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ANTI-PERIODIC BOUNDARY VALUE PROBLEMS FOR NONLINEAR HIGHER ORDER IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with the anti-periodic boundary value problems for nonlinear higher order impulsive differential equations

$$\begin{cases} x^{(n)}(t)=f(t,x(t),x'(t),\cdots,x^{(n-1)}(t)),\ t\in[0,T],\ t\neq t_k,\ k=1,\cdots,p,\\ \Delta x^{(i)}(t_k)=I_{i,k}(x(t_k),x'(t_k),\cdots,x^{(n-1)}(t_k)),\ k=1,\cdots,p, i=0,\cdots,n-1,\\ x^{(i)}(0)=-x^{(i)}(T),\ i=0,\cdots,n-1. \end{cases}$$

We obtain sufficient conditions for the existence of at least one solution. Examples are presented to illustrate the main results.

1. Introduction

There exist many papers concerned with the solvability of periodic boundary value problems or the existence of periodic solutions for higher order ordinary differential equations with or without impulses effects.

For examples, in [1,2], the existence of periodic solutions of the the equation

$$x^{(n)}(t) = f(t,x(t),x'(t),\cdots,x^{(n-1)}(t)) \tag{1}$$

has been studied by many authors [1-7] under a variety of conditions on f. The authors proved that equation (1) has at least one periodic solution if some conditions

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imposed on f are satisfied. In [3-7], the authors studied the existence of periodic solutions of the following differential equations

$$x^{(2n)}(t)+\sum_{i=1}^{n-1}a_ix^{(2i)}(t)+f(t,x(t))=0,\ \ t\in R, \tag{2}$$

and

$$x^{(2n+1)}(t)+\sum_{i=1}^{n}a_ix^{(2i-1)}(t)+f(t,x(t))=0,\ \ t\in R, \tag{3}$$

respectively. Equations (2) and (3) are special cases of equation (1).

In paper [12-15], the problems

$$\begin{cases} x'(t) = f(t,x(t)),\ t\in[0,T],\ t\neq t_k,\ k=1,\cdots,m,\\ \Delta x(t_k) = I_k(x(t_k)),\ k=1,\cdots,m,\\ x(0) = x(T), \end{cases} \tag{4}$$

and

$$\begin{cases} x''(t) = f(t,x(t),x'(t)),\ t\in[0,T],\ t\neq t_k,\ k=1,\cdots,m,\\ \Delta x(t_k) = I_k(x(t_k),x'(t_k)),\ k=1,\cdots,m,\\ \Delta x'(t_k) = I_k(x(t_k),x'(t_k)),\ k=1,\cdots,m,\\ x(0) = x(T), x'(0)=x'(T) \end{cases} \tag{5}$$

were studied. The existence results for these problems were established by using lower and upper solutions methods. Some recent studies on the existence of periodic solutions and their stability for functional or ordinary differential equations, which are arise in many applications, with or without impulses effects can be found in [34-44] and the references therein. The general theory of impulsive differential equations (IDE) and systems can be found in [27, 31-33].

The existence of solutions for anti-periodic boundary value problems for first order impulsive ordinary differential equations was studied in [8-11, 16, 21]. When the impulses are absent, i.e., $I_k=0$ for $k=1,\cdots,m$, anti-periodic boundary value problems for first order ordinary differential equations were studied in [9-11], anti-periodic problems for nonlinear differential equations in Hilbert spaces, and for nonlinear evolution equations have been studied in papers [20,28-30]. Also, anti-periodic boundary conditions for partial differential equations and abstract differential equations are considered in [17-19]. Anti-periodic boundary value problems for higher order differential equations are studied in [22]. Notice that anti-periodic boundary value problems appear in physics in a variety of situations, see, for example, [19-21, 24-26].

In paper [16], Luo, Shen and Nieto studied the problem

$$(6)\qquad \begin{cases} x'(t) = f(t, x(t)),\ t \in [0,T],\ t \neq t_k,\ k = 1, \cdots, m, \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1, \cdots, m, \\ x(0) = -x(T), \end{cases}$$

and the following results were obtained.

Theorem. *Suppose $\lambda > 0$. Assume that there is function $\psi : [0, +\infty) \to (0, +\infty)$ and a function $\rho \in L^1([0,T])$ with*

$$|f(t,x) + \lambda x| \le \rho(t)\psi(|x|),$$

and there exist $b_k \ge 0$ such that

$$|I_k(x)| \le b_k|x| \text{ and } \sum_{k=1}^{m} b_k < 1 + e^{-\lambda T}.$$

Furthermore, suppose

$$(7)\qquad \sup_{c \ge 0} \frac{c}{\psi(c)} > \frac{||\rho||_{L^1}}{1 + e^{-\lambda T} - \sum_{k=1}^{m} b_k}.$$

Then (6) has at least one solution.

Under the assumptions

$$(8)\qquad f(t,u) - f(t,v) \ge -\lambda(u - v) + M(u - v)$$

and that there are a pair of coupled lower and upper solutions for (4), and I_k are nondecreasing, and other assumptions, the existence result was also proved by authors in [8,10] using lower and upper solutions methods and monotone iterative technique.

We note that equation (7) or (8) implies that $f(t, x)$ grows at most linearly in x. So the problem have not been solved when $f(t, x)$ is supper for x. Furthermore, there exist no paper concerned with the solvability of anti-periodic problems for higher order impulsive differential equations.

In this paper, we are concerned with the existence of solutions of the anti-periodic boundary value problems for nonlinear impulsive functional differential equations

$$
(9)\qquad \begin{cases} x^{(n)}(t) = f(t, x(t), x'(t), \cdots, x^{(n-1)}(t)), \\ \qquad t \in [0,T],\ t \neq t_k,\ k = 1, \cdots, p, \\ \Delta x^{(i)}(t_k) = I_{i,k}(x(t_k), x'(t_k), \cdots, x^{(n-1)}(t_k)), \\ \qquad k = 1, \cdots, p, i = 0, \cdots, n-1, \\ x^{(i)}(0) = -x^{(i)}(T),\ i = 0, \cdots, n-1. \end{cases}
$$

where $n \geq 2$, $T > 0$, $0 < t_1 < \cdots < t_m < T$ are constants, f is an impulsive Carathedeodory function, $I_{i,k}$ are continuous functions. The purpose is to establish existence results for solutions of (9). We do not require the assumption that f is at most linear nor $I_{i,k}$ are nondecreasing. Finally, some examples illustrate the main results.

2. Main Results and Proofs

In this section, we establish the main results. To define solutions of (9), we introduce the Banach space.

Let $u : J = [0,T] \to R$, and $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, for $k = 0, \cdots, p$, define the function $u_k : (t_k, t_{k+1}) \to R$ by $u_k(t) = u(t)$. We will use the following Banach space

$$
X = \left\{ \begin{array}{l} u : J \to R, u_k^{(i)} \in L^1(t_k, t_{k+1}), k = 0, \cdots, p, i = 0, \cdots, n-1, \\ \text{ the limits exist } \lim\limits_{t \to t_k^-} x(t) = x(t_k), \\ \lim\limits_{t \to t_k^+} x(t),\ \lim\limits_{t \to 0^+} x(t) = x(0),\ \lim\limits_{t \to T^-} x(t) = -x(T) \end{array} \right\}
$$

and

$$
Y = \{u \in X,\ u_k \in W^{1,1}(t_k, t_{k+1}), k = 0, \cdots, p\} \times R^{pn}
$$

with the norms

$$
||u||_X = \max\{ \sup_{t \in (t_k, t_{k+1})} |u_k(t)|,\ k = 0, \cdots, p\}
$$

for $u \in X$ and

$$
||y||_Y = \max \left\{ ||u_k||_{W^{1,1}(t_k, t_{k+1})},\ k = 0, \cdots, p,\ \max_{1 \leq k \leq pn} \{|x_k|\} \right\}
$$

for $y = \{u, x_1, \cdots, x_{pn}\} \in Y$.

A function F is an impulsive Carathedeodory function if

$*$ $F(\bullet, u_0, u_1, \cdots, u_n)$ is measurable for each $u \in R$;

$*$ $F(t, \bullet, \cdots, \bullet)$ is continuous for a.e. $t \in J \setminus \{t_1, \cdots, t_p\}$;
$*$ for each $r > 0$ there is $h_r \in L^1(J)$ so that

$$|F(t, u_0, u_1, \cdots, u_n)| \le l_r(t), \ a.e.t \in J \setminus \{t_1, \cdots, t_p\}$$

and every u satisfying $||(u_0, u_1, \cdots, u_n)|| > r$;
$*$ and for each $(t, u_0, u_1, \cdots, u_n) \in (J \setminus \{t_1, \cdots, t_p\}) \times R^{n+1}$ the limits exist

$$\lim_{t \to 0^+} F(t, u_0, u_1, \cdots, u_n) = F(0, u_0, u_1, \cdots, u_n),$$
$$\lim_{t \to t_k^-} F(t, u_0, u_1, \cdots, u_n) = F(t_k, u_0, u_1, \cdots, u_n)$$

and

$$\lim_{t \to t_k^+} F(t, u_0, u_1, \cdots, u_n), \ \ k = 1, \cdots, p,$$
$$\lim_{t \to T^+} F(t, u_0, u_1, \cdots, u_n) = F(T, u_0, u_1, \cdots, u_n)$$

exist.

By a solution of (9) we mean a function $u \in X$ satisfying (9).

The following abstract existence theorem is also used in this paper, whose proof can be see in [23].

Lemma 2.1. *Let X and Y be Banach spaces. Suppose $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero with* Ker$L = \{0\}$, *$N : X \to Y$ is $L-$compact on any open bounded subset of X. If $0 \in \Omega \subset X$ is a open bounded subset and $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0, 1]$, then there is at least one $x \in \Omega$ so that $Lx = Nx$.*

Now, we define the linear operator $L : D(L) \subseteq X \to Y$ and the nonlinear operator $N : X \to Y$ by

$$Lx(t) = \begin{pmatrix} x^{(n)}(t) \\ \Delta x(t_1) \\ \cdot \\ \cdot \\ \cdot \\ \Delta x(t_p) \\ \cdot \\ \cdot \\ \cdot \\ \Delta x^{(n-1)}(t_1) \\ \cdot \\ \cdot \\ \cdot \\ \Delta x^{(n-1)}(t_p) \end{pmatrix} \quad \text{for } x \in D(L)$$

where $D(L) = \{u \in X,\ u_k \in C^n(t_k, t_{k+1}),\ k = 0, 1, \cdots, m\}$ and

$$Nx(t) = \begin{pmatrix} f(t, x(t), x'(t), \cdots, x^{(n-1)}(t)) \\ I_{0,1}(x(t_k), x'(t_k), \cdots, x^{(n-1)}(t_k)) \\ \cdot \\ \cdot \\ \cdot \\ I_{0,p}(x(t_k), x'(t_k), \cdots, x^{(n-1)}(t_k)) \\ \cdot \\ \cdot \\ \cdot \\ I_{n-1,1}(x(t_k), x'(t_k), \cdots, x^{(n-1)}(t_k)) \\ \cdot \\ \cdot \\ \cdot \\ I_{n-1,p}(x(t_k), x'(t_k), \cdots, x^{(n-1)}(t_k)) \end{pmatrix} \text{ for } x \in X.$$

It is easy to see that L is a Fredholm operator of index zero with Ker$L = \{0\}$, N is $L-$compact on any open bounded subset of X and that $x \in X$ is a solution of problem (9) if and only if x is a solution of the operator equation $Lx = Nx$.

We set the following assumptions which should be used in Theorem 2.1.

(**A**) $I_{n-1,k}(x_0, \cdots, x_{n-1})(2x_{n-1} + I_{n-1,k}(x_0, \cdots, x_{n-1}) \geq 0$ for all $x \in R$.

(**B**) There are numbers $\alpha_{i,k} \geq 0$ such that $|I_{i,k}(x_0, \cdots, x_{n-1})| \leq \alpha_{i,k}|x_i|$ for all $i = 0, \cdots, n-2$ and $k = 1, \cdots, p$ with $\sum_{k=1}^{p} \alpha_{i,k} < \frac{1}{2}$ for all $i = 0, \cdots, n-2$.

(**C**) There are functions $h : [0, T] \times R^n \to R$ and $g_i : [0, T] \times R \to R$ such that

(i) $f(t, x_0, \cdots, x_{n-1}) = h(t, x_0, \cdots, x_{n-1}) + \sum_{i=0}^{n-1} g_i(t, x_i)$ holds for all $(t, x_0, \cdots, x_{n-1}) \in [0, T] \times R^n$.

(ii) $g_i(t, x)$ satisfies that $g_i(\bullet, x) \in X$ for every $x \in R$ and $g_i(t, \bullet)$ is continuous for every $t \in [0, T]$.

(iii) h satisfies that $h(\bullet, x_0, \cdots, x_{n-1}) \in C_p^0$ for every $(x_0, \cdots, x_{n-1}) \in R^n$ and $g_i(t, \bullet, \cdots, \bullet)$ is continuous for every $t \in [0, T]$.

(iv) There are constants $m \geq 0$ and $\beta > 0$ so that

$$h(t, x_0, \cdots, x_{n-1})x_{n-1} \geq \beta |x_{n-1}|^{m+1}$$

holds for all $(t, x_0, \cdots, x_{n-1}) \in [0, T] \times R^n$.

(v) $\lim_{|x| \to +\infty} \sup_{t \in [0,T]} \frac{|g_i(t,x)|}{|x|^m} = r_i \in [0, +\infty)$ for $i = 0, \cdots, n-1$.

Theorem 2.1. *Suppose* (B), (A) *and* (C) *hold. Then problem* (9) *has at least one solution if*

$$r_0 + \sum_{k=1}^{n-1} r_k \left(\frac{3}{2}\right)^{m(n-k-2)} \prod_{i=k}^{n-2} \left(\frac{1}{1 - 2\sum_{k=1}^{p} \alpha_{i,k}}\right)^m < \beta. \tag{10}$$

Proof. Let $\lambda \in (0,1)$. Suppose x is a solution of the system

$$\begin{cases} x^{(n)}(t) = \lambda f(t, x(t), x'(t), \cdots, x^{(n-1)}(t)), \\ \qquad t \in [0,T],\ t \neq t_k,\ k = 1, \cdots, p, \\ \Delta x^{(i)}(t_k) = \lambda I_{i,k}(x(t_k), \cdots, x^{(n-1)}(t)), \\ \qquad k = 1, \cdots, p, i = 0, \cdots, n-1, \\ x^{(i)}(0) = -x^{(i)}(T),\ i = 0, \cdots, n-1. \end{cases} \tag{11}$$

For $i = 0, \cdots, n-2$, from $x^{(i)}(0) = -x^{(i)}(T)$, we get

$$x^{(i)}(0) = -\frac{1}{2}\left(\int_0^T x^{(i+1)}(s)ds + \sum_{k=1}^{p} I_{i,k}(x(t_k), \cdots, x^{(n-1)}(t))\right).$$

Then

$$\begin{aligned} |x^{(i)}(0)| &\leq \frac{1}{2}\left(\int_0^T |x^{(i+1)}(s)|ds + \sum_{k=1}^{p} |I_{i,k}(x(t_k), \cdots, x^{(n-1)}(t))|\right) \\ &\leq \frac{1}{2}\left(\int_0^T |x^{(i+1)}(s)|ds + \sum_{k=1}^{p} \alpha_{i,k}|x^{(i)}(t_k)|\right). \end{aligned}$$

So,

$$\begin{aligned} |x^{(i)}(t)| &\leq \left|\int_0^t x^{(i+1)}(s)ds\right| + \sum_{0<t_k<t} |I_{i,k}(x(t_k), \cdots, x^{(n-1)}(t))| \\ &\quad + \frac{1}{2}\left(\int_0^T |x^{(i+1)}(s)|ds + \sum_{k=1}^{p} \alpha_{i,k}|x^{(i)}(t_k)|\right) \\ &\leq \frac{3}{2}\int_0^T |x^{(i+1)}(s)|ds + 2\sum_{k=1}^{p} \alpha_{i,k}|x^{(i)}(t_k)|. \end{aligned}$$

We get

$$||x^{(i)}||_\infty \leq \frac{3}{2-4\sum_{k=1}^{p}\alpha_{i,k}}\int_0^T |x^{(i+1)}(s)|ds \text{ for } i=0,\cdots,n-2.$$

Hence we get

$$||x^{(i)}||_\infty \leq \left(\frac{3}{2}\right)^{n-2-i}\prod_{k=i}^{n-2}\frac{1}{1-2\sum_{k=1}^{p}\alpha_{i,k}}\int_0^T |x^{(n-1)}(s)|ds \text{ for } i=0,\cdots,n-2.$$

We divide the remainder of the proof into two steps.

Step 1. Prove that there is a constant $M>0$ so that $\int_0^T |x^{(n-1)}(s)|^{m+1}ds \leq M$.

Transforming the first equation of (9) to

$$x^{(n)}(t)x^{(n-1)}(t) = \lambda f(t,x(t),x'(t),\cdots,x^{(n-1)}(t))x^{(n-1)}(t). \tag{12}$$

Integrating it from 0 to T, we get

$$\begin{aligned}
&-\frac{1}{2}\sum_{k=1}^{p}\left[\left(x^{(n-1)}(t_k^+)\right)^2-\left(x^{(n-1)}(t_k^-)\right)^2\right]\\
&= \lambda\int_0^T f(s,x(t),x'(t),\cdots,x^{(n-1)}(t))x^{(n-1)}(s)ds\\
&= \lambda\left(\int_0^T h(s,x(t),x'(t),\cdots,x^{(n-1)}(t))x^{(n-1)}(s)ds+\int_0^T g_0(s,x(s))x(s)ds\right.\\
&\quad\left.+\sum_{i=1}^{n-1}\int_0^T g_i(s,x^{(i)}(s))x^{(n-1)}(s)ds+\int_0^T r(s)x^{(n-1)}(s)ds\right).
\end{aligned}$$

It follows from (A) that

$$\begin{aligned}
&\left(x^{(n-1)}(t_k^+)\right)^2-\left(x^{(n-1)}(t_k^-)\right)^2\\
&= \left(x^{(n-1)}(t_k^+)-x^{(n-1)}(t_k^-)\right)\left(x^{(n-1)}(t_k^+)+x^{(n-1)}(t_k^-)\right)\\
&= \Delta x^{(n-1)}(t_k^-)\left(2x^{(n-1)}(t_k^-)+\Delta x^{(n-1)}(t_k^-)\right)\\
&= I_k(x(t_k^-),\cdots,x^{(n-1)}(t_k^-))\left(2x^{(n-1)}(t_k^-)+I_k(x(t_k^-),\cdots,x^{(n-1)}(t_k^-))\right)\\
&\geq 0.
\end{aligned}$$

So

$$\int_0^T h(s, x(t), x'(t), \cdots, x^{(n-1)}(t))x^{(n-1)}(s)ds + \int_0^T g_0(s, x(s))x^{(n-1)}(s)ds$$
$$+\sum_{i=1}^{n-1}\int_0^T g_i(s, x^{(i)}(s))x^{(n-1)}(s)ds + \int_0^T r(s)x^{(n-1)}(s)ds \le 0.$$

It follows from (C) that

$$\begin{aligned}
&\beta\int_0^T |x^{(n-1)}(s)|^{m+1}ds\\
\le\ & -\int_0^T g_0(s, x(s))x^{(n-1)}(s)ds - \sum_{i=1}^{n-1}\int_0^1 g_i(s, x^{(i)}(s))x^{(n-1)}(s)ds\\
&-\int_0^T r(s)x^{(n-1)}(s)ds\\
\le\ & \int_0^T |g_0(s, x(s))||x^{(n-1)}(s)|ds + \sum_{i=1}^{n-1}\int_0^T |g_i(s, x^{(i)}(s))||x^{(n-1)}(s)|ds\\
&+\int_0^T |r(s)||x^{(n-1)}(s)|ds.
\end{aligned}$$

Let $\epsilon > 0$ satisfy that

$$(r_0+\epsilon) + \sum_{k=1}^{n}(r_k+\epsilon)\left(\frac{3}{2}\right)^{m(n-k-2)}\prod_{i=k}^{n-2}\left(\frac{1}{1-2\sum_{k=1}^{p}\alpha_{i,k}}\right)^m < \beta. \tag{13}$$

For such $\epsilon > 0$, there is $\delta > 0$ so that for every $i = 0, 1, \cdots, n$,

$$|g_i(t, x)| < (r_i+\epsilon)|x|^m \text{ uniformly for } t \in [0, T] \text{ and } |x| > \delta. \tag{14}$$

Let, for $i = 1, \cdots, n$, $\Delta_{1,i} = \{t :\ t \in [0, T],\ |x^{(i)}(\alpha_i(t))| \le \delta\}$, $\Delta_{2,i} = \{t : t \in [0, T],\ |x^{(i)}(\alpha_i(t))| > \delta\}$, $g_{\delta,i} = \max_{t\in[0,T],|x|\le\delta}|g_i(t, x)|$, and $\Delta_1 = \{t \in [0, T], |x(t)| \le \delta\}$, $\Delta_2 = \{t \in [0, T], |x(t)| > \delta\}$. Then we get

$$\begin{aligned}
&\beta\int_0^T |x^{(n-1)}(s)|^{m+1}ds\\
\le\ & (r_0+\epsilon)\int_0^T |x^{(n-1)}(s)|^{m+1}ds + \sum_{k=1}^{n-1}(r_k+\epsilon)\int_0^T |x^{(k)}(s))|^m|x^{(n-1)}(s)|ds\\
&+\int_0^T |r(s)||x^{(n-1)}(s)|ds + g_{\delta,0}\int_0^T |x^{(n-1)}(s)|ds + \sum_{k=1}^{n-1} g_{\delta,k}\int_0^T |x^{(n-1)}(s)|ds
\end{aligned}$$

$$
\begin{aligned}
\le\ & (r_0+\epsilon)\int_0^T |x^{(n-1)}(s)|^{m+1}ds \\
&+\sum_{k=1}^{n}(r_k+\epsilon)\left(\int_0^T |x^{(k)}(s))|^{m+1}ds\right)^{m/(m+1)}\left(\int_0^T |x^{(n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
&+\left(\int_0^T |r(s)|ds\right)^{m/(m+1)}\left(\int_0^T |x^{(n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
&+\sum_{k=0}^{n-1} g_{\delta,k}\int_0^T |x^{(n-1)}(s)|ds \\
=\ & (r_0+\epsilon)\int_0^T |x^{(n-1)}(s)|^{m+1}ds \\
&+\sum_{k=1}^{n-1}(r_k+\epsilon)\left(\int_0^T |x^{(i)}(u)|^{m+1}du\right)^{m/(m+1)}\left(\int_0^T |x^{(n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
&+\left(\int_0^T |r(s)|ds\right)^{m/(m+1)}\left(\int_0^T |x^{(n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
&+T^{m/(m+1)}\sum_{k=0}^{n-1} g_{\delta,k}\left(\int_0^T |x^{(n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
\le\ & (r_0+\epsilon)\int_0^T |x^{(n-1)}(s)|^{m+1}ds \\
&+\sum_{k=1}^{n-1}(r_k+\epsilon)\left(\int_0^T |x^{(i)}(u)|^{1+m}|du\right)^{m/(m+1)}\left(\int_0^T |x^{(n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
&+\left(\int_0^T |r(s)|ds\right)^{m/(m+1)}\left(\int_0^T |x^{(n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
&+T^{m/(m+1)}\sum_{k=0}^{n-1} g_{\delta,k}\left(\int_0^T |x^{(n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
=\ & \left((r_0+\epsilon)+\sum_{k=1}^{n-1}(r_k+\epsilon)\left(\frac{3}{2}\right)^{m(n-k-2)}\prod_{i=k}^{n-2}\left(\frac{1}{1-2\sum_{k=1}^{p}\alpha_{i,k}}\right)^{m}\right)\int_0^T |x^{(n-1)}(s)|^{m+1}ds \\
&+\left(\int_0^T |r(s)|\right)^{m/(m+1)}\left(\int_0^T |x^{(n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
&+T^{m/(m+1)}M^{1/(m+1)}\sum_{k=0}^{n-1} g_{\delta,k}.
\end{aligned}
$$

It follows from (14) that there is a constant $M>0$ so that $\int_0^T |x^{(n-1)}(s)|^{m+1}ds \le M$.

Step 2. Prove that there is a constant $M_1 > 0$ so that $||x^{(n-1)}||_\infty \le M_1$.
It follows from Step 1 that there is $\xi \in [0, T]$ so that $|x^{(n-1)}(\xi)| \le (M/T)^{1/(m+1)}$.

Case 1. If $t < \xi$, integrating it from t to ξ, we get, using (A), that

$$\left(x^{(n-1)}(t)\right)^2 = (x^{(n-1)}(\xi))^2 - \frac{1}{2}\sum_{t \le t_k < \xi}\left[\left(x^{(n-1)}(t_k^+)\right)^2 - \left(x^{(n-1)}(t_k^-)\right)^2\right]$$
$$-\lambda\int_t^\xi f(s, x(s), \cdots, \cdots, x^{(n-1)}(s))x^{(n-1)}(s)ds$$
$$\le (M/T)^{2/(m+1)} - \lambda\int_t^\xi f(s, x(s), \cdots, \cdots, x^{(n-1)}(s))x^{(n-1)}(s)ds$$
$$\le (M/T)^{2/(m+1)} - \lambda\left(\int_t^\xi h(s, x(s), \cdots, x^{(n-1)}(s))x^{(n-1)}(s)ds\right.$$
$$+\int_t^\xi g_0(s, x(s))x^{(n-1)}(s)ds$$
$$\left.+\sum_{i=1}^{n-1}\int_t^\xi g_i(s, x^{(i)}(s))x^{(n-1)}(s)ds + \int_t^\xi r(s)x^{(n-1)}(s)ds\right)$$
$$\le (M/T)^{2/(m+1)} - \int_t^\xi g_0(s, x(s))x^{(n-1)}(s)ds$$
$$-\sum_{i=1}^{n-1}\int_t^\xi g_i(s, x^{(i)}(s))x^{(n-1)}(s)ds - \int_t^\xi r(s)x^{(n-1)}(s)ds$$
$$\le (M/T)^{2/(m+1)} + \int_0^T |g_0(s, x(s))||x^{(n-1)}(s)|ds$$
$$+\sum_{i=1}^{n-1}\int_0^T |g_i(s, x^{(i)}(s))||x^{(n-1)}(s)|ds$$
$$+\int_0^T |r(s)||x^{(n-1)}(s)|ds$$
$$\le (M/T)^{2/(m+1)} + \left[\left((r_0+\epsilon) + \sum_{k=1}^{n-1}(r_k+\epsilon)\left(\frac{3}{2}\right)^{m(n-k-2)}\right.\right.$$
$$\left.\prod_{i=k}^{n-2}\left(\frac{1}{1-2\sum_{k=1}^{p}\alpha_{i,k}}\right)^m\right)\times$$
$$\left.\int_0^T |x^{(n-1)}(s)|^{m+1}ds + \left(\int_0^T |r(s)|ds\right)^{m/(m+1)}\left(\int_0^T |x^{(n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right]$$

$$
\begin{aligned}
&+\sum_{k=0}^{n-1} g_{\delta,k} T^{m/(m+1)}\left(\int_0^T |x^{(n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\
&\leq (M/T)^{2/(m+1)} + \left[\left((r_0+\epsilon)+\sum_{k=1}^{n-1}(r_k+\epsilon)\left(\frac{3}{2}\right)^{m(n-k-2)}\right.\right.\\
&\quad \left.\prod_{i=k}^{n-2}\left(\frac{1}{1-2\sum_{k=1}^{p}\alpha_{i,k}}\right)^m\right) M\\
&\quad \left.+\left(\int_0^T |r(s)|ds\right)^{m/(m+1)} M^{1/(m+1)}\right] + T^{m/(m+1)}M^{1/(m+1)}\sum_{k=0}^{n-1} g_{\delta,k}\\
&=: M_2.
\end{aligned}
$$

Hence one sees that

$$[x^{(n-1)}(t)]^2 \leq M_2, \text{ for } t \in [0,\xi].$$

This implies $[x^{(n-1)}(0)]^2 \leq M_3$. So $[x^{(n-1)}(T)]^2 = [x^{(n-1)}(0)]^2 \leq M_3$. For $t \in [\xi, T]$, we have

$$
\begin{aligned}
\left(x^{(n-1)}(t)\right)^2 &= (x^{(n-1)}(T))^2 - \frac{1}{2}\sum_{\xi \leq t_k < t}\left[\left(x^{(n-1)}(t_k^+)\right)^2 - \left(x^{(n-1)}(t_k^-)\right)^2\right]\\
&\quad -\lambda \int_t^T f(s, x(s), \cdots, x^{(n-1)}(s))x^{(n-1)}(s)ds.
\end{aligned}
$$

Similar to above discussion, we get that there is $M_4 > 0$ so that $[x^{(n-1)}(t)]^2 \leq M_4$ for $t \in [\xi, T]$. All above discussion implies that there is $M_1 > 0$ so that $|x^{(n-1)}(t)| \leq M_1$. Thus $||x^{(n-1)}||_\infty \leq M_1$. So

$$
\begin{aligned}
|x^{(k)}(t)| &\leq \left(\frac{3}{2}\right)^{m(n-k-2)} \prod_{i=k}^{n-2}\left(\frac{1}{1-2\sum_{k=1}^{p}\alpha_{i,k}}\right)^m \int_0^T |x^{(n-1)}(s)|ds\\
&\leq \left(\frac{3}{2}\right)^{m(n-k-2)} \prod_{i=k}^{n-2}\left(\frac{1}{1-2\sum_{k=1}^{p}\alpha_{i,k}}\right)^m M_1 T.
\end{aligned}
$$

Hence there is $M_0 > 0$ so that

$$||x|| \leq \max\{||x||_\infty, \cdots, ||x^{(n-1)}||_\infty\} \leq M_0.$$

It follows from Lemma 2.1 that equation $Lx = Nx$ has at least one solution, which is a solution of problem (9). The proof is complete.

We now set the following assumptions which will be used in Theorem 2.2.

(A')**.** $I_{n-1,k}(x_0, \cdots, x_{n-1})(2x_{n-1} + I_{n-1,k}(x_0, \cdots, x_{n-1}) \leq 0$ for all $x \in R$.

(C')**.** There are functions $h : [0,T] \times R^n \to R$ and $g_i : [0,T] \times R \to R$ such that

(i) $f(t, x_0, \cdots, x_{n-1}) = h(t, x_0, \cdots, x_{n-1}) + \sum_{i=0}^{n-1} g_i(t, x_i)$ holds for all $(t, x_0, \cdots, x_{n-1}) \in [0, T] \times R^{n+1}$.

(ii) $g_i(t, x)$ satisfies that $g_i(\bullet, x) \in X$ for every $x \in R$ and $g_i(t, \bullet)$ is continuous for every $t \in [0, T]$.

(iii) h satisfies that $h(\bullet, x_0, \cdots, x_{n-1}) \in X$ for every $(x_0, \cdots, x_{n-1}) \in R^n$ and $h(t, \bullet, \cdots, \bullet)$ is continuous for every $t \in [0, T]$.

(iv) There are constants $m \geq 0$ and $\beta > 0$ such that

$$h(t, x_0, \cdots, x_{n-1})x_{n-1} \leq -\beta|x_{n-1}|^{m+1}$$

holds for all $(t, x_0, \cdots, x_{n-1}) \in [0, T] \times R^n$.

(v) $\lim_{|x|\to+\infty} \sup_{t\in[0,T]} \frac{|g_i(t,x)|}{|x|^m} = r_i \in [0, +\infty)$ for $i = 0, \cdots, n-1$.

Theorem 2.2. *Suppose* (B), (A') *and* (C') *hold. Then problem* (9) *has at least one solution if*

$$r_0 + \sum_{k=1}^{n} r_k \left(\frac{3}{2}\right)^{m(n-k-2)} \prod_{i=k}^{n-2} \left(\frac{1}{1 - 2\sum_{k=1}^{p} \alpha_{i,k}}\right)^m < \beta. \tag{15}$$

Proof. The proof is similar to that of Theorem 2.1. We consider system (10), from assumption (A'), it is easy to get

$$\int_0^T h(s, x(s), \cdots, x^{(n-1)}(s)))x^{(n-1)}(s)ds + \int_0^T g_0(s, x(s))x^{(n-1)}(s)ds$$
$$+\sum_{i=1}^{n} \int_0^T g_i(s, x^{(i)}(s))x^{(n-1)}(s)ds + \int_0^T r(s)x^{(n-1)}(s)ds \geq 0.$$

The remainder of the proof is similar to that of Theorem 2.1 and is omitted.

3. Examples

In this section, we give two examples, which can not be solved by the results in previous papers, to illustrate the main results.

Example 3.1. Consider the problem

$$\begin{cases} x^{(n)}(t) = a[x^{(n-1)}(t)]^{2m+1} + \sum_{k=0}^{n-1} p_k(t)[x^{(k)}(t)]^{2m+1} + r(t), \\ \qquad t \in [0, T],\ t \neq t_k,\ k = 1, \cdots, p, \\ \Delta x^{(i)}(t_k) = \alpha_{i,k}x^{(i)}(t_k),\ k = 1, \cdots, p, i = 0, \cdots, n-1, \\ x^{(i)}(0) = -x^{(i)}(T),\ i = 0, \cdots, n-1, \end{cases} \tag{16}$$

then (16) has at least one solution if

$$\begin{cases} a > 0, \\ \sum_{k=1}^{p} \alpha_{i,k} < \frac{1}{2},\ i = 0, \cdots, n-2, \\ \alpha_{n-1,k}(2 + \alpha_{n-1,k}) \geq 0,\ k = 1, \cdots, p \\ ||p_0||_\infty + \sum_{k=1}^{n-1} \left(\frac{3}{2}\right)^{m(n-k-2)} \prod_{i=k}^{n-2} \left(\frac{1}{1 - 2\sum_{k=1}^{p} \alpha_{i,k}}\right)^m ||p_k||_\infty < a. \end{cases}$$

Example 3.2. Consider the problem

$$(17) \quad \begin{cases} x^{(n)}(t) = a[x^{(n-1)}(t)]^{2m+1} + \sum_{k=0}^{n-1} p_k(t)[x^{(k)}(t)]^{2m+1} + r(t), \\ \qquad t \in [0, T],\ t \neq t_k,\ k = 1, \cdots, p, \\ \Delta x^{(i)}(t_k) = \alpha_{i,k} x^{(i)}(t_k),\ k = 1, \cdots, p, i = 0, \cdots, n-1, \\ x^{(i)}(0) = -x^{(i)}(T),\ i = 0, \cdots, n-1. \end{cases}$$

Then (17) has at least one solution if

$$\begin{cases} a < 0, \\ \sum_{k=1}^{p} \alpha_{i,k} < \frac{1}{2},\ i = 0, \cdots, n-2, \\ \alpha_{n-1,k}(2 + \alpha_{n-1,k}) \leq 0,\ k = 1, \cdots, p \\ ||p_0||_\infty + \sum_{k=1}^{n-1} \left(\frac{3}{2}\right)^{m(n-k-2)} \prod_{i=k}^{n-2} \left(\frac{1}{1 - 2\sum_{k=1}^{p} \alpha_{i,k}}\right)^m ||p_k||_\infty < -a. \end{cases}$$

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