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SHARPNESS AND GENERALIZATION OF JORDAN'S INEQUALITY AND ITS APPLICATION

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Abstract. In this paper we sharpen and generalize the Jordan's inequality, our results unify and optimize some corresponding known results in the recent papers. As application, the obtained results are used to improve the well-known L. Yang's inequality.

1. Introduction

The following inequality is known as the Jordan's inequality [1]:
If $0 < x \leqslant \frac{\pi}{2}$, then

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} < 1. \tag{1}$$

The Jordan's inequality plays an important role in the trigonometry, calculus, approximationtechnique and the theory of limit etc. Owing to various applications, this inequality have been given considerable attention in the literature (see [2–4]).

In 2003, Debnath and Zhao [5] presented the following sharpness of Jordan's inequality

Theorem A. *If* $0 < x \leqslant \frac{\pi}{2}$, *then*

$$\frac{\sin x}{x} \geqslant \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2). \tag{2}$$

Recently, a reverse of the inequality (2) was given by Zhu [6] as follows:

Theorem B. *If* $0 < x \leqslant \frac{\pi}{2}$, *then*

$$\frac{\sin x}{x} \leqslant \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2). \tag{3}$$

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In fact, the inequality (3) is a sharpness of the right hand side of Jordan's inequality, since the inequality (3) is equivalent to the inequality:

$$\frac{\sin x}{x} \leqslant 1 - \frac{4\pi - 8}{\pi^3} x^2. \tag{4}$$

An interesting analogue of the inequalities (2) and (3) was established by Deng [7], i.e.

Theorem C. *If* $0 < x \leqslant \frac{\pi}{2}$, *then*

$$\frac{2}{\pi} + \frac{2}{3\pi^4}(\pi^3 - 8x^3) \leqslant \frac{\sin x}{x} \leqslant \frac{2}{\pi} + \frac{\pi - 2}{\pi^4}(\pi^3 - 8x^3). \tag{5}$$

It is worth noticing that the constant factors $1/\pi^3$, $(\pi - 2)/\pi^3$, $2/3\pi^4$ and $(\pi - 2)/\pi^4$ are best possible in the inequalities (2), (3) and (5) respectively, the detailed accounts can be found in [6] and [7].

In this paper, by introducing two parameters θ and λ $(0 < \theta \leqslant \pi,\ \lambda \geqslant 2)$ we give an unified sharpness and generalization of the above inequalities (see Theorem 1 below), which possesses the best possible coefficients. In section 4, we show that the obtained result can be used for improving the well-known L. Yang's inequality, where an interesting result established by Debnath and Zhao in [5] is generalized and sharpened.

2. Lemmas

In order to prove the main results in Section 3 and Section 4, we need the following Lemmas.

Lemma 1. *If* $0 < x \leqslant \pi$, *then*

$$6\sin x - 2x - 4x\cos x - x^2 \sin x > 0, \tag{6}$$

$$x\cos x < \sin x < \frac{2}{3}x + \frac{1}{3}x\cos x. \tag{7}$$

Proof. Define the following functions on the interval $(0, \pi]$ by

$$\phi(x) = 6\sin x - 2x - 4x\cos x - x^2 \sin x,$$

and

$$\psi(x) = \sin x - \frac{2}{3}x - \frac{1}{3}x\cos x.$$

Differentiating with respect to x gives

$$\phi'(x) = 2\cos x + 2x\sin x - x^2\cos x - 2, \quad \phi''(x) = x^2\sin x.$$

Obviously, $\phi''(x) > 0$ for all $x \in (0,\pi)$, this implies that $\phi'(x)$ is strictly increasing on $(0,\pi)$, and so we have $\phi'(x) > \phi'(0) = 0$ for all $x \in (0,\pi)$. Consequently, we infer that $\phi(x)$ is strictly increasing on $(0,\pi)$, this yields $\phi(x) > \phi(0) = 0$ for all $x \in (0,\pi]$. The inequality (6) is proved.

Note that

$$\psi'(x) = \frac{1}{3}(2\cos x + x\sin x - 2) = -\frac{2}{3}\sin x(\tan\frac{x}{2} - \frac{x}{2}),$$

and using the well-known inequality [2]:

$$\tan x > x > \sin x \ \ (0 < x < \frac{\pi}{2}), \tag{8}$$

we have $\psi'(x) < 0$ for all $x \in (0,\pi)$. It shows that $\psi(x)$ is strictly decreasing on $(0,\pi)$. Hence we get $\psi(x) < \psi(0) = 0$ for all $x \in (0,\pi]$, which is the right-hand inequality of (7).

The left-hand inequality of (7) can be deduced directly from (8) in the case when $0 < x < \frac{\pi}{2}$, in addition, it is evidently valid for $\frac{\pi}{2} \leqslant x \leqslant \pi$, since here, $x\cos x \leqslant 0$ and $\sin x \geqslant 0$.

The proof of Lemma 1 is complete. ■

Lemma 2. *Let* $A \geqslant 0,\ B \geqslant 0$ *and let* $A + B \leqslant \theta,\ \theta \in [0,\pi]$. *Then*

$$\sin^2\theta \leqslant \cos^2 A + \cos^2 B - 2\cos\theta\cos A\cos B \leqslant 4\sin^2\frac{\theta}{2}. \tag{9}$$

Proof. Since

$$\begin{aligned}
&\cos^2\theta + \cos^2 A + \cos^2 B - 2\cos\theta\cos A\cos B \\
&= (\cos\theta - \cos A\cos B)^2 + \cos^2 A + \cos^2 B - \cos^2 A\cos^2 B \\
&= (\cos\theta - \cos A\cos B)^2 - \sin^2 A\sin^2 B + 1 \\
&= (\cos\theta - \cos(A+B))(\cos\theta - \cos(A-B)) + 1 \\
&= 4\sin\tfrac{\theta+A+B}{2}\sin\tfrac{\theta-A-B}{2}\sin\tfrac{\theta+A-B}{2}\sin\tfrac{\theta-A+B}{2} + 1.
\end{aligned} \tag{10}$$

From the hypotheses $A \geqslant 0,\ B \geqslant 0,\ A + B \leqslant \theta,\ \theta \in [0,\pi]$, we infer that

$$\frac{\theta+A+B}{2},\ \frac{\theta-A-B}{2},\ \frac{\theta+A-B}{2},\ \frac{\theta-A+B}{2} \in [0,\pi].$$

Thus

$$\cos^2\theta + \cos^2 A + \cos^2 B - 2\cos\theta\cos A\cos B \geqslant 1,$$

which yields the left-hand inequality of (9).

Note that $f(x) = \sin x$ is a continuous and concave function defined on $[0, \pi]$, using the Jensen's inequality and the Arithmetic-geometric mean inequality [8], we have

$$\begin{aligned}
&\sin\frac{\frac{\theta+A+B}{2}+\frac{\theta-A-B}{2}+\frac{\theta+A-B}{2}+\frac{\theta-A+B}{2}}{4}\\
&\geqslant \frac{1}{4}\left(\sin\frac{\theta+A+B}{2}+\sin\frac{\theta-A-B}{2}+\sin\frac{\theta+A-B}{2}+\sin\frac{\theta-A+B}{2}\right)\\
&\geqslant \left(\sin\frac{\theta+A+B}{2}\sin\frac{\theta-A-B}{2}\sin\frac{\theta+A-B}{2}\sin\frac{\theta-A+B}{2}\right)^{\frac{1}{4}},
\end{aligned}$$

this yields

$$\sin\frac{\theta+A+B}{2}\sin\frac{\theta-A-B}{2}\sin\frac{\theta+A-B}{2}\sin\frac{\theta-A+B}{2} \leqslant \sin^4\frac{\theta}{2}. \tag{11}$$

Combining (10) and (11), we obtain

$$\cos^2 A + \cos^2 B - 2\cos\theta\cos A\cos B \leqslant 1 + 4\sin^4\frac{\theta}{2} - \cos^2\theta = 4\sin^2\frac{\theta}{2}.$$

The Lemma 2 is proved.

3. Sharpness and Generalization of Jordan's Inequality

In this section we establish the following sharp and generalized version of Jordan's inequality.

Theorem 1. *Let* $0 < x \leqslant \theta,\ \theta \in (0, \pi],$ *and let* $\lambda \geqslant 2.$ *Then*

$$\begin{aligned}
&\frac{\sin\theta}{\theta} + \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\left(1 - \frac{x^\lambda}{\theta^\lambda}\right)\\
&+\left(1 - \frac{\sin\theta}{\theta} - \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\right)\left(1 - \frac{x}{\theta}\right)^\lambda\\
&\leqslant \frac{\sin x}{x} \leqslant \frac{\sin\theta}{\theta} + \left(1 - \frac{\sin\theta}{\theta}\right)\left(1 - \frac{x^\lambda}{\theta^\lambda}\right).
\end{aligned} \tag{12}$$

where the coefficients $\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)$, $1-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)$ *and* $1-\frac{\sin\theta}{\theta}$ *are best possible.*

Proof. By Lemma 1 and the hypothesis $\lambda \geqslant 2$, we have

$$(1-x/\theta)^{\lambda} \leqslant (1-x/\theta)^{2},$$

and

$$\begin{aligned}&1-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right) \geqslant 1-\frac{\sin\theta}{\theta}-\frac{1}{2}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\\ &=\frac{3}{2}\left(\frac{2}{3}+\frac{\cos\theta}{3}-\frac{\sin\theta}{\theta}\right) > 0.\end{aligned}$$

It follows that

$$\begin{aligned}&\left(1-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\right)\left(1-\frac{x}{\theta}\right)^{\lambda}\\ &\leqslant \left(1-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\right)\left(1-\frac{x}{\theta}\right)^{2}.\end{aligned}$$

Now in order to prove the left-hand inequality of (12), we need only to prove that

$$\begin{aligned}\frac{\sin x}{x} &\geqslant \frac{\sin\theta}{\theta}+\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\left(1-\frac{x^{\lambda}}{\theta^{\lambda}}\right)\\ &+\left(1-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\right)\left(1-\frac{x}{\theta}\right)^{2},\end{aligned} \tag{13}$$

which is equivalent to

$$\begin{aligned}&\frac{\sin x}{x}-1+\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\frac{x^{\lambda}}{\theta^{\lambda}}\\ &-\left(1-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\right)\left(\frac{x^{2}}{\theta^{2}}-\frac{2x}{\theta}\right) \geqslant 0.\end{aligned} \tag{14}$$

Define a function $f:(0,\theta] \longrightarrow \mathbb{R}$ by

$$\begin{aligned}f(x) &= \frac{\sin x}{x^{2}}-\frac{1}{x}+\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\frac{x^{\lambda-1}}{\theta^{\lambda}}\\ &-\left(1-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\right)\left(\frac{x}{\theta^{2}}-\frac{2}{\theta}\right).\end{aligned}$$

Differentiating with respect to x gives

$$\begin{aligned} f'(x) &= \frac{1}{x^2} + \frac{\cos x}{x^2} - \frac{2\sin x}{x^3} + \frac{\lambda-1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\frac{x^{\lambda-2}}{\theta^\lambda} \\ &\quad -\frac{1}{\theta^2}\left(1 - \frac{\sin\theta}{\theta} - \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\right), \\ f''(x) &= \frac{6\sin x - 2x - 4x\cos x - x^2\sin x}{x^4} \\ &\quad +\frac{(\lambda-1)(\lambda-2)}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\frac{x^{\lambda-3}}{\theta^\lambda}. \end{aligned}$$

We conclude from Lemma 1 and $\lambda \geqslant 2$ that $f''(x) > 0$ for all $x \in (0,\theta]$. It implies that $f'(x)$ is strictly increasing on $(0,\theta]$, consequently $f'(x) < f'(\theta) = 0$ for all $x \in (0,\theta)$. Thus we assert that $f(x)$ is strictly decreasing on $(0,\theta)$, it follows that $f(x) \geqslant f(\theta) = 0$ for all $x \in (0,\theta]$, this yields $xf(x) \geqslant 0$ for all $x \in (0,\theta]$, which leads to the inequality (14). Hence the left hand side of (12) is proved.

The right-hand inequality of (12) is equivalent to

$$\frac{\sin x}{x} - 1 + \left(1 - \frac{\sin\theta}{\theta}\right)\frac{x^\lambda}{\theta^\lambda} \leqslant 0. \tag{15}$$

Consider the function $g : (0,\theta] \longrightarrow \mathbb{R}$,

$$g(x) = \frac{\sin x}{x^2} - \frac{1}{x} + \left(1 - \frac{\sin\theta}{\theta}\right)\frac{x^{\lambda-1}}{\theta^\lambda}. \tag{16}$$

Differentiating with respect to x gives

$$g'(x) = \frac{1}{x^2} + \frac{\cos x}{x^2} - \frac{2\sin x}{x^3} + (\lambda-1)\left(1 - \frac{\sin\theta}{\theta}\right)\frac{x^{\lambda-2}}{\theta^\lambda},$$

$$g''(x) = \frac{6\sin x - 2x - 4x\cos x - x^2\sin x}{x^4} + (\lambda-1)(\lambda-2)\left(1 - \frac{\sin\theta}{\theta}\right)\frac{x^{\lambda-3}}{\theta^\lambda}.$$

By Lemma 1 and $\lambda \geqslant 2$ we conclude $g''(x) \geqslant 0$ for all $x \in (0,\theta]$. Hence $g'(x)$ is increasing on $(0,\theta]$.

In addition, since

$$\lim_{x\to 0} g'(x) = \begin{cases} -\dfrac{1}{6}, & \lambda > 2 \\ -\dfrac{1}{\theta^3}\left(\sin\theta - \theta + \dfrac{1}{6}\theta^3\right), & \lambda = 2 \end{cases},$$

$$g'(\theta) = \frac{3}{\theta^2}\left(\frac{2}{3} + \frac{1}{3}\cos\theta - \frac{\sin\theta}{\theta}\right) + \frac{\lambda - 2}{\theta^2}\left(1 - \frac{\sin\theta}{\theta}\right),$$

it follows from Taylor's formula and Lemma 1 respectively that $\lim_{x\to 0} g'(x) < 0$ and $g'(\theta) > 0$.

Therefore by the continuity of $g'(x)$ we conclude that there exists $\xi \in (0,\theta)$ such that $g'(x) < 0$ for all $x \in (0,\xi)$ and $g'(x) > 0$ for all $x \in (\xi,\theta)$. This implies that $g(x)$ is strictly decreasing on $(0,\xi)$ and increasing strictly on (ξ,θ). We get $g(x) \leqslant 0$ for all $x \in (0,\theta]$, since $g(x) \to 0$ $(x \to 0)$ and $g(\theta) = 0$. Thus we deduce that $xg(x) \leqslant 0$ for all $x \in (0,\theta]$, the right hand side inequality of (12) is proved.

Next, we need to show that the coefficients $\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)$, $1 - \frac{\sin\theta}{\theta} - \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)$ and $1 - \frac{\sin\theta}{\theta}$ are best possible in the inequality (12).

Consider the inequality (12) in a general form as

$$\begin{aligned} &\frac{\sin\theta}{\theta} + k_1\left(1 - \frac{x^\lambda}{\theta^\lambda}\right) + k_2\left(1 - \frac{x}{\theta}\right)^\lambda \\ &\leqslant \frac{\sin x}{x} \leqslant \frac{\sin\theta}{\theta} + \mu_1\left(1 - \frac{x^\lambda}{\theta^\lambda}\right) + \mu_2\left(1 - \frac{x}{\theta}\right)^\lambda. \end{aligned} \tag{17}$$

Firstly, let $x \to 0$ in (17), we find $k_1 + k_2 \leqslant 1 - \frac{\sin\theta}{\theta}$ and $\mu_1 + \mu_2 \geqslant 1 - \frac{\sin\theta}{\theta}$. Now it is easy to see that the best possible coefficients k_1, k_2, μ_1 and μ_2 are given by the following inequality with the maximal k_1 and minimal μ_1

$$\begin{aligned} &\frac{\sin\theta}{\theta} + k_1\left(1 - \frac{x^\lambda}{\theta^\lambda}\right) + \left(1 - \frac{\sin\theta}{\theta} - k_1\right)\left(1 - \frac{x}{\theta}\right)^\lambda \\ &\leqslant \frac{\sin x}{x} \leqslant \frac{\sin\theta}{\theta} + \mu_1\left(1 - \frac{x^\lambda}{\theta^\lambda}\right) + \left(1 - \frac{\sin\theta}{\theta} - \mu_1\right)\left(1 - \frac{x}{\theta}\right)^\lambda. \end{aligned}$$

Based on the above inequality and the property of functional limit, we deduce that

$$\begin{aligned} k_1 &\leqslant \lim_{x\to\theta}\left(\frac{\sin x}{x} - \frac{\sin\theta}{\theta} - \left(1 - \frac{\sin\theta}{\theta} - k_1\right)\left(1 - \frac{x}{\theta}\right)^\lambda\right)\Big/ \\ &\left(1 - \frac{x^\lambda}{\theta^\lambda}\right) = \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right), \end{aligned}$$

and

$$\lim_{x\to 0}\left(\frac{\sin x}{x} - \frac{\sin\theta}{\theta} - \mu_1\left(1 - \frac{x^\lambda}{\theta^\lambda}\right) - \left(1 - \frac{\sin\theta}{\theta} - \mu_1\right)\left(1 - \frac{x}{\theta}\right)^\lambda\right)\Big/ x \leqslant 0,$$

$$\iff \frac{\lambda}{\theta}\left(1 - \frac{\sin\theta}{\theta} - \mu_1\right) \leqslant 0 \iff \mu_1 \geqslant 1 - \frac{\sin\theta}{\theta}.$$

Therefore, the best possible values of k_1, k_2, μ_1 and μ_2 in (17) are

$$k_1 = \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right), \quad k_2 = 1 - \frac{\sin\theta}{\theta} - \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right),$$
$$\mu_1 = 1 - \frac{\sin\theta}{\theta}, \quad \mu_2 = 0.$$

The proof of Theorem 1 is complete.

Putting $\lambda = 2$ and $\lambda = 3$ (together with $\theta = \frac{\pi}{2}$) in (12) respectively, we obtain

Corollary 1. *If* $0 < x \leqslant \frac{\pi}{2}$, *then*

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{\pi - 3}{\pi^3}(\pi - 2x)^2 \leqslant \frac{\sin x}{x} \leqslant \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2), \tag{18}$$

$$\frac{2}{\pi} + \frac{2}{3\pi^4}(\pi^3 - 8x^3) + \frac{3\pi - 8}{3\pi^4}(\pi - 2x)^3 \leqslant \frac{\sin x}{x} \leqslant \frac{2}{\pi} + \frac{\pi - 2}{\pi^4}(\pi^3 - 8x^3). \tag{19}$$

Remark 1. From the inequalities (18) and (19), it is easy to see that the inequality (12) is an unified generalization of the inequality (2), (3) and (5), especially, as a consequence of the inequality (12), the above inequalities have also sharpened the inequality (2) and (5). So the inequality (12) can be considered as a generalization and sharpness of Jordan's inequality.

In inequality (12), putting $\lambda = 2$ and integrating both sides of the inequality gives

Corollary 2. *If* $0 < \theta \leqslant \pi$, *then*

$$\frac{1}{3}\theta + \frac{5}{6}\sin\theta - \frac{1}{6}\theta\cos\theta < \int_0^{\theta} \frac{\sin x}{x}\, dx < \frac{2}{3}\theta + \frac{1}{3}\sin\theta. \tag{20}$$

Specially, taking $\theta = \frac{\pi}{2}$ in (20), an estimate of $\int_0^{\frac{\pi}{2}} (\sin x/x)dx$ is obtained as follows

$$\frac{\pi + 5}{6} < \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\, dx < \frac{\pi + 1}{3}. \tag{21}$$

4. Application to the Improvement of L. Yang's Inequality

It is well-known that the L. Yang's inequality plays an important role in the theory of distribution of values of functions, it can be stated as follows [9]:

Theorem D. *Let* $A_1 > 0,\ A_2 > 0, A_1 + A_2 \leqslant \pi$*, and let* $0 \leqslant \mu \leqslant 1$*. Then*

$$\cos^2 \mu A_1 + \cos^2 \mu A_2 - 2\cos\mu\pi \cos\mu A_1 \cos\mu A_2 \geqslant \sin^2 \mu\pi. \tag{22}$$

In [5], an improvement of L. Yang's inequality was given by Debnath and Zhao, i.e.

Theorem E. *Let* $A_i > 0\ (i = 1, 2, \cdots, n,\ n \geqslant 2)$ *with* $\sum_{i=1}^{n} A_i \leqslant \pi$*, let* $0 \leqslant \mu \leqslant 1$*, and let* n *be a natural number. Then*

$$\begin{aligned} &\binom{n}{2} \mu^2(3-\mu^2)^2 \cos^2 \frac{\mu\pi}{2} \leqslant (n-1)\sum_{k=1}^{n} \cos^2 \mu A_k \\ &-2\cos\mu\pi \sum_{1\leqslant i<j\leqslant n} \cos\mu A_i \cos\mu A_j \leqslant \binom{n}{2} \mu^2\pi^2. \end{aligned} \tag{23}$$

We give here a further sharpness and generalization of the inequality (23) by using the improved Jordan's inequality (12).

Theorem 2. *Let* $A_i \geqslant 0\ (i = 1, 2, \cdots, n,\ n \geqslant 2)$ *with* $\sum_{i=1}^{n} A_i \leqslant \theta$*,* $\theta \in [0, \pi]$*, let* $\lambda \geqslant 2$*, and let* n *be a natural number. Then*

$$\begin{aligned} &\binom{n}{2}\left(\left(\pi - 2 - \frac{2}{\lambda}\right)\left(1-\frac{\theta}{\pi}\right)^{\lambda} - \frac{2}{\lambda}\left(\frac{\theta}{\pi}\right)^{\lambda} + \frac{2}{\lambda} + 2\right)^2 \left(\frac{\theta}{\pi}\cos\frac{\theta}{2}\right)^2 \\ &\leqslant (n-1)\sum_{k=1}^{n} \cos^2 A_k - 2\cos\theta \sum_{1\leqslant i<j\leqslant n} \cos A_i \cos A_j \\ &\leqslant \binom{n}{2}\left(2\left(\frac{\theta}{\pi}\right)^{\lambda+1} - \theta\left(\tfrac{\theta}{\pi}\right)^{\lambda} + \theta\right)^2. \end{aligned} \tag{24}$$

Proof. Let $H_{ij} = \cos^2 A_i + \cos^2 A_j - 2\cos\theta \cos A_i \cos A_j$. It follows from Lemma 2 that

$$\sin^2\theta \leqslant H_{ij} \leqslant 4\sin^2\frac{\theta}{2} \quad (1 \leqslant i < j \leqslant n). \tag{25}$$

Taking the sum for all inequalities in (25), we obtain

$$\sum_{1\leqslant i<j\leqslant n} \sin^2\theta \leqslant \sum_{1\leqslant i<j\leqslant n} H_{ij} \leqslant \sum_{1\leqslant i<j\leqslant n} 4\sin^2\frac{\theta}{2}. \tag{26}$$

From the following obvious identities:

$$\sum_{1\leqslant i<j\leqslant n} \sin^2\theta = 4\binom{n}{2}\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}, \quad \sum_{1\leqslant i<j\leqslant n} 4\sin^2\frac{\theta}{2} = 4\binom{n}{2}\sin^2\frac{\theta}{2}$$

and

$$\sum_{1\leqslant i<j\leqslant n} H_{ij} = (n-1)\sum_{k=1}^{n}\cos^2 A_k - 2\cos\theta \sum_{1\leqslant i<j\leqslant n}\cos A_i\cos A_j,$$

we deduce that

$$\begin{aligned} &4\binom{n}{2}\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} \leqslant (n-1)\sum_{k=1}^{n}\cos^2 A_k \\ &-2\cos\theta\sum_{1\leqslant i<j\leqslant n}\cos A_i\cos A_j \leqslant 4\binom{n}{2}\sin^2\frac{\theta}{2}. \end{aligned} \tag{27}$$

On the other hand, utilizing the Theorem 1 together with $0<\frac{\theta}{2}\leqslant\frac{\pi}{2}$, we obtain

$$\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}+\frac{1}{\lambda}\left(\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}-\cos\frac{\pi}{2}\right)\left(1-\frac{\theta^\lambda}{\pi^\lambda}\right)+\left(1-\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}-\frac{1}{\lambda}\left(\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}-\cos\frac{\pi}{2}\right)\right)\left(1-\frac{\theta}{\pi}\right)^\lambda$$

$$\leqslant\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\leqslant\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}+\left(1-\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}\right)\left(1-\frac{\theta^\lambda}{\pi^\lambda}\right),$$

this yields

$$\begin{aligned} &0<\frac{\theta}{\pi}\left(\left(\frac{\pi}{2}-\frac{1}{\lambda}-1\right)\left(1-\frac{\theta}{\pi}\right)^\lambda-\frac{1}{\lambda}\left(\frac{\theta}{\pi}\right)^\lambda+\frac{1}{\lambda}+1\right) \\ &\leqslant\sin\frac{\theta}{2}\leqslant\left(\frac{\theta}{\pi}\right)^{\lambda+1}-\frac{\theta}{2}\left(\frac{\theta}{\pi}\right)^\lambda+\frac{\theta}{2}. \end{aligned} \tag{28}$$

It is easy to see that the inequality (28) is also valid for $\theta=0$ (It becomes the identity).

Combining the inequalities (27) and (28) yield the inequality (24). This completes the proof of Theorem 2.

Note that when $\sum_{i=1}^{n} A_i\leqslant\pi$ and $0\leqslant\mu\leqslant 1$, it implies $\sum_{i=1}^{n}\mu A_i\leqslant\mu\pi$ and $\mu\pi\in[0,\pi]$. By using the Theorem 2 with the substitution $A_i\to\mu A_i$ $(i=1,2,\cdots,n)$ and $\theta\to\mu\pi$ in (24), we get the following inequality.

Corollary 3. *Let* $A_i>0$ $(i=1,2,\cdots,n,\ n\geqslant 2)$ *with* $\sum_{i=1}^{n} A_i\leqslant\pi$, *let*

$0 \leqslant \mu \leqslant 1$, $\lambda \geqslant 2$, *and let* n *be a natural number. Then*

$$\begin{aligned} &\binom{n}{2}\left(2\lambda - 2\mu^{\lambda} + 2 + (\lambda\pi - 2\lambda - 2)(1-\mu)^{\lambda}\right)^2 \left(\tfrac{\mu}{\lambda}\cos\tfrac{\mu\pi}{2}\right)^2 \\ &\leqslant (n-1)\sum_{k=1}^{n}\cos^2\mu A_k - 2\cos\mu\pi \sum_{1\leqslant i<j\leqslant n}\cos\mu A_i \cos\mu A_j \qquad (29) \\ &\leqslant \binom{n}{2}\left(\mu\pi + \mu^{\lambda+1}(2-\pi)\right)^2. \end{aligned}$$

In particular, putting $\lambda = 2$ in (29), a sharp version of inequality (23) is derived as follows

Corollary 4. *Let* $A_i > 0$ $(i = 1, 2, \cdots, n,\ n \geqslant 2)$ *with* $\sum_{i=1}^{n} A_i \leqslant \pi$, *let* $0 \leqslant \mu \leqslant 1$, *and let* n *be a natural number. Then*

$$\begin{aligned} &\binom{n}{2}\mu^2\left(3 - \mu^2 + (\pi-3)(1-\mu)^2\right)^2\cos^2\tfrac{\mu\pi}{2} \\ &\leqslant (n-1)\sum_{k=1}^{n}\cos^2\mu A_k - 2\cos\mu\pi \sum_{1\leqslant i<j\leqslant n}\cos\mu A_i \cos\mu A_j \qquad (30) \\ &\leqslant \binom{n}{2}\mu^2\left(\pi - (\pi-2)\mu^2\right)^2. \end{aligned}$$

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