diagonal matrix.

In case k is odd, from (2) and since $char(R) \neq 2$,

$$0 = (a-b) - (e_{ii} - e_{jj})(a-b)(e_{ii} - e_{jj})$$

and again right multiplying by e_{ii} and left multiplying by e_{jj}:

$$0 = e_{jj}(a-b)e_{ii} + e_{jj}(a-b)e_{ii}$$

that is $a-b$ is a diagonal matrix as above.

Let now φ is an automorphism of $M_2(F)$, the same conclusion holds for $\varphi(a-b)$, since as above, for all $r_1, r_2 \in M_2(F)$

$$0 = [\varphi(a)\varphi([r_1, r_2]) + \varphi([r_1, r_2])\varphi(b), \varphi([r_1, r_2])]_k.$$

Therefore $\varphi(a-b)$ must be a diagonal matrix. In particular choose $\varphi(x) = (1 + e_{ij})x(1 - e_{ij})$ for $i \neq j$. Thus the (i, j) entry of the matrix $\varphi(a-b)$ must be zero, that is $a_{jj} - b_{jj} = a_{ii} - b_{ii}$ for all $i \neq j$, which means that $a-b$ is a central element. ■

As a natural consequence we obtain the following:

Corollary 1. *Let R be a non-commutative prime ring, $a \in R$, I a two-sided ideal of R, $k \geq 1$ a fixed integer.*

If $[a[r_1, r_2], [r_1, r_2]]_k = 0$, for any $r_1, r_2 \in I$, then either $a \in Z(R)$ or $char(R) = 2$ and R satisfies the standard identity s_4.

Corollary 2. *Let R be a non-commutative prime ring, $b \in R$, I a two-sided ideal of R, $k \geq 1$ a fixed integer.*

If $[[r_1, r_2]b, [r_1, r_2]]_k = 0$, for any $r_1, r_2 \in I$, then either $b \in Z(R)$ or $char(R) = 2$ and R satisfies the standard identity s_4.

Now we will consider the Engel condition on Lie ideals:

Theorem 2. *Let R be a prime ring, with extended centroid C, g a non-zero generalized derivation of R, L a non-central Lie ideal of R, $k \geq 1$ a fixed* *integer. If* $[g(u), u]_k = 0$*, for all* u*, then either* $g(x) = ax$*, with* $a \in C$ *or* R *satisfies the standard identity* s_4*. Moreover in the latter case either* $char(R) = 2$ *or* $char(R) \neq 2$ *and* $g(x) = ax + xb$ *, with* $a, b \in Q$ *and* $a - b \in C$*.*

Proof. Since L is a non-central Lie ideal, by [4, pages 4-5] we have that either $char(R) = 2$ and R satisfies s_4, or there exists a two-sided ideal I of R such that $[I, I] \subseteq L$. In this last case we get that $[g([r_1, r_2]), [r_1, r_2]]_k = 0$ for any $r_1, r_2 \in I$.

Denote $g(x) = ax + d(x)$, for $a \in Q$, the Martindale quotient ring of R, and d a derivation of U.

If d is an inner derivation induced by an element $c \in Q$, it follows that

$$[(a + c)[r_1, r_2] - [r_1, r_2]c, [r_1, r_2]]_k = 0$$

for any $r_1, r_2 \in I$, and by theorem 1 we have that one of the following holds:

(i) $char(R) = 2$ and R satisfies s_4, and we are done;

(ii) $a+c$ and c are central elements, that is $a, c \in C$, so that $d = 0$ and $g(x) = ax$;

(iii) $char(R) \neq 2$, R satisfies s_4 and $(a + c) - (-c) = a + 2c \in C$, which means that $g(x) = a'x + xb'$, with $a' = a + c$, $b' = -c$ and $a' - b' \in C$.

Let now d an outer derivation. Since

$$0 = [a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)], [x_1, x_2]]_k \tag{3}$$

is an identity for I, by Kharchenko's result in [6], it follows that $[a[r_1, r_2], [r_1, r_2]]_k = 0$ for any $r_1, r_2 \in I$ and we end up, by Corollary 1, that either $char(R) = 2$ and R satisfies s_4, or $a \in C$. In this last case, from (3), we have that

$$[[d(x_1), x_2] + [x_1, d(x_2)], [x_1, x_2]]_k$$

is an identity for I and again by Kharchenko's theorem in [6], it follows that $[[x_1, x_3], [x_1, x_2]]_k$ is an identity for I. This implies obviously that R is a P.I.-ring satisfying $[[x_1, x_3], [x_1, x_2]]_k$. Thus there exists a field F such that R and $M_t(F)$, the ring of $t \times t$ matrices over F, satisfy the same polynomial identities. If $t = 1$ R is commutative, which is a contradiction since L is not central. Moreover in case $t = 2$ and $char(R) = 2$ we are also done.

Suppose $t = 2$ and $char(R) \neq 2$. Pick $x_1 = e_{12}$, $x_2 = e_{21}$ and $x_3 = e_{22}$. By calculation we have the contradiction $0 = [[x_1, x_3], [x_1, x_2]]_k = (-2)^k e_{12}$.

Assume now that $t \geq 3$ and choose $x_1 = e_{13}$, $x_2 = e_{31}$, $x_3 = e_{32}$. Also in this case we get the contradiction $0 = [[x_1, x_3], [x_1, x_2]]_k = (-1)^k e_{12}$. ■

2. Engel Condition on Right Ideals

Now we extend the previous results to a non-zero right ideal of R and prove the following:

Theorem. *Let R be a prime ring, g a non-zero generalized derivation of R, I a non-zero right ideal of R such that $[I, I]I \neq 0$, $k \geq 1$.*

If $[g([r_1, r_2]), [r_1, r_2]]_k = 0$, for any $r_1, r_2 \in I$, then either $g(x) = cx$, for suitable $c \in R$, such that $(c - \gamma)I = 0$ for a suitable $\gamma \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and $eRCe$ satisfies s_4. In the latter case either $char(R) = 2$ or $char(R) \neq 2$ and $g(x) = cx + xb$, for suitable $c, b \in R$ and there exists $\gamma \in C$ such that $(c - b + \gamma)I = 0$.

We begin this section with:

Lemma 1. *Let R be a prime ring, g a non-zero generalized derivation of R, I a non-zero right ideal of R, $k \geq 1$ a fixed integer such that $[g([r_1, r_2]), [r_1, r_2]]_k = 0$, for any $r_1, r_2 \in I$. Then R satisfies a non-trivial generalized polynomial identity, except when $g(x) = ax$, with $a \in Q$ and there exists $\lambda \in C$ such that $(a-\lambda)I = 0$.*

Proof. Consider the generalized derivation g assuming the form $g(x) = ax + d(x)$, for an usual derivation d of R. We divide the proof into two cases:

Case 1. Suppose that the derivation d is inner, induced by some element $q \in Q$, that is $d(x) = [q, x]$.

Thus we have, for all $r_1, r_2 \in I$

$$[a[r_1, r_2] + d([r_1, r_2])), [r_1, r_2]]_k = [(a + q)[r_1, r_2] - [r_1, r_2]q, [r_1, r_2]]_k = 0$$

and denote $a + q = c$, so that

$$[c[r_1, r_2] - [r_1, r_2]q, [r_1, r_2]]_k = 0.$$

If both c and q are central elements we conclude that $g(x) = ax$, $a \in C$. Thus consider that one of q and c is non-central.

Let $u \in I$ such that $\{cu, u\}$ are linearly C-independent. If $qu = \beta u$ for some $\beta \in C$, then R satisfies

$$\sum_{i+j=k-1} [ux_1, ux_2]^i (c[ux_1, ux_2] - [ux_1, ux_2]\beta)[ux_1, ux_2]^j$$
$$+[ux_1, ux_2]^k (c[ux_1, ux_2] - [ux_1, ux_2]q)$$

which is a non-trivial GPI. On the other hand

$$[c[ux_1, ux_2] - [ux_1, ux_2]q, [ux_1, ux_2]]_k$$

is a non-trivial GPI also in case $\{q, qu\}$ are linearly C-independent.

Let now $cu = \alpha u$ for some $\alpha \in C$. Then R satisfies

$$[\alpha[ux_1, ux_2] - [ux_1, ux_2]q, [ux_1, ux_2]]_k$$

which is again a non-trivial GPI for R.

Case 2. Let now d be an outer derivation. Since I satisfies

$$[a[x_1, x_2] + d([x_1, x_2])), [x_1, x_2]]_k$$

it also satisfies

$$[(a-\lambda)[x_1, x_2] + d([x_1, x_2]), [x_1, x_2]]_k$$

for any $\lambda \in C$.

Note that, if there exists $\lambda \in C$ such that $(a-\lambda)I = 0$, then $[d([x_1, x_2]), [x_1, x_2]]_k$ is a differential identity for I. In this case, by [9], one of the following holds:

– $[x_1, x_2]x_3$ is an identity for I, so R is a GPI-ring;
– $char(R) = 2$ and $s_4(I, I, I, I)I = 0$ and again R is GPI;
– $d = 0$ and so $g(x) = ax$ for $(a - \lambda)I = 0$, and again we are done.

Consider the case when $(a - \alpha)I \neq 0$, for all $\alpha \in C$. We note that, under this assumption, there exists $u \in I$ such that $au \neq \alpha u$, for all $\alpha \in C$. In fact, if suppose that $\{ay, y\}$ are linearly C-dependent, for all $y \in I$, then, by Lemma 3 in [11], there exists $\beta \in C$ such that $(a - \beta)I = 0$, a contradiction.

Since I and IU satisfy the same differential identities,

$$[a[x_1, x_2] + d([x_1, x_2]), [x_1, x_2]]_k$$

is an identity for IU, that is

$$[a[ux_1, ux_2] + d([ux_1, ux_2]), [ux_1, ux_2]]_k$$

is an identity for U. Thus U satisfies the following

$$[a[ux_1, ux_2] + [d(u)x_1 + ud(x_1), x_2] + [x_1, d(u)x_2 + ud(x_2)], [ux_1, ux_2]]_k \,.$$

Since d is an outer derivation, by Kharchenko's result in [6], U satisfies the identity

$$[a[ux_1, ux_2] + [d(u)x_1 + uy_1, x_2] + [x_1, d(u)x_2 + uy_2], [ux_1, ux_2]]_k \,.$$

which is a non-trivial GPI for R, since au and u are linearly C-independent. ■

Remark 1. Without loss of generality R is simple and equal to its own socle, $IR = I$.

In fact by Lemma 1, R is GPI and so RC has non-zero socle H with non-zero right ideal $J = IH$ [15]. Note that H is simple, $J = JH$ and J satisfies the same basic conditions as I [13]. Now just replace R by H, I by J and we are done.

Remark 2. It is well known that all the following statements hold (see [12]):

(1) If $[x_1, x_2]x_3$ is an identity for I, then there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and $eRCe$ is commutative;

(2) if $char(R) = 2$ and I satisfies $s_4(x_1, x_2, x_3, x_4)x_5$ then there exists $e^2 = e \in soc(RC)$ such that $IC = eRC$ and $s_4(x_1, .., x_4)$ is an identity for $eRCe$;

Remark 3. Since $R = H$ is a regular ring, then for any $a_1, .., a_n \in I$ there exists $h = h^2 \in R$ such that $\sum_{i=1}^{n} a_i R = hR$. Then $h \in IR = I$ and $a_i = ha_i$ for each $i = 1, .., n$.

In order to continue our line of investigation, we need the following:

Lemma 2. *Let R be a prime ring, $a \in R$, I a non-zero right ideal of R, $k \geq 1$, such that $[I, I]I \neq 0$. If $[a[r_1, r_2], [r_1, r_2]]_k = 0$ for all $r_1, r_2 \in I$, then either $(a - \gamma)I = 0$ for a suitable $\gamma \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$, $char(R) = 2$ and $s_4(x_1, x_2, x_3, x_4)$ is an identity for $eRCe$.*

Proof. Suppose by contradiction that there exist $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9 \in I$ such that

- $[c_1, c_2]c_3 \neq 0$;
- if $char(R) = 2$, $s_4(c_4, c_5, c_6, c_7)c_8 \neq 0$;
- $\{c_9, ac_9\}$ are linearly C-independent.

By Remark 3, there exists an idempotent element $h \in IH = IR$ such that $hR = \sum_{i=1}^{9} c_i R$ and $c_i = hc_i$, for any $i = 1, .., 9$. Since $[a[hx_1, hx_2], [hx_1, hx_2]]_k$ is satisfied by $R = H$, left multiplying by $(1 - h)$, we get that R satisfies $(1 - h)a[hx_1, hx_2]^{k+1}$. By [2] it follows that either $(1 - h)ah = 0$ or $[hx_1, hx_2]hx_3$ is a generalized identity for R. Since this last contradicts with $[c_1, c_2]c_3 \neq 0$, we have that $ah = hah$. Moreover $[a[x_1, x_2], [x_1, x_2]]_k$ is also satisfied by hRh.

By Corollary 1, again since $[c_1, c_2]c_3 \neq 0$, we get either $ah \in Ch$ or $char(R) = 2$ and hRh satisfies s_4.

In the last case we get a contradiction since $s_4(c_4, c_4, c_6, c_7)c_8 \neq 0$ when $char(R) = 2$. In the first case, if $ah \in Ch$, then there exists $\lambda \in C$ such that $ahc_9 = (\lambda)hc_9$, that is $ac_9 = \lambda c_9$, a contradiction again. ∎

Lemma 3. *Let $R = M_n(F)$ the ring of $n \times n$ matrices over the field F. Let $b \in R$ and I a non-zero right ideal of R such that $s_4(I, I, I, I)I \neq 0$. If $[[r_1, r_2]b, [r_1, r_2]]_k = 0$, for all $r_1, r_2 \in I$, then $b \in F$.*

Proof. We denote again e_{ij} the usual matrix unit with 1 in the (i,j)-entry and zero elsewhere and write $b = \sum b_{ij}e_{ij}$, with b_{ij} elements of F. Moreover assume $I = eR$ for some $e = \sum_{i=1}^{t} e_{ii}$ and $t \geq 3$.

Since $s_4(I, I, I, I)I \neq 0$, there exist $c_1, c_2, c_3, c_4, c_5 \in I$ such that $s_4(c_1, c_2, c_3, c_4)c_5 \neq 0$. Let $[x, y] = [e_{ij}, e_{ji}] = e_{ii} - e_{jj} \in [I, I]$, for $1 \leq i, j \leq t$ and $i \neq j$. Then $0 = [(e_{ii} - e_{jj})b, (e_{ii} - e_{jj})]_k$ and right multiplying by e_{rr}, for $r \neq i, j$, we have $0 = (e_{ii} - e_{jj})^{k+1}be_{rr}$. Left multiplying by e_{ii} we have that $b_{ir} = 0$ for all $r \neq i, j$. Choose now another index $l \neq j$ such that $1 \leq l \leq t$ and $l \neq i$. As above we get the condition $0 = (e_{ii} - e_{ll})^{k+1}be_{rr}$ for all $r \neq i, l$ and once again, left multiplying by e_{ii}, we have $b_{ir} = 0$ for all $r \neq i, l$. In particular, since $j \neq l$, one has that $b_{ij} = 0$. All this says that, if you fix an index $i \leq t$, it follows that $b_{ir} = 0$ for any $r \neq i$.

Let now $i, j \leq t$ be different indeces and $r > t$, $s \neq i, j, r$. For $[x, y] = [e_{ij}, e_{jr} + e_{ji}] = e_{ir} + e_{ii} - e_{jj} \in [I, I]$,

$$0 = [(e_{ir} + e_{ii} - e_{jj})b, e_{ir} + e_{ii} - e_{jj}]_k$$

and right multiplying by e_{ss}

$$0 = (e_{ir} + e_{ii} - e_{jj})^{k+1}be_{ss} = (e_{ir} + e_{ii} + (-1)^{k+1}e_{jj})be_{ss}.$$

Since we have proved above that $b_{is} = 0$ and $b_{js} = 0$, in this last case we get $b_{rs} = 0$ for all $r > t$ and $s \neq i, j, r$. As above, since $t \geq 3$, by repeating this process for any couple $(i \neq j)$, we get that $b_{rs} = 0$ for all $r > t$ and $s \neq r$.

The previous argument says that $b = \sum_{i=1,n} b_{ii}e_{ii}$.

Let $r \neq s$ be both $\leq t$ and f be the F-automorphism of R defined by $f(x) = (1 - e_{rs})x(1 + e_{rs})$. Thus we have that $f(x) \in I$, for all $x \in I$ and $[[r_1, r_2]f(b), [r_1, r_2]]_k = 0$, for all $r_1, r_2 \in I$. Since $f(b) = (1 - e_{rs})b(1 + e_{rs}) = b + b_{rr}e_{rs} - b_{ss}e_{rs}$ we have that $b_{rr} = b_{ss}$ for all $r, s \leq t$, that is $b = \beta e + \sum_{i=t+1,n} b_{ii}e_{ii}$, for a suitable $\beta \in F$.

This means that there exists $\beta \in F$ such that $(b - \beta)I = 0$. Denote $b - \beta = p$, $pI = 0$. Since $[[r_1, r_2]p, [r_1, r_2]]_k = 0$, for all $r_1, r_2 \in I$, we have that $[r_1, r_2]^{k+1}p = 0$. In this case, by the assumption that $s_4(c_1, c_2, c_3, c_4,)c_5 \neq 0$ and by [2] we have $p = 0$ that is $b \in F$. ■

Lemma 4. *Let R be a prime ring, $b \in R$ and I a non-zero right ideal of R such that $s_4(I, I, I, I)I \neq 0$. If $[[r_1, r_2]b, [r_1, r_2]]_k = 0$, for all $r_1, r_2 \in I$, then $b \in C$.*

Proof. We consider the only case when R satisfies a non-trivial generalized polynomial identity, as a reduction of Lemma 1.

Thus the Martindale quotients ring Q of R is a primitive ring with non-zero socle $H = Soc(Q)$. H is a simple ring with minimal right ideals. Let D the associated division ring of H, it is well known that D is a simple central algebra finite dimensional over $C = Z(Q)$. Thus $H \otimes_C F$ is a simple ring with minimal right ideals, with F the central closure of C. Let b an element of R which induces the derivation d. Moreover $[[r_1, r_2]b, [r_1, r_2]]_k = 0$, for all $r_1, r_2 \in IH \otimes_C F$ (see for instance [1, theorem 2]). Notice that if C is finite, we choose $F = C$.

Suppose that there exist $c_1, c_2 \in IH$ and such that $[b, c_1]c_2 \neq 0$. Moreover we know that $[[r_1, r_2]b, [r_1, r_2]]_k = 0$, for all $r_1, r_2 \in IH$. Since H is regular, by Litoff's theorem (see [3]), there exists $g^2 = g \in IH$, such that $c_1, c_2 \in g(IH \otimes_C F)$, and $e^2 = e \in H \otimes_C F$, such that

$$g, bg, gb, c_1, c_2, bc_1, c_1b \in e(H \otimes_C F)e \cong M_n(F) \quad \text{and} \quad n \geq 3.$$

Let $x_1, x_2 \in ge(H \otimes_C F)e \subseteq (IH \otimes_C F) \cap M_n(F)$, then

$$0 = [[x_1, x_2]b, [x_1, x_2]]_k e = [[x_1, x_2]ebe, [x_1, x_2]]_k.$$

By Lemma 3 we have that $[ebe, ge(H \otimes_C F)e]ge(H \otimes_C F)e = 0$. In particular $[ebe, gc_1]gc_2 = 0$ and hence $[b, c_1]c_2 = 0$ a contradiction. This means that $[b, IH]IH = 0$ and so there exists $\beta \in C$ such that $(b - \beta)I = 0$. Denote $b' = (b - \beta)$, so $b'I = 0$ and, for all $r_1, r_2 \in IH$, $0 = [[r_1, r_2]b', [r_1, r_2]]_k = [r_1, r_2]^{k+1}b'$. Since $s_4(I, I, I, I)I \neq 0$, it follows from [2] that $b' = 0$, that is $b \in C$. ■

Theorem 3. *Let R be a prime ring, $a, b \in R$, I a non-zero right ideal of R such that $[I, I]I \neq 0$, $k \geq 1$.*

If $[a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]]_k = 0$, for any $r_1, r_2 \in I$, then either there exist $\alpha, \beta \in C$ such that $(a - \alpha)I = 0$ and $b = \beta$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and $eRCe$ satisfies s_4. Moreover in the latter case either $char(R) = 2$ or there exists $\gamma \in C$ such that $(a - b + \gamma)I = 0$ and $char(R) \neq 2$.

Proof. First suppose that there exist $c_1, .., c_5 \in I$ such that $s_4(c_1, c_2, c_3, c_4)c_5 \neq 0$.

Of course we are done if there exists $\alpha \in C$ such that $(a - \alpha)I = 0$. In fact in this case we have that for $a' = (a - \alpha)$:

$$0 = [a'[x_1, x_2] + [x_1, x_2]b, [x_1, x_2]]_k = [[x_1, x_2]b, [x_1, x_2]]_k$$

for all $x_1, x_2 \in I$ and we conclude by lemma 4. Therefore suppose that there exists $c_6 \in I$ such that $\{ac_6, c_6\}$ are linearly C-independent. Again there exists an idempotent element $h \in IR$ such that $hR = \sum_{i=1}^{6} c_i R$ and $c_i = hc_i$, for all $i = 1, .., 6$. Of course

$$[a[hx_1h, hx_2h] + [hx_1h, hx_2h]b, [hx_1h, hx_2h]]_k$$

is satisfied by R. Thus, a fortiori,

$$h[a[hx_1h, hx_2h] + [hx_1h, hx_2h]b, [hx_1h, hx_2h]]_k h$$

is satisfied by R and so also

$$[(hah)[hx_1h, hx_2h] + [hx_1h, hx_2h](hbh), [hx_1h, hx_2h]]_k.$$

Therefore, by applying the theorem 1 to the ring hRh, we have that $hah, hbh \in Ch$, since $s_4(hRh, hRh, hRh, hRh)hRh \neq 0$.

Moreover

$$(E1) \qquad [a[hr_1, hr_2] + [hr_1, hr_2]b, [hr_1, hr_2]]_k = 0$$

for any $r_1, r_2 \in R$. Left multiplying the (E1) by $(1-h)$ we get $(1-h)a[hr_1, hr_2]^{k+1} = 0$ and by [2] it follows that $(1-h)ah = 0$, since $[hR, hR]hR \neq 0$. This implies that $ah = hah \in Ch$, so $(a-\alpha)h = 0$ for a suitable $\alpha \in C$ and this contradicts with $(a-\alpha)hc_6 = (a-\alpha)c_6 \neq 0$.

Now suppose that $s_4(I, I, I, I)I = 0$. By remark 2, there exists an idempotent $e^2 = e \in soc(RC)$ such that $I = eRC$ and $s_4(eRCe, eRCe, eRCe, eRCe) = 0$.If $char(R) = 2$ we are done. Consider that case when $char(R) \neq 2$.

Again we repeat the same above argument: since $[a[x_1, x_2]+[x_1, x_2]b, [x_1, x_2]]_k$ is satisfied by eRe, by Theorem 1 we have that either $eae, ebe \in Ce$, or $(eae-ebe) \in Ce$, since $char(R) \neq 2$. Moreover, as above we have that $(1-e)ae = 0$ that is $ae = eae$.

Also we have that

$$(E2) \qquad [a[er_1e, er_2e] + [er_1e, er_2e]b, [er_1e, er_2e]]_k = 0$$

for all $r_1, r_2 \in R$. Right multiplying the (E2) by $(1-e)$ it follows that $[er_1e, er_2e]^{k+1} b(1-e) = 0$, that is again $eb(1-e) = 0$ by [2], since $[eR, eR]eR \neq 0$ and so $eb = ebe$.

Case 1. If $ae, eb \in Ce$ we may repeat the same proof of the first part of this lemma and conclude that $(a-\alpha)e = 0$, for a suitable $\alpha \in C$, that is $(a-\alpha)I = 0$ and $b \in C$.

Case 2. If $(ae-eb) \in Ce$, consider $h = e + er(1-e)$ for an arbitrary element $r \in R$. Notice that $h^2 = h$ and $eR = hR$. Moreover $[a[x_1, x_2]+[x_1, x_2]b, [x_1, x_2]]_k$

is satisfied by $hRCh$ and also $s_4(hRCh, hRCh, hRCh, hRCh) = 0$. This means that we may repeat the same above argument replacing $I = eRC$ with $I = hRC$. Therefore, as we have seen before, we are done in any case, unless when $ah - hb \in Ch$. Hence, to complete the proof we have to analyze this last case. We have that $ah - hb \in Ch$ means

$$(E3) \qquad a(e + er(1-e)) - (e + er(1-e))b = \lambda(e + er(1-e))$$

for all $r \in R$ and λ depending on the choice of r. The (E3) says

$$ae + aer(1-e) - eb - er(1-e)b = \lambda(e + er(1-e))$$

and right multiplying by e we have

$$ae - eb - er(1-e)be = \lambda e.$$

Since $ae - eb \in Ce$, it follows that for all $r \in R$ there exists $\lambda \in C$, depending on the choice of r, such that $er(1-e)be = \lambda e$.

If, for any $r \in R$, $er(1-e)be = 0$ then $(1-e)be = 0$, hence $be = ebe = eb$, that is $(ae - be) \in Ce$ and so $(a-b)I = \alpha I$, for a suitable $\alpha \in C$, and we are done.

Thus suppose that there exists $r_0 \in R$ such that $er_0(1-b)e = \mu e \neq 0$, for $0 \neq \mu \in C$.

Choose $r = [r_0, ye]$ for all $y \in R$. There exists a suitable $\gamma \in C$ such that:

$$\gamma e = e[r_0, ye](1-e)be = eyer_0(1-e)be = \mu eye \qquad (E4).$$

Since (E4) means that $eye \in Ce$ for all $y \in R$, it follows that $[eRC, eRC]eRC = [I, I]I = 0$, a contradiction. ■

Theorem 4. *Let R be a prime ring, g a non-zero generalized derivation of R, I a non-zero right ideal of R such that $[I, I]I \neq 0$, $k \geq 1$.*

If $[g([r_1, r_2]), [r_1, r_2]]_k = 0$, for any $r_1, r_2 \in I$, then either $g(x) = cx$, for suitable $c \in R$, such that $(c - \gamma)I = 0$ for a suitable $\gamma \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and $eRCe$ satisfies s_4. Moreover in the latter case either $char(R) = 2$ or $char(R) \neq 2$, $g(x) = cx + xb$, for suitable $c, b \in R$ and there exists $\gamma \in C$ such that $(c - b + \gamma)I = 0$.

Proof. As we have already remarked, every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$, for some $a \in U$ and a derivation d on U.

If $d = 0$, $g(x) = ax$ and we conclude by Lemma 2. Thus we suppose that $d \neq 0$.

For $u \in I$, U satisfies the following differential identity

$$[a[ux_1, ux_2] + d([ux_1, ux_2]), [ux_1, ux_2]]_k \,.$$

In light of Kharchenko's theory ([6], [13]), we divide the proof into two cases:

Case 1. Let d the inner derivation induced by the element $q \in U$, that is $d(x) = [q, x]$, for all $x \in U$. Thus I satisfies the generalized polynomial identity

$$[a[x_1, x_2] + q[x_1, x_2] + [x_1, x_2]q, [x_1, x_2]]_k$$

$$= [(a + q)[x_1, x_2] - [x_1, x_2]q, [x_1, x_2]]_k \,.$$

If denote $-q = b$ and $a + q = c$, the generalized derivation g is defined as $g(x) = cx + xb$, and we get the conclusion thanks to Theorem 3.

Case 2. Let now d an outer derivation of U. Since $[I, I]I \neq 0$, there exist $c_1, c_2, c_3 \in I$ such that $[c_1, c_2]c_3 \neq 0$. By the reguarity of R there exists $e^2 = e \in IR$ such that $eR = c_1R + c_2R + c_3R$ and $c_i = ec_i$ for $i = 1, 2, 3$. By

$$[a[ex_1, ex_2] + d([ex_1, ex_2]), [ex_1, ex_2]]_k = 0$$

we have that

$$[a[ex_1, ex_2] + [d(e)x_1 + ed(x_1), ex_2] + [ex_1, d(e)x_2 + ed(x_2)], [ex_1, ex_2]]_k = 0.$$

Since d is an outer derivation, by Kharchenko's result in [6], R satisfies the identity

$$[a[ex_1, ex_2] + [d(e)x_1 + ey_1, ex_2] + [ex_1, d(e)x_2 + ey_2], [ex_1, ex_2]]_k \,.$$

Since for $y_1 = y_2 = 0$, U satisfies the blended component

$$[a[ex_1, ex_2] + [d(e)x_1, ex_2] + [ex_1, d(e)x_2], [ex_1, ex_2]]_k$$

it follows that U satisfies also the following

$$[[ey_1, ex_2] + [ex_1, ey_2], [ex_1, ex_2]]_k \,.$$

Again for $y_1 = x_2$ U satisfies $[[ex_1, ey_2], [ex_1, ex_2]]_k$. In particular :

$$0 = [[ex_1, ey_2(1 - e)], [ex_1, ex_2]]_k = [ex_1, ex_2]^k ex_1 ey_2(1 - e) = 0$$

that is $[ex_1, ex_2]^k e = 0$. By [2] we have that $[eR, eR]eR = 0$ a contradiction. ■

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