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ON THE EXISTENCE OF PERIODIC SOLUTION FOR NEUTRAL DELAY COMPETITIVE SYSTEM

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Abstract. In this paper, the $n-$species neutral delay competitive differential system with periodic coefficients is investigated by means of an abstract continuous theorem of $k-$set contractive operator and some analysis techniques. Sufficient conditions are obtained for periodic solution.

1. Introduction

We consider the periodic neutral delay competitive differential equations

$$\dot{x}_i(t) = x_i(t)\Big[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t-\tau_{ij}(t)) - \sum_{j=1}^{n} b_{ij}(t)\dot{x}_i(t-\sigma_{ij}(t))\Big], \quad i = 1, \cdots, n, \tag{1}$$

where the functions $r_i(t)$, $a_{ij}(t)$, $b_{ij}(t)$, $\tau_{ij}(t)$, $\sigma_{ij}(t)$ are continuous periodic functions with period $\omega > 0$ and $r_i(t) \geq 0$, $a_{ij}(t) \geq 0$, $b_{ij}(t) \geq 0$, $\tau_{ij}(t){\geq}0$ and $\sigma_{ij}(t) \geq 0$ $(i, j = 1, \cdots, n)$ for all $t{\geq}0$. Furthermore, $\tau_{ij}(t)$ are continuously differentiable with $\tau'_{ij}(t) < 1$ for all $t{\geq}0$. Single species population models have been studied by a lot of authors (see [1-8]). The simplest and most widely adopted single species growth model is the well-known Logistic equation

$$\dot{x}(t) = rx(t)\Big[1 - \frac{x(t)}{K}\Big], \tag{2}$$

where r is called the intrinsic growth rate of the species x, K is interpreted as the environment capacity for x, and $r[1 - x(t)/K]$ is the per capita growth rate of x at

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time t. Smith [5] argued that the per capita growth rate in (2) should be replaced by $r[1-(x(t)+\rho\dot{x}(t))/K]$. This leads to the equation

$$\dot{x}(t) = rx(t)\Big[1 - \frac{x(t)+\rho\dot{x}(t)}{K}\Big]. \tag{3}$$

Gopalsamy et al. [6] studied a periodic version of (3) that takes the form

$$\dot{x}(t) = r(t)x(t)\left[1 - \frac{x(t-m\omega)+c(t)\dot{x}(t-m\omega)}{K(t)}\right], \tag{4}$$

where $r(t),\ c(t)$ and $K(t)$ are positive continuous periodic functions of period ω, and m is a positive integer.

In 1993, Kuang in [3] proposed an open problem (open problem 9.2) to obtain sufficient conditions for the existence of positive periodic solutions to the following equation

$$\frac{dx}{dt} = r(t)x(t)\Big[a(t) - \beta(t)x(t) - b(t)x(t-\tau(t)) - c(t)\dot{x}(t-\tau(t))\Big], \tag{5}$$

where $r(t),\ a(t),\ \beta(t),\ b(t),\ c(t),\ \tau(t)$ are continuous periodic functions with period $\omega > 0$. Li [9-11], by use of Mawhin's continuation theorem, studied system (5) and

$$\frac{dN}{dt} = N(t)\Big[a(t) - \sum_{j=1}^{n} b_j(t)N(t-\sigma_j) - \sum_{i=1}^{n} c_i(t)\dot{N}(t-\tau_i)\Big] \tag{6}$$

and system

$$\begin{aligned}\frac{dN_i}{dt} &= N_i(t)\Big[r_i(t) - \sum_{j=1}^{n} \alpha_{ij}(t)N_j(t-\tau_{ij}) - \sum_{j=1}^{n} \beta_{ij}(t)\dot{N}_j(t-\sigma_{ij})\Big],\\ i &= 1,2,\cdots,n,\end{aligned} \tag{7}$$

where $\sigma_j,\ \tau_i,\ \tau_{ij}$ and σ_{ij} $(i,j=1,2,\cdots,n)$ are constants. But, Li did not verify the important assumption that operator $\mathcal{N} : \bar{\Omega} \to X$ was $L-$compact in all the papers [9-11]. So the main theorems in [9-11] may not be true. For example, under the transformation $N(t) = e^{x(t)}$, Li rewrote system (6) in the following form

$$Lx = \mathcal{N}x,$$

where $L = \frac{d}{dt}$, $\mathcal{N}x = a(t) - \sum\limits_{j=1}^{n} b_j(t)e^{x(t-\sigma_j)} - \sum\limits_{i=1}^{n} c_i(t)\dot{x}(t-\tau_i)e^{x(t-\tau_i)}$. Even if $\sigma_j = \tau_j$ $(j=1,2,\cdots,n)$, according to the definition of operator $K_p,\ P,\ Q$ and

Banach space $X = \{x : x \in C^1(R, R),\ x(t+\omega) \equiv x(t)\}$ in [10], one can find that

$$
\begin{aligned}
K_P(I-Q)\mathcal{N}: \quad & X \to X, \\
K_P(I-Q)\mathcal{N}x = & \int_0^t \left[a(s) - \sum_{j=1}^{n}(b_j(s) - \dot{c}_j(s))e^{x(t-\tau_j)}\right]ds \\
& -\sum_{j=1}^{n}\left[c_j(t)e^{x(t-\tau_j)} - c_j(0)e^{x(-\tau_j)}\right] \\
& -\frac{1}{\omega}\int_0^{\omega}\int_0^t \left[a(s) - \sum_{j=1}^{n}(b_j(s) - \dot{c}_j(s))e^{x(s-\tau_j)}\right]dsdt \\
& -\left(\frac{t}{\omega} - \frac{1}{2}\right)\int_0^{\omega}\left[a(t) - \sum_{j=1}^{n}(b_j(t) - \dot{c}_j(t))e^{x(t-\tau_j)}\right]dt.
\end{aligned}
$$

As the right side of the above formula contains

$$\sum_{j=1}^{n}\left[c_j(t)e^{x(t-\tau_j)} - c_j(0)e^{x(-\tau_j)}\right],$$

one can not verify that $K_P(I-Q)\mathcal{N}(\bar{\Omega})$ is relatively compact in X for any bounded set $\Omega \subset X$.

It is easy to see that system (7) is a special case of system (1), The purpose of this article is to establish criteria to guarantee the existence of positive periodic solutions to system (1). By using the continuation theory for k-set contractions [12-18], we obtain a new result.

2. Preparation

In order to study system (1), we should make some preparations. Let E be a Banach space. For a bounded subset $A \in E$, denote by

$$
\begin{aligned}
\alpha_E(A) = \inf\Big\{\delta > 0 \Big| & \text{there is finite number of subsets } A_i \subset A, \\
& \text{such that } A = \bigcup_{i=1} A_i \text{ and } \operatorname{diam}(A_i) \le \delta\Big\}
\end{aligned}
$$

the (Kuratoski) measure of noncompactness, where $\operatorname{diam}(A_i)$ denotes the diameter of set A_i. Let $X,\ Y$ be two Banach spaces and Ω a bounded open subset of X. A continuous and bounded map $x : \bar{\Omega} \to Y$ is called $k-$set contractive if for any bounded set $A \subset \Omega$ we have

$$\alpha_Y(N(A)) \le k\alpha_X(A).$$

Also, for a Fredholm operator $L : X \to Y$ with index zero, according to [15,16] we may define

$$l(L) = \sup\{r \geq 0 | r\alpha_X(A) \leq \alpha_Y(L(A)), \text{for all bounded subset } A \subset X\}.$$

Lemma 1. ([17, 18]) *Let $L : X \to Y$be a Fredholm operator with index zero, and $a \in Y$ a fix point. Suppose that $N : \Omega \to Y$ is a $k-$set contractive with $k < l(L)$, where $\Omega \subset X$ is bounded, open, and symmetric about $0 \in \Omega$. Furthermore, we assume that*

(a) $Lx \neq \lambda Nx + \lambda r.$*for* $x \in \partial\Omega, \lambda \in (0, 1)$, *and*

(b) $[QN(x) + Qr, x][QN(-x) + Qr, x] < 0$, *for* $x \in \ker L \bigcap \partial\Omega$, *where* $[\cdot, \cdot]$ *is a bilinear form on* $Y \times X$ *and* Q *is the project of* Y *onto coker*(L).

Then there is $x \in \bar{\Omega}$ *such that*

$$Lx - Nx = r.$$

In order to use Lemma 1 for system(1), By the transformation $x_i(t) = e^{u_i(t)}$, system (1) can be rewritten as

$$\begin{aligned} \dot{u}_i(t) \; &= \; r_i(t) - \sum_{j=1}^{n} a_{ij}(t) e^{u_j(t-\tau_{ij}(t))} \\ &- \sum_{j=1}^{n} b_{ij}(t)\dot{u}_j(t-\sigma_{ij}(t)) e^{u_j(t-\sigma_{ij}(t))}, \; i = 1, 2, \cdots, n. \end{aligned} \tag{8}$$

Let $u(t) = (u_1(t), \cdots, u_n(t))^T$. We write $u > 0$ if $u_i > 0, \; i = 1, \cdots, n$. We take

$$C^0_\omega = \left\{u(t) = (u_1(t), \cdots, u_n(t))^T \in C^0(R, R^n) : u_i(t+\omega) = u_i(t), i = 1, \cdots, n.\right\}$$

with the norm defined by $|u|_0 = \max_{t\in[0,\omega]} |u(t)|$, and

$$C^1_\omega = \left\{u(t) = (u_1(t), \cdots, u_n(t))^T \in C^1(R, R^n) : u_i(t+\omega) = u_i(t), i = 1, \cdots, n.\right\}$$

with the norm $|u|_1 = \max_{t\in[0,\omega]}\{|u|_0, |\dot{u}|_1\}$. Then C^0_ω, C^1_ω are all Banach spaces. Let $L : C^1_\omega \to C^0_\omega$ be defined by $Lu = \frac{du}{dt}$, and $N : C^1_\omega \to C^0_\omega$ defined by

$$Nu = \begin{bmatrix} -\sum_{j=1}^{n} a_{1j}(t) e^{u_j(t-\tau_{1j}(t))} - \sum_{j=1}^{n} b_{1j}(t)\dot{u}_j(t-\sigma_{1j}(t)) e^{u_j(t-\sigma_{1j}(t))} \\ \vdots \\ -\sum_{j=1}^{n} a_{nj}(t) e^{u_j(t-\tau_{nj}(t))} - \sum_{j=1}^{n} b_{nj}(t)\dot{u}_j(t-\sigma_{nj}(t)) e^{u_j(t-\sigma_{nj}(t))} \end{bmatrix}. \tag{9}$$

It is easy to see from [15] that L is a Fredholm operator with index zero. Now, system (8) has an ω-periodic solution if and only if $Lu = Nu + r$ for $u \in C^1_\omega$, where $r =: r(t)$.

In this paper, the notations

$$f^L = \min_{t\in[0,\omega]} f(t), \quad f^M = \max_{t\in[0,\omega]} f(t), \ \bar{f} = \frac{1}{\omega}\int_0^\omega f(t)dt.$$

will be used.

Lemma 2. ([17]). *The differential operator L is a Fredholm operator with index zero and satisfies $l(L) \geq 1$.*

Lemma 3. *Let γ_1, γ_2 be two positive constants and $\Omega = \{u|u \in C^1_\omega, |u|_0 < \gamma_1, |\dot{u}|_1 < \gamma_2\}$. If $k = \max\{\sum_{i=1}^n |b_{ij}|_0 e^{\gamma_1}\} < 1$, then $N : \Omega \to C^0_\omega$ is a $k-$contractive map.*

Proof. Let $A \subset \bar{\Omega}$ be a bounded subset and let $\eta = \alpha_{C^1_\omega(A)}$. Then, for any $\varepsilon > 0$, there is a finite family of subsets A_i satisfying $A = \bigcup_{i=1} A_i$ with $\mathrm{diam}(A_i) \leq \eta+\varepsilon$. Now let $V_i(t, u, z, w) = V_i(t, u_1, \cdots, u_n, z_1, \cdots, z_n, w_1, \cdots, w_n), i = 1, \cdots, n$ and

$$\begin{bmatrix} V_1(t,u,z,w) \\ \vdots \\ V_n(t,u,z,w) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n a_{1j}(t)e^{u_j} + \sum_{j=1}^n b_{1j}(t)w_j e^{z_j} \\ \vdots \\ \sum_{j=1}^n a_{nj}(t)e^{u_j} + \sum_{j=1}^n b_{nj}(t)w_j e^{z_j} \end{bmatrix}. \tag{10}$$

Since $V_i(t, u, z, w)$ are uniformly continuous on any compact subset of $R \times R^{3n+1}$, A and A_i are precompact in C^0_ω. It follows that there is a finite family of subsets A_{ij} of A_i such that $A_i = \bigcup_{j=1} A_{ij}$ with

$$\begin{aligned} &|V_i(t, u(t-\tau_{ij}(t)), u(t-\sigma_{ij}(t)), \dot{v}(t-\sigma_{ij}(t)))\ | \\ &- V_i(t, v(t-\tau_{ij}(t)), v(t-\sigma_{ij}(t)), \dot{v}(t-\sigma_{ij}(t)))| \\ = &\Big|\Big[\sum_{j=1}^n a_{ij}(t)e^{u_j(t-\tau_{ij}(t))} + \sum_{j=1}^n b_{ij}(t)\dot{v}_j(t-\sigma_{ij}(t))e^{u_j(t-\sigma_{ij}(t))}\Big] \\ &-\Big[\sum_{j=1}^n a_{ij}(t)e^{v_j(t-\tau_{ij}(t))} + \sum_{j=1}^n b_{ij}(t)\dot{v}_j(t-\sigma_{ij}(t))e^{v_j(t-\sigma_{ij}(t))}\Big]\Big| \\ \leq &\sum_{j=1}^n a_{ij}(t)\Big|e^{u_j(t-\tau_{ij}(t))} - e^{v_j(t-\tau_{ij}(t))}\Big| \end{aligned}$$

$$+\sum_{j=1}^{n} b_{ij}(t)|\dot{v}_j(t-\sigma_{ij}(t))|\left|e^{u_j(t-\sigma_{ij}(t))}-e^{v_j(t-\sigma_{ij}(t))}\right|$$
$$\leq \sum_{j=1}^{n}\left(a_{ij}^M e^{\gamma_1}+b_{ij}^M\gamma_2 e^{\gamma_1}\right)\sup_{s\in[0,\omega]}|u_j(s)-u_j(s)|\leq\varepsilon,$$

for any $u, v \in A_{ij}$. Therefore we have

$$\begin{aligned}|Nu_i-Nv_i|_0 =& \sup_{t\in[0,\omega]}|V_i(t,u(t-\tau_{ij}(t)),u(t-\sigma_{ij}(t)),\dot{u}(t-\sigma_{ij}(t)))\\
&-V_i(t,v(t-\tau_{ij}(t)),v(t-\sigma_{ij}(t)),\dot{v}(t-\sigma_{ij}(t)))|\\
\leq& \sup_{t\in[0,\omega]}|V_i(t,u(t-\tau_{ij}(t)),u(t-\sigma_{ij}(t)),\dot{u}(t-\sigma_{ij}(t)))\\
&-V_i(t,u(t-\tau_{ij}(t)),u(t-\sigma_{ij}(t)),\dot{v}(t-\sigma_{ij}(t)))|\\
&+\sup_{t\in[0,\omega]}|V_i(t,u(t-\tau_{ij}(t)),u(t-\sigma_{ij}(t)),\dot{v}(t-\sigma_{ij}(t)))\\
&-V_i(t,v(t-\tau_{ij}(t)),v(t-\sigma_{ij}(t)),\dot{v}(t-\sigma_{ij}(t)))|\\
\leq& \sum_{j=1}^{n}|b_{ij}|_0 e^{\gamma_1}\,|\dot{u}_j(t-\sigma_{ij}(t))-\dot{v}_j(t-\sigma_{ij}(t))|+\varepsilon\\
\leq& k\eta+(k+1)\varepsilon.\end{aligned}$$

As ε is arbitrary small, it is easy to see that

$$\alpha_{C_\omega^0}(N(A))\leq k\alpha_{C_\omega^1}((A)).$$

Lemma 4. *Let* $\Phi\in C_\omega^0,\ \tau\in C_\omega^1$, *and* $\dot{\tau}(t)<1$, *then* $\Phi(\nu(t))\in C_\omega^0$, *where* $\nu(t)$ *is the inverse function of* $t-\tau(t)$.

Proof. We need only to prove that $\nu(a+\omega)=\nu(a)+\omega$, for arbitrary $a\in R$. By the condition $\tau'<1$, it is easy to see that the equation $t-\tau(t)=a$ and $t-\tau(t)=a+\omega$ exist a unique solution t_1, t_2 respectively. That is

$$t_1-\tau(t_1)=a,\ t_2-\tau(t_2)=a+\omega,$$

i.e.,

$$\nu(a)=t_1=a+\tau(t_1)\text{ and }\nu(a+\omega)=t_2. \tag{11}$$

As

$$\begin{aligned}\omega+a+\tau(t_1)-\tau(\omega+a+\tau(t_1)) &= \omega+a+\tau(t_1)-\tau(a+\tau(t_1))\\
&=\omega+a+\tau(t_1)-\tau(t_1)\\
&=\omega+a.\end{aligned}$$

It follows that $t_2 = \omega + a + \tau(t_1)$. So by (11), we have $\nu(a+\omega) = \nu(a) + \omega$ for $\forall a \in R$. ■

Throughout this paper, we assume that $\tau_{ij} \in C^2_\omega,\ \sigma_{ij} \in C^2_\omega,\ \dot{\tau}_{ij} < 1,\ \dot{\sigma}_{ij} < 1\ (i,j = 1,\cdots,n)$. So $t - \tau_{ij}(t)$ or $t - \sigma_{ij}(t)$ has a unique inverse, and we set $\nu_{ij}(t),\ \mu_{ij}(t)$ to represent the inverse of function $t-\tau_{ij}(t),\ t-\sigma_{ij}(t)$, respectively.

3. Main Results

Theorem 1. *Suppose* $b_{ij} \in C^1_\omega, \bar{r}_i > 0,\ i,j = 1,\cdots,n,$ *and*

$$\begin{aligned}
a_{ij}(t) &> \dot{b}^0_{ij}(t),\\
\Phi_{ij}(t) &=: \frac{a_{ij}(\nu_{ij}(t))}{1-\dot{\tau}_{ij}(\nu_{ij}(t))} - \frac{\dot{b}^0_{ij}(\mu_{ij}(t))}{1-\dot{\sigma}_{ij}(\mu_{ij}(t))} > 0,\\
\Psi_{ij}(t) &=: \frac{a_{ij}(\nu_{ij}(t))}{1-\dot{\tau}_{ij}(\nu_{ij}(t))} - \frac{\theta_2 a_{ij}(\mu_{ij}(t)) + \theta_1 \dot{b}^0_{ij}(\mu_{ij}(t))}{1-\dot{\sigma}_{ij}(\mu_{ij}(t))} > 0,
\end{aligned}$$

$i,j = 1,\cdots,n$. *Further, we assume that*

$$k = \max_{1\le i\le n}\Big\{\sum_{j=1}^n |b_{ij}|_0,\ \sum_{j=1}^n |b^0_{ij}|_0\Big\} e^M < 1,$$

where

$$\begin{aligned}
M &= \max_{1\le i\le n}\Big\{\Big|\log\frac{\bar{r}_i}{\sum_{j=1}^n \bar{a}_{ij}} Big|, M_i, N_1\Big\},\\
M_i &= 2\bar{r}_i\omega + A_{ii} + \sum_{j=1}^n B_{ij}|b^0_{ij}|_0,\\
A_{ii} &= \max_{1\le i\le n}\Big\{\bar{r}_i\Big(\frac{a_{ii}(\nu_{ii}(\xi_i))}{1-\dot{\tau}_{ii}(\nu_{ii}(\xi_i))} - \frac{\theta_2 a_{ii}(\mu_{ii}(\xi_i)) + \theta_1\dot{b}^0_{ii}(\mu_{ii}(\xi_i))}{1-\dot{\sigma}_{ii}(\mu_{ii}(\xi_i))}\Big)^{-1}\Big\},\\
B_{ij} &= \max_{1\le i,j\le n}\Big\{\frac{\bar{r}_i}{a_{ij}(\xi_i) - \dot{b}^0_{ij}(\xi_i)}\Big\},\ b^0_{ij} = \frac{b_{ij}(t)}{1-\dot{\sigma}_{ij}},\\
N_1 &= \max_{1\le i\le n}\Big\{A_{ii} + \omega\Big(\bar{r}_i + \sum_{j=1}^n \bar{a}_{ij}\Big)\Big(1 - e^{M_j}\sum_{j=1}^n |b^0_{ij}|_0 e^{M_j}\Big)^{-1}\Big\},
\end{aligned}$$

θ_1, θ_2 *are positive satisfying* $\theta_1 + \theta_2 = 1$. *Then Eqs.(1) has at least one positive* ω*-periodic solution.*

Proof. We consider the operator equation

$$Lu = \lambda Nu + \lambda r, \ \lambda \in (0,1).$$

We have

$$\begin{bmatrix} \dot{u}_1(t) \\ \vdots \\ \dot{u}_n(t) \end{bmatrix} = \lambda \begin{bmatrix} r_1(t) - \sum_{j=1}^{n} a_{1j}(t) e^{u_j(t-\tau_{1j}(t))} \\ \vdots \\ r_n(t) - \sum_{j=1}^{n} a_{nj}(t) e^{u_j(t-\tau_{nj}(t))} \end{bmatrix.\qquad (12)$$

$$\begin{matrix} -\sum_{j=1}^{n} b_{1j}(t)\dot{u}_j(t-\sigma_{1j}(t)) e^{u_j(t-\sigma_{1j}(t))} \\ \vdots \\ -\sum_{j=1}^{n} b_{nj}(t)\dot{u}_j(t-\sigma_{nj}(t)) e^{u_j(t-\sigma_{nj}(t))} \end{matrix}\Bigg].$$

Suppose that $u(t) = (u_1(t), \cdots, u_n(t))^T$ is any arbitrary solution of system (12) for a certain $\lambda \in (0,1)$, then we have

$$\begin{aligned} \dot{u}_i(t) = \lambda \Big[& r_i(t) - \sum_{j=1}^{n} a_{ij}(t) e^{u_j(t-\tau_{ij}(t))} \\ & - \sum_{j=1}^{n} b_{ij}(t)\dot{u}_j(t-\sigma_{ij}(t)) e^{u_j(t-\sigma_{ij}(t))} \Big]. \end{aligned} \qquad (13)$$

Integrating system (13) over the interval $[0, \omega]$, we obtain

$$\begin{aligned} \int_0^{\omega} \Big[& r_i(t) - \sum_{j=1}^{n} a_{ij}(t) e^{u_j(t-\tau_{ij}(t))} \\ & - \sum_{j=1}^{n} b_{ij}(t)\dot{u}_j(t-\sigma_{ij}(t)) e^{u_j(t-\sigma_{ij}(t))} \Big] dt = 0. \end{aligned} \qquad (14)$$

Clearly

$$\begin{aligned} & \int_0^{\omega} b_{ij}(t)\dot{u}_j(t-\sigma_{ij}(t)) e^{u_j(t-\sigma_{ij}(t))} dt \\ = & \frac{b_{ij}(\mu_{ij}(t)}{1-\dot{\sigma}_{ij}(\mu_{ij}(t))} e^{u_j(t-\sigma_{ij}(t))}\Big|_0^{\omega} - \int_0^{\omega} e^{u_j(t-\sigma_{ij}(t))} db_{ij}^0(t) \\ = & -\int_0^{\omega} \dot{b}_{ij}^0(t) e^{u_j(t-\sigma_{ij}(t))} dt. \end{aligned} \qquad (15)$$

From (14) and (15), we have

$$\bar{r}_i\omega = \int_0^\omega \left[\sum_{j=1}^n a_{ij}(t) e^{u_j(t-\tau_{ij}(t))} - \sum_{j=1}^n \dot{b}_{ij}^0(t) e^{u_j(t-\sigma_{ij}(t))} \right] dt. \tag{16}$$

Let $t - \tau_{ij}(t) = s,\ i.e.,\ t = \nu_{ij}(s)$, and $t - \sigma_{ij}(t) = s,\ i.e.,\ t = \mu_{ij}(s)\ (i, j = 1, \cdots, n)$. According to Lemma 4, we have

$$\begin{aligned}
\int_0^\omega a_{ij}(t) e^{u_j(t-\tau_{ij}(t))} dt &= \int_{-\tau_{ij}(0)}^{\omega-\tau_{ij}(\omega)} \frac{a_{ij}(\nu_{ij}(s))}{1 - \dot{\tau}_{ij}(\nu_{ij}(s))} e^{u_j(s)} ds \\
&= \int_0^\omega \frac{a_{ij}(\nu_{ij}(t))}{1 - \dot{\tau}_{ij}(\nu_{ij}(t))} e^{u_j(t)} dt, \\
\int_0^\omega \dot{b}_{ij}^0(t) e^{u_j(t-\sigma_{ij}(t))} dt &= \int_{-\sigma_{ij}(0)}^{\omega-\sigma_{ij}(\omega)} \frac{\dot{b}_{ij}^0(\mu_{ij}(t))}{1 - \dot{\sigma}_{ij}(\mu_{ij}(t))} e^{u_j(t)} dt \\
&= \int_0^\omega \frac{\dot{b}_{ij}^0(\mu_{ij}(t))}{1 - \dot{\sigma}_{ij}(\mu_{ij}(t))} e^{u_j(t)} dt.
\end{aligned}$$

So from (16), we get

$$\bar{r}_i\omega = \int_0^\omega \sum_{j=1}^n \left[\frac{a_{ij}(\nu_{ij}(t))}{1 - \dot{\tau}_{ij}(\nu_{ij}(t))} - \frac{\dot{b}_{ij}^0(\mu_{ij}(t))}{1 - \dot{\sigma}_{ij}(\mu_{ij}(t))} \right] e^{u_j(t)} dt. \tag{17}$$

On the other hand

$$\begin{aligned}
&\int_0^\omega \left| \frac{d}{dt} \left[u_i(t) + \lambda \sum_{j=1}^n b_{ij}^0(t) e^{u_j(t-\sigma_{ij}(t))} \right] \right| dt \\
&= \lambda \int_0^\omega \left| \left[r_i(t) - \sum_{j=1}^n a_{ij}(t) e^{u_j(t-\tau_{ij}(t))} + \sum_{j=1}^n \dot{b}_{ij}^0(t) e^{u_j(t-\sigma_{ij}(t))} \right] \right| dt \\
&= \lambda \int_0^\omega \left| \left[r_i(t) - \left(\frac{a_{ij}(\nu_{ij}(t))}{1 - \dot{\tau}_{ij}(\nu_{ij}(t))} - \frac{\dot{b}_{ij}^0(\mu_{ij}(t))}{1 - \dot{\sigma}_{ij}(\mu_{ij}(t))} \right) e^{u_j(t)} \right] \right| dt \\
&\le \int_0^\omega r_i(t) dt + \int_0^\omega \sum_{j=1}^n \left[\frac{a_{ij}(\nu_{ij}(t))}{1 - \dot{\tau}_{ij}(\nu_{ij}(t))} - \frac{\dot{b}_{ij}^0(\mu_{ij}(t))}{1 - \dot{\sigma}_{ij}(\mu_{ij}(t))} \right] e^{u_j(t)} dt \\
&= 2\bar{r}_i\omega.
\end{aligned} \tag{18}$$

In addition, we have

$$
\begin{aligned}
\bar{r}_i\omega &= \theta_1 \int_0^\omega \left[\sum_{j=1}^n a_{ij}(t) e^{u_j(t-\tau_{ij}(t))} - \sum_{j=1}^n \dot{b}_{ij}^{\,0}(t) e^{u_j(t-\sigma_{ij}(t))} \right] dt \\
&\quad + \theta_2 \int_0^\omega \left[\sum_{j=1}^n a_{ij}(t) e^{u_j(t-\tau_{ij}(t))} - \sum_{j=1}^n \dot{b}_{ij}^{\,0}(t) e^{u_j(t-\sigma_{ij}(t))} \right] dt \\
&= \int_0^\omega \left[(\theta_1+\theta_2) \sum_{j=1}^n a_{ij}(t) e^{u_j(t-\tau_{ij}(t))} \right. \\
&\quad \left. - \theta_2 \sum_{j=1}^n a_{ij}(t) e^{u_j(t-\sigma_{ij}(t))} - \theta_1 \sum_{j=1}^n \dot{b}_{ij}^{\,0}(t) e^{u_j(t-\sigma_{ij}(t))} \right] dt \qquad (19) \\
&\quad + \int_0^\omega \theta_2 \sum_{j=1}^n \left(a_{ij}(t) - \dot{b}_{ij}^{\,0}(t) \right) e^{u_j(t-\sigma_{ij}(t))} dt \\
&= \int_0^\omega \sum_{j=1}^n \left[\frac{a_{ij}(\nu_{ij}(t))}{1-\dot{\tau}_{ij}(\nu_{ij}(t))} - \frac{\theta_2 a_{ij}(\mu_{ij}(t)) + \theta_1 \dot{b}_{ij}^{\,0}(\mu_{ij}(t))}{1-\dot{\sigma}_{ij}(\mu_{ij}(t))} \right] e^{u_j(t)} dt \\
&\quad + \int_0^\omega \theta_2 \sum_{j=1}^n \left[a_{ij}(t) - \dot{b}_{ij}^{\,0}(t) \right] e^{u_j(t-\sigma_{ij}(t))} dt.
\end{aligned}
$$

From (19) there is $\xi_i \in [0,\omega]$ such that

$$
\begin{aligned}
\bar{r}_i &= \sum_{j=1}^n \left[\frac{a_{ij}(\nu_{ij}(\xi_i))}{1-\dot{\tau}_{ij}(\nu_{ij}(\xi_i))} - \frac{\theta_2 a_{ij}(\mu_{ij}(\xi_i)) + \theta_1 \dot{b}_{ij}^{\,0}(\mu_{ij}(\xi_i))}{1-\dot{\sigma}_{ij}(\mu_{ij}(\xi_i))} \right] e^{u_j(\xi_i)} \\
&+ \theta_2 \textstyle\sum_{j=1}^n \left[a_{ij}(\xi_i) - \dot{b}_{ij}^{\,0}(\xi_i) \right] e^{u_j(\xi_i-\sigma_{ij}(\xi_i))}, \; i = 1, \cdots, n,
\end{aligned} \qquad (20)
$$

which implies

$$
\begin{aligned}
u_j(\xi_i) &\le \bar{r}_i \left(\frac{a_{ij}(\nu_{ij}(\xi_i))}{1-\dot{\tau}_{ij}(\nu_{ij}(\xi_i))} - \frac{\theta_2 a_{ij}(\mu_{ij}(\xi_i)) + \theta_1 \dot{b}_{ij}^{\,0}(\mu_{ij}(\xi_i))}{1-\dot{\sigma}_{ij}(\mu_{ij}(\xi_i))} \right)^{-1} \\
&= A_{ij}, \; i,j = 1, \cdots, n.
\end{aligned} \qquad (21)
$$

$$
e^{u_j(\xi_i-\sigma_{ij}(\xi_i))} \le \frac{\bar{r}_i}{a_{ij}(\xi_i) - \dot{b}_{ij}^{\,0}(\xi_i)} \le B_{ij}, \; i,j = 1, \cdots, n. \qquad (22)
$$

It follows from (18), (20) and (21) that

$$
u_i(t) + \lambda \sum_{j=1}^n b_{ij}^{\,0}(t) e^{u_j(t-\sigma_{ij}(t))} \le u_i(\xi_i) + \lambda \sum_{j=1}^n b_{ij}^{\,0}(\xi_i) e^{u_j(\xi_i-\sigma_{ij}(\xi_i))}
$$

$$\int_0^\omega \left| \frac{d}{dt}\left[u_i(t) + \lambda \sum_{j=1}^n b_{ij}^0(t) e^{u_j(t-\sigma_{ij}(t))}\right]\right| dt$$

$$\leq 2\bar{r}_i\omega + A_{ii} + \sum_{j=1}^n B_{ij}|b_{ij}^0|_0, \ i = 1, \cdots, n.$$

Then

$$u_i(t) + \lambda \sum_{j=1}^n b_{ij}^0(t) e^{u_j(t-\sigma_{ij}(t))} < M_i, \ i = 1, \cdots, n. \tag{23}$$

As

$$\lambda \sum_{j=1}^n b_{ij}^0(t) e^{u_j(t-\sigma_{ij}(t))} > 0,$$

it follows that

$$u_i(t) < M_i, \ i = 1, \cdots, n. \tag{24}$$

From (13) and (23), we have

$$\int_0^\omega |\dot{u}_i(t)| dt \leq \bar{r}_i\omega + \sum_{j=1}^n \bar{a}_{ij}\omega e^{M_j} + e^{M_j} \sum_{j=1}^n b_{ij}(t) \left|\dot{u}_j(t - \sigma_{ij}(t))\right| dt$$

$$\leq \bar{r}_i\omega + \sum_{j=1}^n \bar{a}_{ij}\omega e^{M_j} + e^{M_j} \sum_{j=1}^n \frac{b_{ij}(\mu_{ij})}{1 - \dot{\sigma}(\mu_{ij}(t)} \left|\dot{u}_j(t)\right| dt,$$

i.e.,

$$\int_0^\omega |\dot{u}_i(t)| dt \leq \omega\left(\bar{r}_i + \sum_{j=1}^n \bar{a}_{ij}\right)\left(1 - e^{M_j}\sum_{j=1}^n |b_{ij}^0|_0 e^{M_j}\right)^{-1}. \tag{25}$$

It follows from (21), we have

$$|u_i(t)| \leq |u_i(\xi_i)| + \int_0^\omega |\dot{u}_i(t)| dt$$

$$\leq A_{ii} + \omega\left(\bar{r}_i + \sum_{j=1}^n \bar{a}_{ij}\right)\left(1 - e^{M_j}\sum_{j=1}^n |b_{ij}^0|_0 e^{M_j}\right)^{-1} =: N_1.$$

By (13) and (26), we get

$$|\dot{u}_i|_0 \leq |r_i|_0 + \sum_{j=1}^n |a_{ij}|_0 e^{M_j} + \sum_{j=1}^n |b_{ij}|_0 e^{M_j} |\dot{u}_j|_0, \ \ i = 1, \cdots, n.$$

It follows that

$$|\dot{u}_i|_0 \le \frac{|r_i|_0 + \sum_{j=1}^{n} |a_{ij}|_0 e^{M_j}}{1 - \sum_{j=1}^{n} |b_{ij}|_0 e^{M_j}} =: N_2, \quad i = 1, \cdots, n. \tag{26}$$

Let $\Omega = \{u \in C^1_\omega : |u|_0 < N_3, \ |\dot{u}|_0 < N_2\}$ and define a bounded bilinear form $[\ \cdot\ ,\ \cdot\ ]$ on $C^0_\omega \times C^1_\omega$ by

$$\begin{bmatrix} [v_1, u_1] \\ \vdots \\ [v_n, u_n] \end{bmatrix} = \begin{bmatrix} \int_0^\omega v_1(t)u_1(t)dt \\ \vdots \\ \int_0^\omega v_n(t)u_n(t)dt \end{bmatrix}.$$

Also we define $Q : v \to coker(L)$ by $v_i \to \int_0^\omega v_i(t)dt, \ i = 1, \cdots, n$. Obviously

$$\{\ u \mid u\ \in kerL \cap \partial\Omega\ \} = \{\ u \mid u \equiv N_3 \text{ or } u \equiv -N_3\}.$$

Without loss of generality, we assume that $u \equiv N_3$. thus

$$\begin{aligned}
&[QN(u) + Q(r), u][QN(-u) + Q(r), u] \\
&= \begin{bmatrix} [QN(u_1) + Q(r_1), u_1][QN(-u_1) + Q(r_1), u_1] \\ \vdots \\ [QN(u_n) + Q(r_n), u_n][QN(-u_n) + Q(r_n), u_n] \end{bmatrix} \\
&= \begin{bmatrix} N_3^2\omega^2 \left[\int_0^\omega r_1(t)dt - e^{N_3} \int_0^\omega \sum_{j=1}^{n} a_{1j}(t)dt\right] \\ \vdots \\ N_3^2\omega^2 \left[\int_0^\omega r_n(t)dt - e^{N_3} \int_0^\omega \sum_{j=1}^{n} a_{nj}(t)dt\right] \\ \times \left[\int_0^\omega r_1(t)dt - e^{-N_3} \int_0^\omega \sum_{j=1}^{n} a_{1j}(t)dt\right] \\ \vdots \\ \times \left[\int_0^\omega r_n(t)dt - e^{-N_3} \int_0^\omega \sum_{j=1}^{n} a_{nj}(t)dt\right] \end{bmatrix}
\end{aligned} \tag{27}$$

$$= \begin{bmatrix} N_3^2\omega^2 \left[\bar{r}_1 - e^{N_3}\sum_{j=1}^{n}\bar{a}_{1j}\right] \times \left[\bar{r}_1 - e^{-N_3}\sum_{j=1}^{n}\bar{a}_{1j}\right] \\ \vdots \\ N_3^2\omega^2 \left[\bar{r}_n - e^{N_3}\sum_{j=1}^{n}\bar{a}_{nj}\right] \times \left[\bar{r}_n - e^{-N_3}\sum_{j=1}^{n}\bar{a}_{nj}\right] \end{bmatrix}.$$

If

$$N_3 > \left|\log \frac{\bar{r}_i}{\sum_{j=1}^{n}\bar{a}_{ij}}\right|, \; i = 1, \cdots, n;$$

then

$$e^{N_3}\sum_{j=1}^{n}\bar{a}_{ij} > \frac{\bar{r}_i}{\sum_{j=1}^{n}\bar{a}_{ij}}\sum_{j=1}^{n}\bar{a}_{ij} = \bar{r}_i,$$

$$e^{-N_3}\sum_{j=1}^{n}\bar{a}_{ij} < \frac{\bar{r}_i}{\sum_{j=1}^{n}\bar{a}_{ij}}\sum_{j=1}^{n}\bar{a}_{ij} = \bar{r}_i.$$

So from (28) we get

(28) $$[QN(u_i) + Q(r_i), u_i][QN(-u_i) + Q(r_i), u_i] < 0, \; i = 1, \cdots, n.$$

From the condition

$$\max_{1\le i\le n}\left\{\sum_{j=1}^{n}|b_{ij}|_0, \; \sum_{j=1}^{n}|b_{ij}^{0}|_0\right\}e^{M_j} < 1,$$

we have that there is a constant $\bar{M} > M$ such that $\sum_{j=1}^{n}|b_{ij}|_0 e^{\bar{M}} < 1$. Applying Lemmas 1 and 3 with $\Omega = \{u | u \in C_\omega^1, |u|_0 < \gamma_1, |\dot{u}|_0 < \gamma_2\}$, we set $\gamma_1 = \bar{M}, \gamma_2 = M$. Then it follows from (23), (28) and (29) that all the conditions of Lemma 1 are satisfied. Hence system (1) has at least one positive ω-periodic solution.

Theorem 2. *If* $\bar{r}_i > 0$ *and* $\Phi_{ij}(t) \le 0, \forall t \in [0, \omega]$, *then system (1) does not have any positive* $\omega-$*periodic solution.*

Proof. We need only to prove that system (8) does not have any positive $\omega-$periodic solution. If (8) have a $\omega-$periodic solution $u(t)$, then integrating two sides of system (8) on the interval $[0, \omega]$, we get

$$\int_0^\omega r_i(t)dt = \int_0^\omega \left[\sum_{j=1}^n a_{ij}(t) e^{u_j(t-\tau_{ij}(t))} - \sum_{j=1}^n \dot{b}_{ij}^0(t) e^{u_j(t-\sigma_{ij}(t))} \right] dt,$$

i.e.,

$$\bar{r}_i \omega = \int_0^\omega \sum_{j=1}^n \left[\frac{a_{ij}(\nu_{ij}(t))}{1-\dot{\tau}_{ij}(\nu_{ij}(t))} - \frac{\dot{b}_{ij}^0(\mu_{ij}(t))}{1-\dot{\sigma}_{ij}(\mu_{ij}(t))} \right] e^{u_j(t)} dt.$$

So there exists a number $\xi \in [0, \omega]$ such that

$$\bar{r}_i \omega = \sum_{j=1}^n \left[\frac{a_{ij}(\nu_{ij}(\xi))}{1-\dot{\tau}_{ij}(\nu_{ij}(\xi))} - \frac{\dot{b}_{ij}^0(\mu_{ij}(\xi))}{1-\dot{\sigma}_{ij}(\mu_{ij}(\xi))} \right] \int_0^\omega e^{u_j(t)} dt.$$

As $\bar{r}_i > 0$, it follows that $\Phi_{ij}(\xi) > 0$ which contradicts $\Phi_{ij}(\xi) \leq 0, \ \forall t \in [0, \omega]$. this contradiction implies that system (8) does not have any positive $\omega-$periodic solution.Theorem 2 is now proved.

Theorem 3. *If $\bar{r}_i = 0$ and $\Phi_{ij}(t) \geq 0$ or $\Phi_{ij}(t) \leq 0$, and furthermore, $\bar{\Phi}_{ij} \neq 0$, then system (1) does not have any positive $\omega-$periodic solution.*

We omit the proof of Theorem 3 since it is similar to that of Theorem 2.

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