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ON MINIMAL HORSE-SHOE LEMMA

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Abstract. The main purpose of this paper is to find the conditions for the Minimal Horse-Shoe Lemma to be hold in some categories of graded modules. Using the Minimal Horse-Shoe Lemma, we prove that the category of p-Koszul modules is closed under direct sums, direct summands, extensions and cokernels.

1. Introduction and Definitions

We know that the Horse-Shoe Lemma plays an important role in Homological algebra. But it is only true for the modules which have projective resolutions. Obviously, a minimal projective resolution has more advantages than the ordinary one in the computation of the Ext group. Thus a natural question arises: does the Horse-Shoe Lemma hold in the minimal case?

Let R be an arbitrary ring with identity 1_R and $M \in Mod(R)$. It is well known that, the projective cover of M does not necessarily exist; and that every finitely generated R-module M has a projective cover if and only if R is semiperfect. As a preparation, we first prove that if a graded algebra $A = \coprod_{i\geq 0} A_i$ is a *nice algebra*, then any graded A-module $M = \coprod_{i\geq 0} M_i \in BGr(A)$ has a graded A-projective cover, which is similar as in the nongraded case. Let $0 \to K \to M \to N \to 0$ be an exact sequence in the category of nice modules with $K \cap JM = JK$, then the Minimal Horse-Shoe Lemma holds, which is the main result in this paper. Finally, as the application of the Minimal Horse-Shoe Lemma, we investigate the category of p-Koszul modules.

Here, first we will give some basic definitions and notations. In section 2, we will show that the graded A-projective covers in the category $BGr(A)$ always

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exist. Moreover, we will give an explicit construction of the graded projective cover for the graded A-module $M = \coprod_{i\geq 0} M_i$ by using the minimal generating spaces $\{S_0, S_1, \ldots\}$ of M. Moreover, we obtain some sufficient conditions for the Minimal Horse-Shoe Lemma to be hold. In the last section, using the Minimal Horse-Shoe Lemma, we prove that the category $K_p(A)$ of p-Koszul modules is closed under direct sums, direct summands, extensions and cokernels.

Let us give some preliminary definitions which will be used in this paper.

Definition 1.1. Let $A = \coprod_{i\geq 0} A_i$ be a graded algebra satisfying that:

1. A_0 is semisimple ;
2. $A_i \cdot A_j = A_{i+j}$,

then A is called a *nice algebra*.

Throughout this paper, we always consider $A = \coprod_{i\geq 0} A_i$ as a nice algebra and $J = \coprod_{i\geq 1} A_i$ to be the graded Jacobson radical of A. Let $Gr(A)$ denote the category consisting of graded A-modules and degree zero morphisms. Let $BGr(A)$ denote the full subcategory of $Gr(A)$ consisting of graded A-modules which are bounded below and degree zero morphisms. Here, $gr(A)$ is the full subcategory of $Gr(A)$ consisting of finitely generated graded modules. Obviously, $gr(A) \subseteq BGr(A)$. We also denote the pure module category by $gr_s(A)$, i.e., if $M \in gr_s(A)$ then M is generated in degree s, where $gr_s(A)$ is a full subcategory of $gr(A)$.

Remark 1.1. Each $M \in BGr(A)$ can be written in the form: $M = \coprod_{i\geq j} M_i$ and $M_i = 0$ for all $i < j$. Since $M[j] = \coprod_{i\geq 0} M_i$, where [ ] is the shift functor. Thus for each $M \in BGr(A)$, we can always assume that $M = \coprod_{i\geq 0} M_i$.

Definition 1.2. Let

$$\cdots \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

be a minimal graded projective resolution of M in $BGr(A)$. If for each $i \geq 0$, it is satisfied that $J \ker f_i = \ker f_i \cap J^2 P_i$, then M is called a *nice module*.

We denote the category of nice modules by $\mathcal{N}(A)$. In the above definition, if we replace the nice algebra A by the Koszul algebra, then it is the definition of *Quasi-Koszul module*, about which one can refer to [7] for more details.

Definition 1.3. ([5]) Let A be a nice algebra and $M \in gr(A)$. Suppose that the minimal graded projective resolution $\mathcal{Q}$ of M is given by

$$\mathcal{Q} : \cdots \to Q_n \xrightarrow{f_n} Q_{n-1} \xrightarrow{f_{n-1}} \cdots \to Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \to 0.$$

If for all $n \geq 0$, Q_n is generated in degree $\delta(n)$, where

$$\delta(n) = \begin{cases} \dfrac{np}{2} + s & \text{if n is even,} \\ \dfrac{(n-1)p}{2} + 1 + s & \text{if n is odd,} \end{cases}$$

and $p \geq 2$, s is a fixed integer, then M is called a *p-Koszul module*.

It is easy to see that the definition of p-Koszul modules is slightly different from that given in [5] where one also used the term d-Koszul module. We denote the category of p-Koszul modules by $K_p(A)$, which is a full subcategory of $gr(A)$.

2. Main Results

In this section, we will give some conditions for the Minimal Horse-Shoe Lemma to be hold. First we have the following result:

Theorem 2.1. *Let $M = \coprod_{i\geq 0} M_i \in BGr(A)$. Then M has a graded A-projective cover.*

Proof. Let S_0, S_1, $\cdots$, S_n, $\cdots$ be the minimal generating spaces of M with $degS_i = i$. Since $M = \coprod_{i\geq 0} M_i$ is a graded A-module, it can also be expressed as

$$M = M_0 \oplus (S_1 + A_1S_0) \oplus (S_2 + A_2S_0 + A_1S_1) \oplus \cdots .$$

Set $S^M = M_0 \oplus S_1 \oplus S_2 \oplus \cdots$, which can be considered as an A_0-module. Let $P = A \otimes_{A_0} S^M$. We will show that P is the graded projective cover of $M = \coprod_{i\geq 0} M_i$. First, we claim that $P = A \otimes_{A_0} S^M$ is a graded A-projective module. It is clear that P is graded, since $P_i = A \otimes_{A_0} S_i$. Let $K \to N \to 0$ be exact in $Gr(A)$. Since $\mathrm{Hom}_A(P, K) = \mathrm{Hom}_A(A \otimes_{A_0} S^M, K) \cong \mathrm{Hom}_{A_0}(S^M, \mathrm{Hom}_A(A, K)) \cong \mathrm{Hom}_{A_0}(S^M, K)$, we obtain that $\mathrm{Hom}_{A_0}(S^M, -)$ is an exact functor. It follows that we have the following commutative diagram,

$$\begin{array}{ccccc} \mathrm{Hom}_{A_0}(S^M, K) & \longrightarrow & \mathrm{Hom}_{A_0}(S^M, N) & \longrightarrow & 0 \\ \cong\big\downarrow & & \cong\big\downarrow & & \\ \mathrm{Hom}_A(P, K) & \longrightarrow & \mathrm{Hom}_A(P, N) & & \end{array}$$

Therefore $\mathrm{Hom}_A(P, K) \to \mathrm{Hom}_A(P, N) \to 0$ is exact and P is a graded A-projective module. Now we define a morphism f as follows,

$$f : P = A \otimes_{A_0} S^M \longrightarrow M$$

via

$$f : \sum a_i \otimes s_i \longmapsto \sum a_i \cdot s_i.$$

It is easy to see that f is an epimorphism. Finally, we claim that $\ker f \subseteq JP = J \otimes_{A_0} S^M$, since $A_i \cdot A_j = A_{i+j}$ for all i, $j \geq 0$. ■

The above theorem means that if $A = \coprod_{i \geq 0} A_i$ is a nice algebra, then there always exist the graded A-projective covers in $BGr(A)$.

Next we will investigate some basic properties of graded projective covers in $BGr(A)$.

Proposition 2.2. *Let $M = \coprod_{i \geq 0} M_i \in BGr(A)$. If P and Q are both graded projective covers of M, then $P \cong Q$.*

Proof. Its proof is similar as in the nongraded case, and we can refer [1] for the details. ■

Proposition 2.2 shows the uniqueness (up to isomorphisms) of the graded projective covers, which is similar as in the nongraded case.

Proposition 2.3. *Let $M = \coprod_{i \geq 0} M_i \in BGr(A)$ and P be the graded projective cover of M. Then M is generated in degree s if and only if P is generated in degree s.*

Proof. “If”: From the proof of Theorem 2.1, we know that $A \otimes_{A_0} S^M$ is one of the graded projective covers of M and by Proposition 2.2, the projective covers of M are unique up to isomorphisms. Hence

$$P \cong A \otimes_{A_0} S^M = A \otimes_{A_0} S_0 \oplus A \otimes_{A_0} S_1 \oplus \cdots$$

Let $P_i = A \otimes_{A_0} S_i$. Further, if M is generated in degree s, then $M = A \cdot M_s = A \cdot S_s$. In this case $P = A \otimes_{A_0} S_s$, evidently P is generated in degree s.

“Only if”: By Theorem 2.1 and Proposition 2.2, $P \cong A \otimes_{A_0} S^M$ and $P_i = A \otimes_{A_0} S_i$. By hypothesis, P is generated in degree s, i.e., $P_j = A_{j-s} \cdot P_s$ for all $j \geq s$. So we have $A \otimes_{A_0} S_j = A_{j-s} \cdot A \otimes_{A_0} S_s$. That is, M is generated in degree s. ■

Corollary 2.4. *Let $M = \coprod_{i \geq 0} M_i \in BGr(A)$ and P be the graded projective cover of M. Then M is generated in degrees $j_1, j_2, \cdots, j_n$ if and only if P is generated in degrees $j_1, j_2, \cdots, j_n$.*

Proof. It is proved similarly to Proposition 2.3. ■

Lemma 2.5. *For every exact sequence* $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *in* $Gr(A)$, *there exists the exact sequence*

$$0 \to K \cap JM \to JM \to JN \to 0$$

in $Gr(A)$.

Proof. Let $\overline{f} = f|_{K \cap JM}$ and $\overline{g} = g|_{JM}$. Since f is an embedding, $\overline{f}$ is a monomorphism. Let $x = \sum a_i n_i \in JN$ with $a_i \in J$ and $n_i \in N$. Since g is an epimorphism, there exists $m_i \in M$ such that $g(m_i) = n_i$. Set $y = \sum a_i m_i$ and we have $\overline{g}(y) = \overline{g}(\sum a_i m_i) = \sum \overline{g}(a_i m_i) = \sum a_i g(m_i) = \sum a_i n_i$, thus $\overline{g}$ is an epimorphism. Let $x = \sum a_i m_i \in ker\overline{g}$ with $a_i \in J$ and $m_i \in M$. So $\overline{g}(x) = g(\sum a_i m_i) = 0$, since g is an epimorphism, there exists $z \in K \cap JM$ such that $\overline{f}(z) = x$. Therefore $\ker(\overline{g}) \subseteq Im\overline{f}$. Conversely, let $x = \overline{f}(y) \in Im\overline{f}$ with $y \in K \cap JM$. Since $\overline{g}(x) = gf(y) = 0$, $x \in Ker\overline{g}$, hence $Im\overline{f} \subseteq \ker \overline{g}$. Then we have the exact sequence:

$$0 \to K \cap JM \to JM \to JN \to 0. \qquad \blacksquare$$

Now we will give some sufficient conditions for the Minimal Horse-Shoe Lemma to be hold:

Lemma 2.6. *Let* $0 \to K \to M \to N \to 0$ *be an exact sequence in* $Gr(A)$, *then the following statements are equivalent:*

1. $K \cap JM = JK$,
2. $A/J \otimes_A K \to A/J \otimes_A M$ *is a monomorphism.*

Proof. Let $K \cap JM = JK$. By Lemma 2.5, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & JK & \longrightarrow & JM & \longrightarrow & JN & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K/JK & \longrightarrow & M/JM & \longrightarrow & N/JN & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Obviously, $A/J \otimes_A K \cong K/JK$ and $A/J \otimes_A M \cong M/JM$. Thus $A/J \otimes_A K \to A/J \otimes_A M$ is a monomorphism, since $0 \to K/JK \to M/JM$ is exact. Therefore the assertion (1) implies the assertion (2). Conversely, suppose that $A/J \otimes_A K \to A/J \otimes_A M$ is a monomorphism. Then $K/JK \to M/JM$ is also a monomorphism. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K/JK & \longrightarrow & M/JM & \longrightarrow & N/JN & \longrightarrow & 0 \\
 & & \Big\downarrow f & & id \Big\downarrow & & id \Big\downarrow & & \\
0 & \longrightarrow & K/K \cap JM & \longrightarrow & M/JM & \longrightarrow & N/JN & \longrightarrow & 0
\end{array}$$

By "Five Lemma", we get $JK = K \cap JM$. ∎

Lemma 2.7. *Let $0 \to \Omega \to P \xrightarrow{f} M \to 0$ be an exact sequence in $Gr(A)$ with $M \in \mathcal{N}(A)$ and P be the graded projective cover of M. Then $A/J \otimes_A \Omega \to A/J \otimes_A JP$ is a monomorphism.*

Proof. Since $\Omega \subseteq JP$, we have the exact sequence:

$$0 \to \Omega \hookrightarrow JP \xrightarrow{\overline{f}} JM \to 0,$$

where $\overline{f} = f|_{JP}$. Since M is a nice module, $J\Omega = \Omega \cap J^2P = \Omega \cap J(JP)$. By Lemma 2.6, $A/J \otimes_A \Omega \to A/J \otimes_A JP$ is a monomorphism. ∎

Now we can state the main theorem as follows.

Theorem 2.8. *Let $0 \to K \to M \to N \to 0$ be an exact sequence in $\mathcal{N}(A)$ with $K \cap JM = JK$. Then the Minimal Horse-Shoe Lemma holds, that is, there is a commutative diagram with exact rows and columns:*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_K^1 & \longrightarrow & P_M^1 & \longrightarrow & P_N^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_K^0 & \longrightarrow & P_M^0 & \longrightarrow & P_N^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where

$$\cdots \to P_K^1 \to P_K^0 \to K \to 0,$$

$$\cdots \to P_M^1 \to P_M^0 \to M \to 0,$$

and

$$\cdots \to P_N^1 \to P_N^0 \to N \to 0$$

are the minimal graded projective resolutions of K, M *and* N *respectively, where* $P_M^i = P_K^i \oplus P_N^i$, *for* $i \geq 0$.

Proof. By Lemma 2.5, the sequence:

$$0 \to K \cap JM \to JM \to JN \to 0$$

is exact.

Since $JK = K \cap JM$, we have the exact sequence:

$$0 \to JK \to JM \to JN \to 0.$$

The following diagram is commutative with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & JK & \longrightarrow & JM & \longrightarrow & JN & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K/JK & \longrightarrow & M/JM & \longrightarrow & N/JN & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $K/JK \cong S^K = M_0 \oplus S_1 \oplus \cdots$, $M/JM \cong S^M$ and $N/JN \cong S^N$ as A_0-modules, we get the exact sequence:

$$0 \to S^K \to S^M \to S^N \to 0$$

in $Gr(A_0)$.

Since A_0 is semisimple, by applying the functor $A \otimes_{A_0} -$ to this sequence, we obtain the following exact sequence:

$$0 \to A \otimes_{A_0} S^K \to A \otimes_{A_0} S^M \to A \otimes_{A_0} S^N \to 0.$$

Assume that $P_K^0 = A \otimes_{A_0} S^K$, $P_M^0 = A \otimes_{A_0} S^M$ and $P_N^0 = A \otimes_{A_0} S^N$ due to the uniqueness (up to isomorphisms) of the graded projective covers. Thus, we get the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega(K) & \longrightarrow & \Omega(M) & \longrightarrow & \Omega(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_K^0 & \longrightarrow & P_M^0 & \longrightarrow & P_N^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The exact sequence

$$0 \to P_K^0 \to P_M^0 \to P_N^0 \to 0$$

splits, since P_N^0 is graded projective. Thus, $P_M^0 = P_K^0 \oplus P_N^0$. Obviously, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega(K) & \longrightarrow & \Omega(M) & \longrightarrow & \Omega(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & JP_K^0 & \longrightarrow & JP_M^0 & \longrightarrow & JP_N^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & JK & \longrightarrow & JM & \longrightarrow & JN & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and, applying the functor $A/J \otimes_A -$ to the above diagram, we get the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & A/J \otimes_A \Omega(K) & \xrightarrow{f} & A/J \otimes_A \Omega(M) & \longrightarrow & A/J \otimes_A \Omega(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A/J \otimes_A JP_K^0 & \longrightarrow & A/J \otimes_A JP_M^0 & \longrightarrow & A/J \otimes_A JP_N^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A/J \otimes_A JK & \longrightarrow & A/J \otimes_A JM & \longrightarrow & A/J \otimes_A JN & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Lemma 2.7, $A/J \otimes_A \Omega(K) \to A/J \otimes_A P_K^0$ is a monomorphism. From the commutativity of the left upper square, we have f as a monomorphism. By Lemma 2.6, $J\Omega(K) = \Omega(K) \cap J\Omega(M)$.

Next we claim that if $M \in \mathcal{N}(A)$ and $P \xrightarrow{f} M \to 0$ with $Kerf \subset JP$, i.e., P is the graded projective cover of M, then $\Omega(M) = \ker f \in \mathcal{N}(A)$. In fact, the minimal graded projective resolution of M is obtained as

$$\cdots \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

such that $J \ker f_i = \ker f_i \cap J^2 P_i$ for all $i \geq 0$, since $M \in \mathcal{N}(A)$. Hence we get a minimal graded projective resolution of $\ker f_0$ as

$$\cdots \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_1 \xrightarrow{f_1} \ker f_0 \to 0$$

such that $J \ker f_i = \ker f_i \cap J^2 P_i$ for all $i \geq 1$. Therefore $\ker f_0 \in \mathcal{N}(A)$.

This implies that $\Omega(K)$, $\Omega(M)$ and $\Omega(N)$ are contained in $\mathcal{N}(A)$.

Replacing K, M and N by $\Omega(K)$, $\Omega(M)$, $\Omega(N)$ respectively, through recursing step by step, we have the Minimal Horse-Shoe Lemma to be hold. ■

Proposition 2.9. *Let $0 \to K \to M \to N \to 0$ be an exact sequence in $\mathcal{N}(A)$ where K, M and N are generated in the same degrees. Then the Minimal Horse-Shoe Lemma holds.*

Proof. Since K, M and N are generated in the same degrees, we can assume $K = A \cdot K_s$, $M = A \cdot M_s$ and $N = A \cdot N_s$. By Theorem 2.8, we only need to show $K \cap JM = JK$. Since $JK \subseteq K$ and $JK \subseteq JM$, $JK \subseteq K \cap JM$. Let x be a nonzero homogeneous element of $K \cap JM$. Then $x \in JM$. It follows that $degx \geq s+1$. Since K is generated in degree s, it means that for any $a \in K$, $dega \geq s+1$ if and only if $a \in JK$. Thus $x \in JK$. So $JK \supseteq K \cap JM$. Hence $JK = K \cap JM$. ■

Proposition 2.10. *Let*

$$0 \to K \to M \to N \to 0$$

be a split exact sequence in $BGr(A)$. Then the Minimal Horse-Shoe Lemma holds.

Proof. Since

$$0 \to K \to M \to N \to 0$$

splits, we have $M = K \oplus N$. Let P_K and P_N are the graded projective covers of K and N respectively. That is, there exist two epimorphisms

$$P_K \xrightarrow{f} K \to 0$$

and

$$P_N \xrightarrow{g} N \to 0$$

such that $kerf \subseteq JP_K$ and $kerg \subseteq JP_N$. Set $P = P_K \oplus P_N$ and define φ as follows:

$$\varphi : P \to M \to 0$$

via

$$\varphi(x \oplus y) = f(x) \oplus g(y)$$

for $x \in P_K$ and $y \in P_N$. Obviously, φ is an epimorphism. Let $\varphi(x+y) = 0$, clearly $f(x) = 0 = g(y)$. Thus $x \in \ker f$ and $y \in \ker g$, $\ker \varphi = \ker f \oplus \ker g$. Since $J \ker \varphi = J(\ker f \oplus \ker g) = J \ker f \oplus J \ker g \subseteq JP_K \oplus JP_N = J(P_K \oplus P_N) = JP$, we have that P is the graded projective cover of M. We get the split exact sequence:

$$0 \to \Omega(K) \to \Omega(M) \to \Omega(N) \to 0.$$

Replacing K, M and N by $\Omega(K)$, $\Omega(M)$, $\Omega(N)$ respectively, through recursing step by step, we have the Minimal Horse-Shoe Lemma to be hold. ■

Lemma 2.11. *Let $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an exact sequence in $gr(A)$ and A be a graded algebra. Then*

(1) *if K and N are generated in degree s, then M is generated in degree s;*

(2) *if M is generated in degree s, then N is generated in degree s;*

(3) *if K and N are generated in degree s and t, respectively, then M is generated in degrees s and t.*

Proof. Consider the exact sequence:

$$0 \to K_n \xrightarrow{\overline{f}} M_n \xrightarrow{\overline{g}} N_n \to 0$$

with $n > s$, where $N = \oplus_{i \geq s} N_i$, $K = \oplus_{i \geq s} K_i$ and $M = \oplus_{i \geq s} M_i$ and $\overline{f} = f\,|_{K_n}$, $\overline{g} = g\,|_{M_n}$. Let $x \in M_n$ be a homogeneous element, then $\overline{g}(x) \in N_n$. Since N is generated in degree s, there exists $n = \sum a_i n_i \in N_n$ with $a_i \in A_{n-s}$ and $n_i \in N_s$, such that $\overline{g}(x) = \sum a_i n_i$. Since

$$0 \to K_s \xrightarrow{\overline{f}} M_s \xrightarrow{\overline{g}} N_s \to 0$$

is exact and $\overline{g}$ is epimorphism, there exists $m_i \in M_s$, such that $\overline{g}(m_i) = n_i$. Let $y = \sum a_i m_i$ be a homogeneous element of M_n. Since $\overline{g}(y) = \overline{g}(\sum a_i m_i) = \sum a_i \overline{g}(m_i) = \sum a_i n_i = \overline{g}(x)$, we have $y - x \in Ker\overline{g} = Im\overline{f}$ and thus $x - y = \overline{f}(\sum b_i k_i)$ with $b_i \in A_{n-s}$ and $k_i \in K_s$. Therefore $x = \sum a_i m_i + \sum b_i \overline{f}(k_i)$ where $m_i, \overline{f}(k_i) \in M_s$. It follows that M is generated in degree s. Thus the assertion (1) holds.

Assume M is generated in degree s. For each integer $n > s$, consider the exact sequence:

$$0 \to K_n \xrightarrow{\overline{f}} M_n \xrightarrow{\overline{g}} N_n \to 0.$$

Let $x \in N_n$ be a homogeneous element and since $\overline{g}$ is epimorphism, there exists $y \in M_n$ such that $\overline{g}(y) = x$ and $y = \sum a_i m_i$ with $m_i \in M_s$ and $a_i \in A_{n-s}$. Hence $x = \overline{g}(y) = \overline{g}(\sum a_i m_i) = \sum a_i \overline{g}(m_i)$ and $\overline{g}(m_i) \in N_s$ and it follows that N is generated in degree s. Then the assertion (2) holds. The assertion (3) can be proved similarly. ■

Proposition 2.12. *Let $0 \to K \to M \to N \to 0$ be an exact sequence in $gr(A)$ such that K and N are generated in the same degrees. If K and N are p-Koszul modules, then the Minimal Horse-Shoe Lemma holds.*

Proof. Since K and N are generated in the same degrees, we can assume that $K = A \cdot K_s$ and $N = A \cdot N_s$. By Lemma 2.11, $M = A \cdot M_s$. Thus we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_K^0 & \longrightarrow & P_M^0 & \longrightarrow & P_N^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where P_K^0, P_M^0 and P_N^0 are the graded projective covers of K, M and N respectively, and $\Omega^1(K)$, $\Omega^1(M)$ and $\Omega^1(N)$ are the first syzygies of K, M and N respectively. Since the exact sequence

$$0 \to P_K^0 \to P_M^0 \to P_N^0 \to 0$$

splits, we have that $P_M^0 = P_K^0 \oplus P_N^0$. Notice that since K and N are p-Koszul modules, P_K^0 and P_N^0 are generated in degree $\delta(0)$. By Lemma 2.11, P_M^0 is generated in degree $\delta(0)$. Since P_K^1 and P_N^1 are the graded projective covers of $\Omega^1(K)$ and $\Omega^1(N)$ respectively, $\Omega^1(K)$ and $\Omega^1(N)$ are generated in degree $\delta(1)$. By Lemma 2.11, $\Omega^1(M)$ is generated in degree $\delta(1)$.

Replacing K, M and N by $\Omega^1(K)$, $\Omega^1(M)$, $\Omega^1(N)$ respectively, through recursing step by step, the result is proved. ■

Though we have found some sufficient conditions for the Minimal Horse-Shoe Lemma to be hold, an interesting but difficult question is how to find some necessary conditions.

3. The Applications of the Minimal Horse-shoe Lemma

In this section, we will study the category $K_p(A)$ of p-Koszul modules in details. As the application of the Minimal Horse-Shoe Lemma, we will prove that the category $K_p(A)$ is closed under direct sums, direct summands, extensions and cokernels.

Proposition 3.1. *Let $M = M_1 \oplus M_2 \in gr(A)$. Then M is a p-Koszul module if and only if M_1 and M_2 are p-Koszul modules.*

Proof. We have the following split exact sequence:

$$0 \to M_1 \to M \to M_2 \to 0.$$

By Proposition 2.10, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^2_{M_1} & \longrightarrow & P^2_{M_1} \oplus P^2_{M_2} & \longrightarrow & P^2_{M_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^1_{M_1} & \longrightarrow & P^1_{M_1} \oplus P^1_{M_2} & \longrightarrow & P^1_{M_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^0_{M_1} & \longrightarrow & P^0_{M_1} \oplus P^0_{M_2} & \longrightarrow & P^0_{M_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where

$$\cdots \to P^2_{M_1} \to P^1_{M_1} \to P^0_{M_1} \to M_1 \to 0,$$

$$\cdots \to P^2_{M_1} \oplus P^2_{M_2} \to P^1_{M_1} \oplus P^1_{M_2} \to P^0_{M_1} \oplus P^0_{M_2} \to M \to 0,$$

and

$$\cdots \to P^2_{M_2} \to P^1_{M_2} \to P^0_{M_2} \to M_2 \to 0,$$

are the minimal graded projective resolutions of M_1, M and M_2 respectively. If M is a p-Koszul module, by definition, $P^i_{M_1} \oplus P^i_{M_2}$ is generated in degree $\delta(i)$ for all $i \geq 0$. Obviously, $P^i_{M_1}$ and $P^i_{M_2}$ are generated in degree $\delta(i)$. Hence M_1 and M_2 are p-Koszul modules. Conversely, if M_1 and M_2 are p-Koszul modules, then $P^i_{M_1}$ and $P^i_{M_2}$ are generated in degree $\delta(i)$ for all $i \geq 0$. Evidently, $P^i_{M_1} \oplus P^i_{M_2}$ is generated in degree $\delta(i)$ for all $i \geq 0$. Therefore M is a p-Koszul module. ∎

By Proposition 3.1 we have

Corollary 3.2. *Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n \in gr(A)$. Then M is a p-Koszul module if and only if all M_i are p-Koszul modules.*

Now we investigate the extension closure of the category $K_p(A)$ of p-Koszul modules.

Proposition 3.3. *Let $0 \to K \to M \to N \to 0$ be an exact sequence in $gr(A)$ such that K and N are in $gr_s(A)$. If K, $N \in K_p(A)$, then $M \in K_p(A)$.*

Proof. Let $0 \to K \to M \to N \to 0$ be an exact sequence in $gr(A)$. Since K and N are in degree $gr_s(A)$ and K and M are p-Koszul modules, by proposition 2.12, we have the following commutative diagram with exact rows and exact columns:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^2_K & \longrightarrow & P^2_K \oplus P^2_N & \longrightarrow & P^2_N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^1_K & \longrightarrow & P^1_K \oplus P^1_N & \longrightarrow & P^1_N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^0_K & \longrightarrow & P^0_K \oplus P^0_N & \longrightarrow & P^0_N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where

$$\cdots \to P^2_K \to P^1_K \to P^0_K \to K \to 0,$$

$$\cdots \to P^2_K \oplus P^2_N \to P^1_K \oplus P^1_N \to P^0_K \oplus P^0_N \to M \to 0,$$

and

$$\cdots \to P^2_N \to P^1_N \to P^0_N \to N \to 0,$$

are the minimal graded projective resolutions of K, M and N respectively. Since K and N are p-Koszul modules, both P^i_K and P^i_N are generated in degree $\delta(i)$ for all $i \geq 0$. Thus $P^i_K \oplus P^i_N$ are generated in degree $\delta(i)$ for all $i \geq 0$. Therefore M is a p-Koszul module. Hence $K_p(A)$ is closed under extensions. ∎

The next corollary shows that the category $K_p(A)$ preserves cokernels.

Corollary 3.4. *Let* $0 \to K \to M \to N \to 0$ *be an exact sequence in* $gr_s(A)$. *If* K *and* M *are* p*-Koszul modules, then* $M/K \cong N$ *is a* p*-Koszul module.*

Proof. Similarly to Proposition 3.3, we have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_K^2 & \longrightarrow & P_K^2 \oplus P_N^2 & \longrightarrow & P_N^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_K^1 & \longrightarrow & P_K^1 \oplus P_N^1 & \longrightarrow & P_N^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_K^0 & \longrightarrow & P_K^0 \oplus P_N^0 & \longrightarrow & P_N^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where

$$\cdots \to P_K^2 \to P_K^1 \to P_K^0 \to K \to 0,$$

$$\cdots \to P_K^2 \oplus P_N^2 \to P_K^1 \oplus P_N^1 \to P_K^0 \oplus P_N^0 \to M \to 0,$$

and

$$\cdots \to P_N^2 \to P_N^1 \to P_N^0 \to N \to 0,$$

are the minimal graded projective resolutions of K, M and N respectively. Since P_K^i and $P_K^i \oplus P_N^i$ are generated in degree $\delta(i)$ for all $i \geq 0$, by Lemma 2.11, P_N^i is also generated in degree $\delta(i)$ for all $i \geq 0$. Hence $M/K \cong N$ is a p-Koszul module. ∎

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REFERENCES

1. F. W. Anderson and K. R. Fuller, *Rings and categories of Modules* $[M]$, Springer-Verlag, 1974.

2. M. Auslander and I. Reiten, Syzygy Modules for Noetherian Rings, *J Algebra*, **183** (1996), 167-185.
3. R. Berger, Koszulity for nonquadratic algebras, *J. Algebra*, **239** (2001), 705-734.
4. A. Beilinson, V. Ginszburg and W. Soergel, Koszul duality patterns in representation theory, *J. Amer. Math. Soc.*, **9** (1996), 473-525.
5. E. L. Green, E. N. Marcos, R. Martinez-Villa and Pu Zhang, D-Koszul algebras, *J. Pure and Appl. Algebra*, **193** (2004), 141-162.
6. E. L. Green, R. Martinez-Villa and I. Reiten, ϕ. Solberg, D. Zacharia, On modules with linear presentations, *J. Algebra*, **205** (1998), 578-604.
7. R. Martinez-Villa and D. Zacharia, Approxiamtions with modules having linear resolution, *J. Algebra*, **266** (2003), 671-697.
8. S. Priddy, Koszul resolutions, *Trans. Amer. Math. Soc.*, **152** (1970), 39-60.
9. M.Scott Osborne, Basic Homological Algebra, GTM 196.
10. C. A. Weible, *An Introduction To Homological Algebra*, Cambridge Studies in Avanced Mathematics, Vol. 38, Cambridge Univ. Press, 1995.

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