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SOME PEČARIĆ'S TYPE INEQUALITIES IN 2-INNER PRODUCT SPACES AND APPLICATIONS

Y. J. Cho*, S. S. Dragomir, C.-S. Lin, S. S. Kim* and Y.-H. Kim

Abstract. Some results related to Pečarić's type generalisation of Bessel's inequality in 2-inner product spaces are given. Applications for determinantal integral inequalities are also provided.

1. Introduction

In 1992, Pečarić [5] proved the following inequality for vectors in complex inner product spaces $(H;(\cdot,\cdot))$.

Theorem 1. *Suppose that* $x, y_1, \ldots, y_n$ *are vectors in* H *and* $c_1, \ldots, c_n$ *are complex numbers. Then the following inequalities*

$$\left|\sum_{i=1}^{n} c_i\,(x, y_i)\right|^2 \le \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{j=1}^{n} |(y_i, y_j)|\right) \le \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \max_{1\le i\le n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right) \tag{1.1}$$

hold.

He also showed that for $c_i = \overline{(x, y_i)}, i \in \{1, ..., n\}$, one gets

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \le \|x\|^2 \sum_{i=1}^{n} |(x, y_i)|^2 \left(\sum_{j=1}^{n} |(y_i, y_j)|\right) \le \|x\|^2 \sum_{i=1}^{n} |(x, y_i)|^2 \max_{1\le i\le n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right), \tag{1.2}$$

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*Corresponding authors.

which improves *Bombieri's inequality* [1]

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\|^2 \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |(y_i, y_j)| \right). \tag{1.3}$$

Note that (1.3) is in its turn a natural generalization of *Bessel's inequality*

$$\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2 \tag{1.4}$$

for any $x \in H$, which holds for the orthornormal vectors $(e_i)_{1 \le i \le n}$.

In this paper we point out some results of Pečarić's type for 2-inner products spaces. Some inequalities of Bombieri type holding in these spaces are also mentioned. Natural applications for determinantal integral inequalities are given as well.

2. Some Preliminary Results in 2-Inner Product Spaces

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [2]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

$(2I_1)$ $(x, x|z) \ge 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent,

$(2I_2)$ $(x, x|z) = (z, z|x)$,

$(2I_3)$ $(y, x|z) = \overline{(x, y|z)}$,

$(2I_4)$ $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,

$(2I_5)$ $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(\cdot, \cdot|\cdot)$ is called a 2-*inner product* on X and $(X, (\cdot, \cdot|\cdot))$ is called a 2-*inner product space* (or 2-*pre-Hilbert space*). Some basic properties of 2-inner product $(\cdot, \cdot|\cdot)$ can be immediately obtained as follows [3]:

(1) If $\mathbb{K} = \mathbb{R}$, then $(2I_3)$ reduces to

$$(y, x|z) = (x, y|z).$$

(2) From $(2I_3)$ and $(2I_4)$, we have

$$(0, y|z) = 0, \quad (x, 0|z) = 0$$

and also

$$(x, \alpha y|z) = \bar{\alpha}(x, y|z). \tag{2.1}$$

(3) Using ($2I_2$)–($2I_5$), we have

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2\text{Re}(x, y|z)$$

and

$$\text{Re}(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)]. \tag{2.2}$$

In the real case $\mathbb{K} = \mathbb{R}$, (2.2) reduces to

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] \tag{2.3}$$

and, using this formula, it is easy to see , for any $\alpha \in \mathbb{R}$, that

$$(x, y|\alpha z) = \alpha^2(x, y|z). \tag{2.4}$$

In the complex case, using (2.1) and (2.2), we have

$$\text{Im}(x, y|z) = \text{Re}[-i(x, y|z)] = \frac{1}{4}[(z, z|x + iy) - (z, z|x - iy)],$$

which, in combination with (2.2), yields

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] + \frac{i}{4}[(z, z|x + iy) - (z, z|x - iy)]. \tag{2.5}$$

Using the above formula and (2.1), we have, for any $\alpha \in \mathbb{C}$, that

$$(x, y|\alpha z) = |\alpha|^2(x, y|z). \tag{2.6}$$

However, for $\alpha \in \mathbb{R}$, (2.6) reduces to (2.4). Also, from (2.6) it follows that

$$(x, y|0) = 0.$$

(4) For any three given vectors $x, y, z \in X$, consider the vector $u = (y, y|z)x - (x, y|z)y$. By ($2I_1$), we know that $(u, u|z) \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u, u|z) \geq 0$ can be rewritten as

$$(y, y|z)[(x, x|z)(y, y|z) - |(x, y|z)|^2] \geq 0. \tag{2.7}$$

For $x = z$, (2.7) becomes

$$-(y, y|z)|(z, y|z)|^2 \geq 0,$$

which implies that

$$(z, y|z) = (y, z|z) = 0 \tag{2.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (2.8) holds too. Thus (2.8) is true for any two vectors $y, z \in X$. Now, if y and z are linearly independent, then $(y, y|z) > 0$ and, from (2.7), it follows

$$|(x, y|z)|^2 \leq (x, x|z)(y, y|z). \tag{2.9}$$

Using (2.8), it is easy to check that (2.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (2.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x-(x, y|z)y$ and z are linearly dependent. In fact, we have the equality in (2.9) if and only if the three vectors x, y and z are linearly dependent.

In any given 2-inner product space $(X, (\cdot, \cdot \mid \cdot))$, we can define a function $\| \cdot | \cdot \|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x, x|z)} \tag{2.10}$$

for all $x, z \in X$.

It is easy to see that this function satisfies the following conditions:

$(2N_1)$ $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,

$(2N_2)$ $\|z|x\| = \|x|z\|$,

$(2N_3)$ $\|\alpha x|z\| = |\alpha|\|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,

$(2N_4)$ $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$.

Any function $\|\cdot|\cdot\|$ defined on $X \times X$ and satisfying the conditions $(2N_1)$-$(2N_4)$ is called a 2-*norm* on X and $(X, \| \cdot | \cdot \|)$ is called a *linear* 2-*normed space*. For recent result devoted to the geometry of linear 2-normed spaces, see [4].

Whenever a 2-inner product space $(X, (\cdot, \cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X, \| \cdot | \cdot \|)$ with the 2-norm defined by (2.10).

Let $(X; (\cdot, \cdot|\cdot))$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. If $(f_i)_{1\leq i\leq n}$ are linearly independent vectors in the 2-inner product space X, and, for a given $z \in X, (f_i, f_j|z) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$ where δ_{ij} is the Kronecker delta (we say that the family $(f_i)_{1\leq i\leq n}$ is $z-$*orthonormal*), then the following inequality is the corresponding Bessel's inequality (see for example [3]) for $z-$orthonormal family $(f_i)_{1\leq i\leq n}$ in the 2-inner product space $(X; (\cdot, \cdot|\cdot))$:

$$\sum_{i=1}^{n} |(x, f_i|z)|^2 \leq \|x|z\|^2 \tag{2.11}$$

for any $x \in X$. For more details on this inequality, see the recent paper [3] and the references therein.

3. Some Inequalities for 2-Norms

We start with the following lemma that is interesting in its own right.

Lemma 1. *Let $(X,(\cdot,\cdot|\cdot))$ be a 2-inner product space on $\mathbb{K}$ and $z_1,\dots,z_n,z\in X$, $\alpha_1,\dots,\alpha_n\in\mathbb{K}$. Then one has the inequalities:*

$$\begin{aligned}
&\left\|\sum_{i=1}^{n}\alpha_i z_i|z\right\|^2\\
(3.1)\qquad &\le\left(\sum_{i=1}^{n}|\alpha_i|^p\left(\sum_{j=1}^{n}|(z_i,z_j|z)|\right)\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|\alpha_i|^q\left(\sum_{j=1}^{n}|(z_i,z_j|z)|\right)\right)^{\frac{1}{q}}\\
&\le\begin{cases}A;\\ B;\\ C;\end{cases}
\end{aligned}$$

where

$$A:=\begin{cases}
\max\limits_{1\le i\le n}|\alpha_i|^2\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|;\\[2ex]
\max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}}\\
\qquad\qquad\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}},\\
\qquad\qquad \text{if } \gamma>1,\ \dfrac{1}{\gamma}+\dfrac{1}{\delta}=1;\\[2ex]
\max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{q}\right)^{\frac{1}{q}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}}\\
\qquad\qquad\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}};
\end{cases}$$

$$B := \begin{cases} \max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}} \\ \times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\beta}\right)^{\frac{1}{\beta q}}, \qquad \text{if } \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\beta}\right)^{\frac{1}{p\beta}} \\ \times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \qquad \text{if } \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1 \\ \quad \text{and } \ \gamma>1,\ \dfrac{1}{\gamma}+\dfrac{1}{\delta}=1; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{q}\right)^{\frac{1}{q}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}} \\ \times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \text{if } \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1; \end{cases}$$

$$C := \begin{cases} \max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{p}\right)^{\frac{1}{p}}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}}; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{p}\right)^{\frac{1}{p}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}} \\ \times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \qquad \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{p}\right)^{\frac{1}{p}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{q}\right)^{\frac{1}{q}}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right), \end{cases}$$

and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. We observe that

$$\begin{aligned} \left\|\sum_{i=1}^{n}\alpha_i z_i|z\right\|^2 &= \left(\sum_{i=1}^{n}\alpha_i z_i, \sum_{j=1}^{n}\alpha_j z_j|z\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\left(z_i, z_j|z\right) = \left|\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\left(z_i, z_j|z\right)\right| \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha_i|\,|\alpha_j|\,|(z_i, z_j|z)| =: M. \end{aligned} \tag{3.2}$$

If one uses the Hölder inequality for double sums, i.e., we recall it

$$\sum_{i,j=1}^{n} m_{ij}a_{ij}b_{ij} \le \left(\sum_{i,j=1}^{n} m_{ij}a_{ij}^{p}\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^{n} m_{ij}b_{ij}^{q}\right)^{\frac{1}{q}}, \tag{3.3}$$

where $m_{ij}, a_{ij}, b_{ij} \ge 0$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$; then

$$\begin{aligned} M &\le \left(\sum_{i,j=1}^{n}|(z_i, z_j|z)|\,|\alpha_i|^p\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^{n}|(z_i, z_j|z)|\,|\alpha_i|^q\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{n}|\alpha_i|^p\left(\sum_{j=1}^{n}|(z_i, z_j|z)|\right)\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|\alpha_i|^q\left(\sum_{j=1}^{n}|(z_i, z_j|z)|\right)\right)^{\frac{1}{q}} \end{aligned} \tag{3.4}$$

and the first inequality in (3.1) is proved.

Observe, by Hölder inequality, that

$$\sum_{i=1}^{n}|\alpha_i|^p\left(\sum_{j=1}^{n}|(z_i, z_j|z)|\right) \le \begin{cases} \max\limits_{1\le i\le n}|\alpha_i|^p\sum\limits_{i,j=1}^{n}|(z_i, z_j|z)|; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i, z_j|z)|\right)^{\beta}\right)^{\frac{1}{\beta}} \\ \qquad \text{if } \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1; \\ \sum\limits_{i=1}^{n}|\alpha_i|^p\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i, z_j|z)|\right); \end{cases}$$

which gives

$$
\left(\sum_{i=1}^{n}|\alpha_i|^p\left(\sum_{j=1}^{n}|(z_i,z_j|z)|\right)\right)^{\frac{1}{p}}
\tag{3.5}
$$
$$
\leq\begin{cases}
\max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}}; \\
\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\beta}\right)^{\frac{1}{\beta p}} \\
\qquad \text{if } \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1; \\
\left(\sum\limits_{i=1}^{n}|\alpha_i|^p\right)^{\frac{1}{p}}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{p}}.
\end{cases}
$$

Similarly, we have

$$
\left(\sum_{i=1}^{n}|\alpha_i|^q\left(\sum_{j=1}^{n}|(z_i,z_j|z)|\right)\right)^{\frac{1}{q}}
\tag{3.6}
$$
$$
\leq\begin{cases}
\max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}}; \\
\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}} \\
\qquad \text{if } \gamma>1,\ \dfrac{1}{\gamma}+\dfrac{1}{\delta}=1; \\
\left(\sum\limits_{i=1}^{n}|\alpha_i|^q\right)^{\frac{1}{q}}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j|z)|\right)^{\frac{1}{q}}.
\end{cases}
$$

Using (3.1) and (3.5)-(3.6), we deduce the 9 inequalities in the second part of (3.2). ■

Remark 1. The case $p = q = 2$, will produce some simpler inequalities which will not be stated here for the sake of brevity.

4. Some Pečarić Type Inequalities for 2-Inner Products

We are now able to point out the following result which complements and generalizes the Bessel inequality (2.11) in 2-inner product spaces.

Theorem 2. *Let $x, y_1, \dots, y_n, z$ be vectors of an inner product space $(X; (\cdot,\cdot))$ and $c_1, \dots, c_n \in \mathbb{K}$. Then we have*

$$\begin{aligned}
&\left|\sum_{i=1}^{n} c_i\,(x, y_i|z)\right|^2 \\
&\le \|x|z\|^2 \left(\sum_{i=1}^{n} |c_i|^p \left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right)\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |c_i|^q \left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right)\right)^{\frac{1}{q}} \qquad (4.1)\\
&\le \|x|z\|^2 \times \begin{cases} G; \\ H; \\ L; \end{cases}
\end{aligned}$$

where

$$G := \begin{cases}
\max\limits_{1\le i\le n} |c_i|^2 \sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|\,; \\[2ex]
\max\limits_{1\le i\le n} |c_i| \left(\sum\limits_{i=1}^{n} |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \\
\qquad \text{if } \gamma > 1,\ \dfrac{1}{\gamma} + \dfrac{1}{\delta} = 1; \\[2ex]
\max\limits_{1\le i\le n} |c_i| \left(\sum\limits_{i=1}^{n} |c_i|^{q}\right)^{\frac{1}{q}} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{p}} \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{q}};
\end{cases}$$

$$H := \begin{cases}
\max\limits_{1\le i\le n} |c_i| \left(\sum\limits_{i=1}^{n} |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \\
\qquad \text{if } \alpha > 1,\ \dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1; \\[2ex]
\left(\sum\limits_{i=1}^{n} |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum\limits_{i=1}^{n} |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\beta}\right)^{\frac{1}{p\beta}} \\
\times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \qquad \text{if } \alpha > 1,\ \tfrac{1}{\alpha} + \tfrac{1}{\beta} = 1 \\
\qquad \text{and } \gamma > 1,\ \tfrac{1}{\gamma} + \tfrac{1}{\delta} = 1; \\[2ex]
\left(\sum\limits_{i=1}^{n} |c_i|^{q}\right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^{n} |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{q}} \\
\times \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \qquad \text{if } \alpha > 1,\ \dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1;
\end{cases}$$

and

$$L := \begin{cases} \max\limits_{1\le i\le n} |c_i| \left(\sum\limits_{i=1}^{n} |c_i|^p\right)^{\frac{1}{p}} \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{p}} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{q}}; \\ \left(\sum\limits_{i=1}^{n} |c_i|^p\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{p}} \\ \times \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \qquad \text{if } \gamma > 1,\ \dfrac{1}{\gamma} + \dfrac{1}{\delta} = 1; \\ \left(\sum\limits_{i=1}^{n} |c_i|^p\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} |c_i|^q\right)^{\frac{1}{q}} \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right); \end{cases}$$

where $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1.$

Proof. We note that

$$\sum_{i=1}^{n} c_i\,(x, y_i|z) = \left(x, \sum_{i=1}^{n} \overline{c_i} y_i | z\right).$$

Using Schwarz's inequality in 2-inner product spaces, we have

$$\left|\sum_{i=1}^{n} c_i\,(x, y_i|z)\right|^2 \le \|x|z\|^2 \left\|\sum_{i=1}^{n} \overline{c_i} y_i | z\right\|^2. \tag{4.2}$$

Finally, using Lemma 1 with $\alpha_i = \overline{c_i},\ z_i = y_i\ (i = 1, \ldots, n)$, we deduce the desired inequality (4.1). ■

Remark 2. If in (4.1) we choose $p = q = 2$, we obtain amongst others, some particular inequalities generalising the version of Pečarić's inequality for 2-inner products, i.e., the inequality

$$\begin{aligned} \left|\sum_{i=1}^{n} c_i\,(x, y_i|z)\right|^2 &\le \|x|z\|^2 \left(\sum_{i=1}^{n} |c_i|^2 \left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right)\right) \\ &\le \left(\sum_{i=1}^{n} |c_i|^2\right) \max_{1\le i\le n} \left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right). \end{aligned} \tag{4.3}$$

For the sake of brevity, we do not present them here.

5. Some Results of Bombieri Type for 2-Inner Products

The following results of Bombieri type hold.

Theorem 3. *Let $x, y_1, \ldots, y_n, z \in X$. Then one has the inequalities:*

$$
\begin{aligned}
\sum_{i=1}^{n} |(x, y_i|z)|^2 &\le \|x|z\| \left[\sum_{i=1}^{n} |(x, y_i|z)|^p \left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right)\right]^{\frac{1}{2p}} \\
&\quad \times \left[\sum_{i=1}^{n} |(x, y_i|z)|^q \left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right)\right]^{\frac{1}{2q}} \\
&\le \|x|z\| \times \begin{cases} Q; \\ R; \\ S; \end{cases}
\end{aligned} \tag{5.1}
$$

where

$$
Q := \begin{cases}
\max\limits_{1\le i\le n} |(x, y_i|z)| \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{2}}; \\[2ex]
\max\limits_{1\le i\le n} |(x, y_i|z)|^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n} |(x, y_i|z)|^{\gamma q}\right)^{\frac{1}{2\gamma q}} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{2p}} \\[2ex]
\times \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\delta}\right)^{\frac{1}{2\delta q}}, \\[2ex]
\qquad \text{if } \gamma > 1,\ \dfrac{1}{\gamma} + \dfrac{1}{\delta} = 1; \\[2ex]
\max\limits_{1\le i\le n} |(x, y_i|z)|^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n} |(x, y_i|z)|^{q}\right)^{\frac{1}{2q}} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{2p}} \\[2ex]
\times \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j|z)|\right)^{\frac{1}{2q}};
\end{cases}
$$

$$R := \begin{cases} \max\limits_{1\le i\le n} |(x,y_i|z)|^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{\alpha p}\right)^{\frac{1}{2\alpha\beta}} \left(\sum\limits_{i,j=1}^{n} |(y_i,y_j|z)|\right)^{\frac{1}{2q}} \\ \times \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \qquad \text{if } \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1; \\ \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{\alpha p}\right)^{\frac{1}{2\alpha p}} \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{\gamma q}\right)^{\frac{1}{2\gamma q}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)|\right)^{\beta}\right)^{\frac{1}{2p\beta}} \\ \times \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)|\right)^{\delta}\right)^{\frac{1}{2\delta q}}, \qquad \text{if } \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1 \\ \qquad \text{and } \gamma>1,\ \dfrac{1}{\gamma}+\dfrac{1}{\delta}=1; \\ \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{q}\right)^{\frac{1}{2q}} \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{\alpha p}\right)^{\frac{1}{2\alpha p}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)|\right)^{\frac{1}{2p}} \\ \quad \times \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)|\right)^{\beta}\right)^{\frac{1}{2p\beta}}, \qquad \text{if } \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1; \end{cases}$$

and

$$S := \begin{cases} \max\limits_{1\le i\le n} |(x,y_i|z)|^{\frac{1}{2}} \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{p}\right)^{\frac{1}{2p}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)|\right)^{\frac{1}{2p}} \\ \times \left(\sum\limits_{i,j=1}^{n} |(y_i,y_j|z)|\right)^{\frac{1}{2q}}; \\ \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{p}\right)^{\frac{1}{2p}} \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{\gamma q}\right)^{\frac{1}{2\gamma q}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)|\right)^{\frac{1}{2p}} \\ \times \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)|\right)^{\delta}\right)^{\frac{1}{2\delta q}}, \qquad \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{p}\right)^{\frac{1}{2p}} \left(\sum\limits_{i=1}^{n} |(x,y_i|z)|^{q}\right)^{\frac{1}{2q}} \max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n} |(y_i,y_j|z)|\right)^{\frac{1}{2}}, \end{cases}$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof follows by Theorem 2 on choosing $c_i = \overline{(x, y_i|z)}$ for $i \in \{1, ..., n\}$ and taking the square root in both sides of the inequalities involved. We omit the details. ∎

Remark 3. We observe, by the last inequality in (5.1), we get

$$\text{(5.2)} \quad \frac{\left(\sum_{i=1}^{n} |(x, y_i|z)|^2\right)^2}{\left(\sum_{i=1}^{n} |(x, y_i|z)|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |(x, y_i|z)|^q\right)^{\frac{1}{q}}} \le \|x|z\|^2 \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right),$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and provided that not all $(x, y_i|z)$ for $i \in \{1, ..., n\}$ are zero.

Remark 4. If in this inequality we choose $p = q = 2$, then we obtain the following Bombieri's type result for 2-inner products

$$\text{(5.3)} \quad \sum_{i=1}^{n} |(x, y_i|z)|^2 \le \|x|z\|^2 \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |(y_i, y_j|z)|\right).$$

6. Applications for Determinantal Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra Σ of subsets of Ω and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are 2-ρ-integrable on Ω, i.e., $\int_\Omega \rho(s) |f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L^2_\rho(\Omega)$ by formula

$$\text{(6.1)} \quad (f, g|h)_\rho := \frac{1}{2} \int_\Omega \int_\Omega \rho(s) \rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s) d\mu(t),$$

where

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}$$

denotes the determinant of the matrix

$$\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t) \end{bmatrix},$$

generating the 2-norm on $L_{\rho}^{2}(\Omega)$ expressed by

$$\|f|h\|_{\rho} := \left(\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(s)\rho(t)\left|\begin{array}{cc} f(s) & f(t) \\ h(s) & h(t)\end{array}\right|^{2} d\mu(s)\,d\mu(t)\right)^{1/2}. \tag{6.2}$$

A simple calculation with integrals reveals that

$$(f,g|h)_{\rho} = \left|\begin{array}{cc} \int_{\Omega}\rho fg d\mu & \int_{\Omega}\rho fh d\mu \\ \int_{\Omega}\rho gh d\mu & \int_{\Omega}\rho h^{2} d\mu \end{array}\right| \tag{6.3}$$

and

$$\|f|h\|_{\rho} = \left|\begin{array}{cc} \int_{\Omega}\rho f^{2} d\mu & \int_{\Omega}\rho fh d\mu \\ \int_{\Omega}\rho fh d\mu & \int_{\Omega}\rho h^{2} d\mu \end{array}\right|^{1/2} \tag{6.4}$$

where, for simplicity, instead of $\int_{\Omega}\rho(s) f(s) g(s)\, d\mu(s)$, we have written $\int_{\Omega}\rho fg d\mu$.

Using the representations (6.3), (6.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, we can get some interesting determinantal integral inequalities.

We give here only two examples.

Proposition 1. *Let $f, g_1, ..., g_n, h \in L_{\rho}^{2}(\Omega)$, where $\rho : \Omega \to [0,\infty)$ is a measurable function on Ω, then we have the inequality*

$$\left(\sum_{i=1}^{n}\left|\begin{array}{cc} \int_{\Omega}\rho fg_i d\mu & \int_{\Omega}\rho fh d\mu \\ \int_{\Omega}\rho g_i h d\mu & \int_{\Omega}\rho h^{2} d\mu \end{array}\right|^{2}\right)^{2}$$

$$\leq \left|\begin{array}{cc} \int_{\Omega}\rho f^{2} d\mu & \int_{\Omega}\rho fh d\mu \\ \int_{\Omega}\rho fh d\mu & \int_{\Omega}\rho h^{2} d\mu \end{array}\right|$$

$$\times \max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}\left|\det\left[\begin{array}{cc} \int_{\Omega}\rho g_j g_i d\mu & \int_{\Omega}\rho g_j h d\mu \\ \int_{\Omega}\rho g_i h d\mu & \int_{\Omega}\rho h^{2} d\mu \end{array}\right]\right|\right\}$$

$$\times\left(\sum_{i=1}^{n}\left|\det\begin{bmatrix}\int_\Omega \rho f g_i d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|^p\right)^{1/p}$$

$$\times\left(\sum_{i=1}^{n}\left|\det\begin{bmatrix}\int_\Omega \rho f g_i d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|^q\right)^{1/q},$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1.$

The proof follows by the inequality for 2-inner products incorporated in (5.2).

Proposition 2. *Let* $f, g_1, ..., g_n, h \in L^2_\rho(\Omega)$, *where* $\rho : \Omega \to [0, \infty)$ *is a measurable function on* Ω, *then we have the inequality*

$$\sum_{i=1}^{n}\begin{vmatrix}\int_\Omega \rho f g_i d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu\end{vmatrix}^2$$

$$\leq \begin{vmatrix}\int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu\end{vmatrix}$$

$$\times \max_{1\le i\le n}\left\{\sum_{j=1}^{n}\left|\det\begin{bmatrix}\int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|\right\}.$$

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Y. J. Cho
Department of Mathematics Education and RINS,
College of Education,
Gyeongsang National University,
Chinju 660-701, Korea
E-mail: yjcho@gsnu.ac.kr

S. S. Dragomir
School of Computer Science and Mathematics,
Victoria University of Technology,
P. O. Box 14428,
MCMC Victoria 8001, Australia
E-mail: sever.dragomir@vu.edu.au
http://rgmia.vu.edu.au/SSDragomirWeb.html

C.-S. Lin
Department of Mathematics,
Bishop's University Lennoxville,
Quebec J1M 1Z7, Canada
E-mail: plin@ubishops.ca

S. S. Kim
Department of Matematics,
Dongeui University,
Pusan 614-714, Korea
E-mail: sskim@dongeui.ac.kr

Y.-H. Kim
Department of Applied Mathematics,
Changwon National University,
Changwon 641-773, Korea
E-mail: yhkim@sarim.changwon.ac.kr