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INTEGRAL OPERATORS ON A SUBSPACE OF HOLOMORPHIC FUNCTIONS ON THE DISC

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Abstract. Let $H(D)$ be an algebra of all holomorphic functions on the open unit disc D and X a subspace of $H(D)$. When g is a function in $H(D)$, put

$$J_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta \text{ and } I_g(f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (z \in D)$$

for f in X. In this paper, we study $J[X] = \{g \in H(D) \ ; \ J_g(f) \in X \text{ for all } f \text{ in } X\}$ and $I[X] = \{g \in H(D) \ ; \ I_g(f) \in X \text{ for all } f \text{ in } X\}$. We apply the results to concrete spaces. For example, we study $J[X]$ and $I[X]$ when X is a weighted Bloch space, a Hardy space or a Privalov space.

1. INTRODUCTION

Let D denote the open unit disc in the complex plane $\mathbb{C}$ and $H = H(D)$ the set of all holomorphic functions on D. For a given g in H, define three operators :

$$(M_g f)(z) = g(z)f(z) \qquad (f \in H, \ z \in D)$$
$$(J_g f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (f \in H, \ z \in D)$$

and

$$(I_g f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (f \in H, \ z \in D).$$

Then $(J_g f)(z) + (I_g f)(z) = (M_g f)(z) - g(0)f(0)$. If $g(z) = z$ then J_g is the Voltera integral operator and if $g(z) = \log 1/(1-z)$ then J_g is the Cesáro operator.

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In this paper we assume that X is a subspace of H which contains constants. X_1 denotes the set $\{f \in H \ ; \ f' \in X\}$. For each subspace X put

$$M[X] = \{g \in H \ ; \ M_g(X) \subseteq X\},$$

$$J[X] = \{g \in H \ ; \ J_g(X) \subseteq X\}$$

and

$$I[X] = \{g \in H \ ; \ I_g(X) \subseteq X\}.$$

We define that $J^{n+1}[X] = J[J^n[X]]$ and $I^{n+1}[X] = I[I^n[X]]$ for $n \geq 1$ where $J^1[X] = J[X]$ and $I^1[X] = I[X]$. For X and Y which are subspaces of H, XY denotes a subspace of H which is generated by a product of a function in X and one in Y. Let Y^n be a subspace of H which is generated by finite n products of functions in a subspace Y of H. For a subspace X of H, $B(X)$ denotes the set of all bounded linear operators on X.

Now we give a lot of examples of X. For $0 < p \leq \infty$, H^p is the usual Hardy space on D, N is the Nevanlinna class and N_+ is the Smirnov class on D. These are F-spaces, and N and N_+ are algebras. It is known that $J[H^p]$ = BMOA (see [2], [1]), $z \notin J[N]$ [5] and $z \notin J[N_+]$ [7]. The Bloch space $\mathcal{B}$ is defined to be a Banach space in H with the norm

$$\|f\| = \sup_{z \in D}(1 - |z|^2)|f'(z)| + |f(0)|.$$

Then $\mathcal{B}$ contains H^∞ properly. Recently, R.Yoneda [8] described $J[\mathcal{B}]$ and he [9] also proved that $I[\mathcal{B}] = H^\infty$. It is well known that $M[H^p] = H^\infty$.

In Section 2, we assume only that X is a subspace of H. Theorem 1 implies that $J[X]^n \subset X$ for any $n \geq 1$. In Section 3, we study $J[X]$ and $I[X]$ when X is an invariant subspace of H or a subalgebra of H. Theorem 2 implies that if $H^\infty X \subset X$ and $J[X]$ contains z then $J[X] \supseteq H_1^\infty$. In Section 4, assuming that X is an F-space we show that $J[X]$ is contained in some weighted Bloch space and $I[X] \subset H^\infty$. In Section 5, we define a weighted Bloch space $\mathcal{B}_\omega$ and we describe $J[\mathcal{B}_\omega]$. In Section 6, we study $J\left[\bigcap_{t<p} H^t\right]$ and $I\left[\bigcap_{t<p} H^t\right]$. In Section 7, we show that $J[N^p]$ is a subalgebra of N^p which contains N_1^p, where N^p is a Privalov space.

2. Subspace

In this section, we study $M[X]$, $J[X]$ and $I[X]$ assuming only that X is a subspace of H.

Lemma 1. *Let X be a subspace of H and f, g in H.*

(1) $I_g I_f = I_{gf} = I_f I_g$ on X

(2) $I_g J_f = J_f M_g$ on X

Proof. For $k \in X$,

$$((I_g I_f)k)(z) = \int_0^z (I_f k)'(\zeta) g(\zeta) d\zeta = \int_0^z k'(\zeta) f(\zeta) g(\zeta) d\zeta = (I_{fg} k)(z)$$

(2) For $k \in X$,

$$\begin{aligned}((I_g J_f)k)(z) &= \int_0^z (J_f k)'(\zeta) g(\zeta) d\zeta = \int_0^z k(\zeta) f'(\zeta) g(\zeta) d\zeta \\ &= (J_f(gk)(z) = ((J_f M_g)k)(z).\end{aligned}$$

Theorem 1. *Let X be a subspace of H with constants. Then $J[X]$ is a subspace of X with constants and $J[X]^n \subset X$.*

Proof. If $g \in J[X]$ then $J_g(1) = g - g(0) \in X$ and so $g \in X$ because $1 \in X$. Hence $J[X]$ is a subspace of X with constants.

Assuming $J[X]^n \subset X$, we will show that $J[X]^{n+1} \subset X$. Suppose that $g \in J[X]$ and $\{g_j\}_{j=1}^n \subset J[X]$. In order to prove that $g \prod_{j=1}^n g_j$ belongs to X, we will use the following equalities.

$$\begin{aligned}&\int_0^z g(\zeta) \left(\prod_{j=1}^n g_j \right)' (\zeta) d\zeta \\ &= g(z) \left(\prod_{j=1}^n g_j \right)(z) - g(0) \left(\prod_{j=1}^n g_j \right)(0) - \int_0^z g'(\zeta) \left(\prod_{j=1}^n g_j \right)(\zeta) d\zeta\end{aligned}$$

and

$$\int_0^z g(\zeta) \left(\prod_{j=1}^n g_j \right)' (\zeta) d\zeta = \sum_{\ell=1}^n \int_0^z (g(\zeta) \prod_{j \neq \ell} g_j(\zeta)) g_\ell'(\zeta) d\zeta.$$

By hypothesis on induction, $\prod_{j=1}^n g_j \in X$ and so $\int_0^z g'(\zeta) \left(\prod_{j=1}^n g_j \right)(\zeta) d\zeta \in X$ because $g \in J[X]$. By hypothesis on induction, for $\ell = 1, \cdots, n$ $g \prod_{j \neq \ell} g_j \in X$ and so $\displaystyle\int_0^z (g(\zeta)\prod_{j\neq\ell} g_j(\zeta))g'_\ell(\zeta)d\zeta \in X$ because $g_\ell \in J[X]$. By the above two equalities, $\displaystyle g\prod_{j=1}^{n} g_j$ belongs to X. This implies that $J[X]^{n+1} \subset X$.

Proposition 1. *Let X be a subspace of H with constants. Then $I[X]$ is a subalgebra of H.*

Proof. If $k \in I[X]$ and $g \in I[X]$ then it is easy to see that $I_k I_g = I_{kg}$ (see Proposition 3). Hence $I_k I_g(X) = I_k(I_g(X)) \subseteq I_k(X) \subseteq X$ and so kg belongs to $I[X]$. It is clear that $I[X]$ is a subspace of H.

Proposition 2. *Suppose X is a subspace of H with constants.*

(1) *$M[X]$ is an algebra in X.*
(2) *$J[X] \cap M[X] = I[X] \cap M[X]$.*
(3) *$J[X] \cap I[X] \subseteq M[X]$.*
(4) *$J[X] \subset M[X]$ if and only if $J[X] \subset I[X]$. Similarly $I[X] \subset M[X]$ if and only if $I[X] \subset J[X]$.*

Proof. (1) is clear. (2) and (3) follow from the equality : $J_g f + I_g f = M_g f - g(0)f(0)$. (4) If $J[X] \subset M[X]$ then by (2) $J[X] \subset I[X]$. Conversely if $J[X] \subset I[X]$ then by (3) $J[X] \subset M[X]$.

3. Invariant Subspace and Subalgebra

In this section, we study $J[X]$ and $I[X]$ when X is an invariant subspace or a subalgebra of H.

Theorem 2. *Suppose that X is a subspace of H with constants and $kX \subset X$ for any k in H^∞.*

(1) *If g_0 is an arbitrary function in $J[X]$, then $J[X]$ contains $\{g \in H\ ;\ |g'(z)| \leq |g'_0(z)|(z \in D)\}$.*
(2) *If $J[X]$ contains z then it contains H_1^∞.*
(3) *Suppose $J[X]$ contains z. If $\{g_n\}$ is in $J[X]$ and $g'_n \to g'$ uniformly on D then g belongs to $J[X]$.*
(4) *$zJ[X] \subset J[X]$ if and only if $J_z(J[X]X) \subset X$.*
(5) *$J[X] \cap H^\infty \subset I[X]$ and hence $I[X]$ contains H_1^∞ if $z \in J[X]$.*

Proof.

(1) If $g \in H$ and $|g'(\zeta)| \leq |g_0'(\zeta)| (\zeta \in D)$, then $g'(g_0')^{-1} \in H^\infty$ and so $fg'(g_0')^{-1} \in X$ for any $f \in X$. Hence for any $f \in X$

$$\int_0^z f(\zeta)g'(\zeta)d\zeta = \int_0^z f(\zeta)g'(\zeta)g_0'(\zeta)^{-1}g_0'(\zeta)d\zeta$$

belongs to X because $fg'(g_0')^{-1} \in X$ and $g_0 \in J[X]$. This implies that g belongs to $J[X]$.

(2) Since $z \in J[X]$, by (1) and the definition of H_1^∞, H_1^∞ is contained in $J[X]$.

(3) If $g_n' \to g'$ uniformly on D, then $(g - g_n)' \in H^\infty$. Hence $f(g - g_n)' \in X$ for any $f \in X$. Therefore g belongs to $J[X]$ because $z \in J[X]$ and

$$\int_0^z f(\zeta)g'(\zeta)d\zeta = \int_0^z f(\zeta)(g(\zeta) - g_n(\zeta))'d\zeta + \int_0^z f(\zeta)g_n'(\zeta)d\zeta.$$

(4) Follows trivially from the following equality :

$$\int_0^z f(\zeta)(\zeta g(\zeta))'d\zeta = \int_0^z f(\zeta)g(\zeta)d\zeta + \int_0^z f(\zeta)\zeta g'(\zeta)d\zeta$$

for $f \in X$ and $g \in J[X]$.

(5) By the equality : $I_g(f) = fg - (fg)(0) - J_g(f)$, if $g \in J[X] \cap H^\infty$ and $f \in X$ then $I_g(f)$ belongs to X because $gX \subset X$.

Proposition 3. *If X is a subalgebra of H which contains constants then $M[X] = X$, $J[X]$ is also a subalgebra of X and $J[X] = I[X] \cap X$.*

Proof. $M[X] = X$ is clear. If both g and h are in $J[X]$, then by Theorem 1 both fh and fg belongs to X for any $f \in X$ because X is an algebra. Hence gh belongs to $J[X]$ by the following equality : $J_{gh}(f) = J_g(fh) + J_h(fg)$ for any $f \in X$. This implies that $J[X]$ is a subalgebra of X by Theorem 1. From (2) of Proposition 2 $J[X] = I[X] \cap X$ follows.

4. F-SPACE

Let X be an F-space in H with an invariant metric d. For each a in D, put for f in X

$$\mathcal{E}_a f = f(a) \text{ and } \mathcal{D}_a f = f'(a).$$

In this section we assume that both $\mathcal{E}_a$ and $\mathcal{D}_a$ are bounded on X. Put

$$S(a) = \sup\{|\mathcal{E}_a(f)| \ ; \ f \in X, \ d(f, 0) \leq 1\}$$

and

$$s(a) = \sup\{|\mathcal{D}_a(f)| \ ; \ f \in X, \ d(f, 0) \leq 1\},$$

then $S(a) < \infty$ and $s(a) < \infty$ if $a \in D$. Suppose v is a nonnegative function on D. For a function f in H put

$$\|f\|_\omega = \sup_{z \in D} \omega(z)|f'(z)| + |f(0)|$$

and

$$\mathcal{B}_\omega = \{f \in H \ ; \ \|f\|_\omega < \infty\}.$$

If ω is bounded, $\mathcal{B}_\omega$ contains all holomorphic functions on the closed unit disc $\bar{D}$.

Proposition 4. *If X is an F-space such that $S(a) < \infty$ and $s(a) < \infty$ for each $a \in D$, then $M[X], J[X]$ and $I[X]$ belongs to $B[X]$.*

Proof. We will prove only that $J[X] \subset B[X]$ because the other statements are similar. By the closed graph theorem, it is enough to prove that for $\phi \in J[X]$ if $f_n \to f$ in X and $J_\phi(f_n) \to F$ then $J_\phi(f) = F$. Since $S(a) < \infty, \ f_n(a) \to f(a) \ (a \in D)$. Since $s(a) < \infty, \ f_n(a)\phi'(a) \to F'(a) \ (a \in D)$. Thus $f(a)\phi'(a) = F'(a)$ and so $J_\phi(f) = F$ because $F(0) = 0$.

Theorem 3. *Let X be an F-space in H with an invariant metric d. Suppose that $\sup_{|a| \leq 1-\varepsilon} S(a) < \infty$ for any $\varepsilon > 0$. Then $J[X] \subset \mathcal{B}_{\omega_0} \cap X$ and $I[X] \subset H^\infty$, where $\omega_0 = 1/sS$.*

Proof. If $g \in J[X]$ then by Proposition 4, for any $f \in X$ $d(J_g f, 0) \leq \|J_g\| d(f, 0)$. Since $J_g f \in X$, by definition of $\mathcal{D}_z$ $|\mathcal{D}_z(J_g f)| \leq s(z) d(J_g f, 0) (z \in D)$. Hence

$$s(z)^{-1}|f(z)||g'(z)| \leq \|J_g\| d(f, 0) \quad (z \in D)$$

and so

$$s^{-1}(z) S^{-1}(z)|g'(z)| \leq \|J_g\| \quad (z \in D).$$

By Theorem 1 g belongs to $\mathcal{B}_{\omega_0} \cap X$ where $\omega_0 = 1/sS$. If $g \in I[X]$ then by Proposition 4, for any $f \in X$ $d(I_g f, 0) \leq \|I_g\| d(f, 0)$. Since $I_g f \in X$, by definition of $\mathcal{D}_z$ $|\mathcal{D}_z(I_g f)| \leq s(z) d(I_g f, 0) \quad (z \in D)$. Hence

$$s(z)^{-1}|f'(z)||g(z)| \leq \|I_g\| d(f, 0) \quad (z \in D)$$

and so

$$|g(z)| \leq \|I_g\| \quad (z \in D).$$

Proposition 5. *Let X be a subspace of H with constants which is of finite dimension. Then $J[X] = I[X] = M[X] = \mathbb{C}$.*

Proof. Suppose $\{f_j\}_{j=1}^n$ is a basis in X with $f_1 \equiv 1$. We will show that $J[X] = \mathbb{C}$. If $g \in J[X]$ then by Theorem 1 $g^\ell \in X$ for any $\ell \geq 0$ and so there exist $\{\alpha_j^\ell\}_1^n \subset \mathbb{C}$ such that $g^\ell = \sum_{j=1}^{n} \alpha_j^\ell f_j$. Hence there exist $\{b_\ell\}_{\ell=0}^n \subset \mathbb{C}$ such that $\sum_{\ell=0}^{n} b_\ell g^\ell = 0$. This implies that g is just constant because g is analytic. Therefore $J[X] = \mathbb{C}$. We will show that $I[X] = \mathbb{C}$. Put $X_1 = \{f' \; ; \; f \in X\}$. If $g \in I[X]$ then by Proposition 1 $g^\ell X_1 \subset X_1$ for any $\ell \geq 1$ and so there exist $\{\alpha_j^\ell\}_1^n \subset \mathbb{C}$ such that $g^\ell f_2' = \sum_{j=2}^{n} \alpha_j^\ell f_j'$. By the same argument above $g f_2'$ is constant. Similary it follows that $\{g f_j'\}_2^n$ are constants and so g is constant because $\{f_j'\}_\ell^n$ is a basis in X_1. Therefore $I[X] = \mathbb{C}$.

5. Weighted Bloch Space

Let ω be a positive bounded function on D. For a function f in H put

$$\|f\|_\omega = \sup_{z \in D} \omega(z)|f'(z)| + |f(0)|$$

and

$$\mathcal{B}_\omega = \{f \in H \; ; \; \|f\|_\omega < \infty\}.$$

Since ω is bounded, $\mathcal{B}_\omega$ contains all holomorphic functions on the closed unit disc $\bar{D}$. $\mathcal{B}_\omega$ is called a weighted Bloch space. A weight ω is called measurable when $\omega(at)$ is measurable on [0,1] for each a in D. Put $\varepsilon(r) = \inf\{\omega(z) \; ; \; |z| \leq r\}$ and $r < 1$.

Lemma 2. *If $\varepsilon(r) > 0$ for $0 \leq r < 1$ then $\mathcal{B}_\omega$ is a Banach space with norm $\| \cdot \|_\omega$.*

Proof. Suppose that $\{f_n\}$ is a Cauchy sequence in $\mathcal{B}_\omega$. For any $\varepsilon > 0$, there exist a positive integer n_0 such that $\|f_n - f_m\|_\omega < \varepsilon$ if $n, m \geq n_0$. Hence if $r < 1$ and $z \in D_r = \{z \; ; \; |z| < r\}$ then

$$|f_n'(z) - f_m'(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)}.$$

By the normal family argument, there exists a function $f' \in H(D_r)$ such that $f_n' \to f'$ uniformly on D_r. Hence as $n \to \infty$,

$$|f'(z) - f_m'(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)} \quad (z \in D_r).$$

Since r is arbitrary, f belongs to $H(D)$ and

$$\omega(z)|f'(z) - f'_m(z)| \le \varepsilon \quad (z \in D)$$

if $m \ge n_0$. Since $f_m(0) \to f(0)$, $\|f - f_m\|_\omega \to 0$.

Theorem 4. *Let ω be a measurable, $\varepsilon(r) > 0$ for $0 \le r < 1$ and $X = \mathcal{B}_\omega$. Then*

$$\mathcal{B}_{\omega S} = J[\mathcal{B}_\omega] \text{ and } I[\mathcal{B}_\omega] \subset H^\infty$$

where $S(z) = \sup\{| f(z) | \ ; \ f \in \mathcal{B}_\omega, \ \| f \|_\omega \le 1\}$. Moreover $\| J_g \| = \| g \|_{\omega S}$ for each g in $J[\mathcal{B}_\omega]$ with $g(0) = 0$.

Proof. By Theorem 1, $J[\mathcal{B}_\omega] \subseteq \mathcal{B}_\omega$. If $g \in J[\mathcal{B}_\omega]$ then $\|J_g f\|_\omega \le \|J_g\| \|f\|_\omega$ $(f \in \mathcal{B}_\omega)$ and so $\omega(z) \mid f(z) \mid \cdot \mid g'(z) \mid \le \| J_g \| \cdot \| f \|_\omega$. Hence

$$\omega(z) S(z) \mid g'(z) \mid \cdot \frac{\mid f(z) \mid}{S(z)} \le \| J_g \| \cdot \| f \|_\omega$$

and so

$$\omega(z) S(z) \mid g'(z) \mid \le \| J_g \| \ .$$

Therefore g belongs to $\mathcal{B}_{\omega S}$ and $\| g \|_{\omega S} \le \| J_g \| + \mid g(0) \mid$. Thus $J[\mathcal{B}_\omega] \subseteq \mathcal{B}_{\omega S}$. Note that $\mathcal{B}_{\omega S} \subseteq \mathcal{B}_\omega$ because $S(z) \ge 1$ $(z \in D)$. Conversely if $g \in \mathcal{B}_{\omega S}$ then

$$\omega(z) |J_g(f)'(z)| = \omega(z) S(z) \mid g'(z) \mid \frac{\mid f(z) \mid}{S(z)} \le \| g \|_{\omega S} \| f \|_\omega$$

and so g belongs to $J[\mathcal{B}_\omega]$. Thus

$$\| g \|_{\omega S} \le \| J_g \| + \mid g(0) \mid \le \| g \|_{\omega S} + \mid g(0) \mid \ .$$

In Theorem 4, if ω is an absolute value of some analytic function and a radial function, R.Yoneda ([8],[9]) showed those under some special technical conditions on ω.

6. Hardy Space

For $0 < p \le \infty$, H^{p-} denotes $\bigcap_{t<p} H^t$ and $H^{\infty-}$ is written as H^ω. For $0 < p < \infty$, when $W = \mid h \mid^p$ for an outer function h in H^p, $H^p(W)$ denotes a weighted Hardy space that is, the closure of H^∞ in $L^p(W d\theta/2\pi)$.

Lemma 3 is well known (cf. [3, Theorem 5.12]). In Proposition 6 it is known ([1],[2]) that $J[H^p]$ = BMOA. Hence our result is weaker than that. However if $J[H^p]$ = BMOA then our result shows that $I[H^p] = H^\infty$.

Lemma 3. *(1) For $0 < p < 1$, if f is a function in H^p then $\int_0^z f(\zeta)d\zeta$ beongs to $H^{p/1-p}$. (2) If f is a function in H^1 then $\int_0^z f(\zeta)d\zeta$ belongs to H^∞.*

Proposition 6. *For $0 < p < \infty$, $H_1^\infty \subset J[H^p] \subset H^\omega$ and $zJ[H^p] \subset J[H^p]$. Moreover $M[H^p] = H^\infty$ and $I[H^p] = J[H^p] \cap H^\infty$.*

Proof. By Lemma 3, $z \in J[H^p]$ and so by (2) of Theorem 2 $H_1^\infty \subset J[H^p]$. Theorem 1 implies that $J[H^p] \subset H^\omega$. By (5) of Theorem 2, $J[H^p] \cap H^\infty \subset I[H^p]$. Theorem 3 implies that $I[H^p] \subset H^\infty$. Hence $I[H^p]\ H^p \subset H^p$ and so (4) of Proposition 2 $I[H^p] \subset J[H^p]$. It is well known that $M[H^p] = H^\infty$. By (2) of Proposition 2 $I[H^p] = J[H^p] \cap H^\infty$. By (4) of Theorem 2, to prove that $zJ[H^p] \subset J[H^p]$ it is sufficient to show that $J_z(J[H^p]H^p) \subset H^p$. Since $J[H^p]H^p \subset H^{p-}$, by Lemma 2, $J_z(J[H^p]H^p) \subset H^p$.

Theorem 5. *For $0 < p < \infty$, $\bigcap_{t<1} H^t \subset J[H^{p-}] \subset H^\omega$ and so $\log(1-z)^{-1}$ belongs to $J[H^{p-}]$. Morover $zJ[H^{p-}] \subset J[H^{p-}]$, $M[H^{p-}] = H^\infty$ and so $J[H^{p-}] \cap H^\infty = I[H^{p-}] \cap H^\infty$. When $p = \infty$, $J[H^\omega] = I[H^\omega] \cap H^\omega$ and $J[H^\omega]$ is a subalgebra of H^ω which contains H_1^∞.*

Proof. By Theorem 1, $J[H^{p-}] \subset H^\omega$. We will show that $\bigcap_{t<1} H_1^t \subset J[H^{p-}]$. If $g \in \bigcap_{t<1} H_1^t$ then g' belongs to H^{1-}. If $f \in H^{p-}$ then f belongs to H^t for any $0 < t < p$. If $0 < s < t/(t+1)$ then $t/s > 1$ and $1/(t/s) + 1/(t/t-s) = 1$. By the Hȍlder inequality,

$$
\begin{aligned}
&\int_0^{2\pi} \mid f(e^{i\theta})g'(e^{i\theta}) \mid^s d\theta/2\pi \\
&\le \left(\int_0^{2\pi} \mid f(e^{i\theta}) \mid^t d\theta/2\pi\right)^{\frac{s}{t}} \left(\int_0^{2\pi} \mid g'(e^{i\theta}) \mid^{\frac{st}{t-s}} d\theta/2\pi\right)^{\frac{s-t}{t}}
\end{aligned}
$$

and so fg' belongs to $\bigcap_{s<t/(t+1)} H^s$. By Lemma 3, $\int_0^z f(\zeta)g'(\zeta)d\zeta$ belongs to $H^{\frac{s}{1-s}}$. As $s \to t/(t+1)$, $s/(1-s) \to t$ and so $\int_0^z f(\zeta)g'(\zeta)d\zeta$ belongs to H^{t-}. As $t \to p$, $\int_0^z f(\zeta)g'(\zeta)d\zeta$ belongs to H^{p-}. Thus $J_g[H^{p-}] \subseteq H^{p-}$ and so $\bigcap_{t<1} H_1^t \subset J[H^{p-}]$. By (4) of Theorem 2, if we show that $J_z(J[H^{p-}]H^{p-}) \subset H^{p-}$ then it follows that $zJ[H^{p-}] \subset J[H^{p-}]$. Since $J[H^{p-}]H^{p-} \subset H^{p-}$, by Lemma 4 $J_z(J[H^{p-}]H^{p-}) \subset H^{p-}$. It is known that $M[H^{p-}] = H^\infty$. The last statement is a result of (2) of Proposition 2.

When $p = \infty$, by Proposition 3 $J[H^\omega] = I[H^\omega] \cap H^\omega$ and $J[H^\omega]$ is a subalgebra of H^ω. Theorem 2 implies $J[H^\omega] \supset H_1^\infty$.

Theorem 6. *Let* $1 \leq p < \infty$ *and* $W = |\ h\ |^p$ *for some outer function* h *in* H^p. *Then* $\{g \in H; g(z) = \int_0^z h(\zeta)k(\zeta)d\zeta$ *and* $k \in H^{\frac{p}{p-1}}\} \subset J[H^p(W)] \subset H^\omega(W)$. $M[H^p(W)] = H^\infty$ *and* $J[H^p(W)] \cap H^\infty = I[H^p(W)]$. *There exists a weight* W *such that* z *does not belong to* $J[H^p(W)]$.

Proof. If $g(z) = \int_0^z h(\zeta)k(\zeta)d\zeta$ and $k \in H^{\frac{p}{p-1}}$, then

$$h(z)\{J_g(h^{-1}f)\}(z) = h(z)\int_0^z f(\zeta)k(\zeta)d\zeta$$

and so $hJ_gh^{-1}f$ belongs to H^p for all $f \in H^p$ by Lemma 3 because $fk \in H^1$. Therefore $\{g \in H; g(z) = \int_0^z h(\zeta)k(\zeta)d\zeta$ and $k \in H^{\frac{p}{p-1}}\} \subset J[H^p(W)]$. By Theorem 1, $J[H^p(W)] \subset \bigcap_{p<\infty} H^p(W)$. In fact, since $g^n h \in H^p$ for any $n \geq 1$, $gh^{1/n} \in H^{np}$ and so g belongs to $H^{np}(W)$. If $\phi \in M(H^p(W))$ then $\phi(h^{-1}H^p) \subset h^{-1}H^p$ and so $\phi H^p \subset H^p$. Hence $\phi \in M(H^p) = H^\infty$. Therefore $M(H^p(W)) = H^\infty$ and so by (2) of Proposition 2 $J[H^p(W)] \cap H^\infty = I[H^p(W)] \cap H^\infty$. For $a \in D$ it is easy to see that

$$\sup\{|\ f(a)\ |\ ;\ f \in H^p(W) \text{ and } \|f\|_{W,p} \leq 1\} = (1- |\ a\ |^2)^{-1/p}\ |\ h(a)\ |^{-p} < \infty$$

and so by Theorem 3 $I[H^p(W)] \subset H^\infty$. Thus $J[H^p(W)] \cap H^\infty = I[H^p(W)]$. If $J_z(H^p(W)) \subseteq H^p(W)$ for any W with $\log W \in L^1(d\theta/2\pi)$ then $J_z(N_+) \subseteq N_+$. For by a theorem of H.Helson [6] N_+ is the union of all $H^p(W)$ as W ranges over the set of weights with sumable $\log W$. Hence there exists a weight W such that $z \notin J[H^p(W))]$. Because it is known that $J_z(N_+) \not\subset N_+$ [7].

7. Privalov Space

We denote by N^p, for $1 \leq p < \infty$, the set of all functions f in H which satisfy

$$\sup_{0<r<1} \int_0^{2\pi} (\log^+ |\ f(re^{i\theta})\ |)^p d\theta < \infty.$$

When $p = 1$, N^p is just N. Then

$$\bigcup_{p>0} H^p \subset \bigcap_{p>1} N^p \text{ and } \bigcup_{p>1} N^p \subset N_+ \subset N^1 = N.$$

Proposition 7. *Let $X = N_+$ or N. Then $J[X]$ is a subalgebra of X and $J[X] = I[X] \cap X$. If $(f')^{-1}$ is in H^∞ then f does not belong to $J[X]$.*

Proof. It is known that N_+ and N are subalgebras of H. Hence the first part of this proposition is a result of Theorem 1 and Proposition 3. By [5] and [7], $z \notin J[X]$ and so the second part follows from (1) of Theorem 2.

In Proposition 7, it is known ([5],[7]) that $z \notin J[X]$. Hence $I[X] \not\ni z$. We don't know whether $J[X] = \not{C}$ and $I[X] = \not{C}$.

Theorem 7. *If $1 < p < \infty$ then $J[N^p]$ is a subalgebra of N^p which contains N_1^p, and $J[N^p] = I[N^p] \cap N^p$.*

Proof. Suppose $1 < p < \infty$ and $g \in N_1^p$. If $f \in N^p$ then

$$\begin{aligned}
&\left\{\int_0^{2\pi} (\log^+ |(J_g f)(re^{i\theta})|)^p d\theta/2\pi\right\}^{1/p} \\
&= \left\{\int_0^{2\pi} \left(\log^+ \left|\int_0^r f(te^{i\theta})g'(te^{i\theta})\right| dt\right)^p d\theta/2\pi\right\}^{1/p} \\
&\leq \left\{\int_0^{2\pi} \left(\log^+ \int_0^{1-} |f(te^{i\theta})g'(te^{i\theta})|dt\right)^p d\theta/2\pi\right\}^{1/p} \\
&\leq \left\{\int_0^{2\pi} \left(\log^+ \sup_{0\leq t<1} |f(te^{i\theta})| + \log^+ \sup_{0\leq t<1} |g'(te^{i\theta})|\right)^p d\theta/2\pi\right\}^{1/p} \\
&\leq \left\{\int_0^{2\pi} \left(\log^+ \sup_{0\leq t<1} |f(te^{i\theta})|\right)^p d\theta/2\pi\right\}^{1/p} \\
&\quad + \left\{\int_0^{2\pi} \left(\log^+ \sup_{0\leq t<1} |g'(te^{i\theta})|\right)^p d\theta/2\pi\right\}^{1/p}.
\end{aligned}$$

Put $u(r,\theta) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} P_r(t-\theta)\log^+ |f(e^{it})|dt$, then $u(r,\theta) \geq \log^+ |f(re^{i\theta})|$. Since $\log^+ |f(e^{it})| \in L^p$, by a theorem of Hardy and Littlewood (cf. [3, Proposition 1.8]), $\sup_{o\leq r<1} u(r,\theta)$ belongs to L^p and so $\log^+ \sup_{0\leq r<1} |f(re^{i\theta})|$ belongs to L^p. Similarly we can prove that $\log^+ \sup_{0\leq r<1} |g'(re^{i\theta})|$ belongs to L^p. Thus $J_g f$ belongs to N^p. Hence $N_1^p \subset J[N^p]$. It is known that N^p is a subalgebra of H. Hence, by Proposition 3 $J[N^p]$ is a subalgebra of N^p and $J[N^p] = I[N^p] \cap N^p$.

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