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ON THE NORM OF A CERTAIN SELF-ADJOINT INTEGRAL OPERATOR AND APPLICATIONS TO BILINEAR INTEGRAL INEQUALITIES

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Abstract. In this paper, the norm of a bounded self-adjoint integral operator $T : L^2(0, \infty) \to L^2(0, \infty)$ is obtained. As applications, a new bilinear integral inequality with a best constant factor and some particular cases such as Hilbert-type inequalities are considered.

1. Introduction

Let H be a real separable Hilbert space. If $T : H \to H$ is a bounded self-adjoint operator, then

$$|(a, Tb)| \leq ||T||||a||||b|| \; (a, b \in H), \tag{1}$$

where the constant factor $||T||$ is the best possible. If T is also a semi-positive definite operator, then inequality (1) can be improved as :

$$|(a, Tb)| \leq \frac{||T||}{\sqrt{2}}(||a||^2||b||^2 + (a, b)^2)^{\frac{1}{2}} \; (a, b \in H), \tag{2}$$

where (a, b) is the inner product of a and b, and $||a|| = \sqrt{(a, a)}$ is the norm of a (see [10]).

One can conclude that the constant factor $||T||/\sqrt{2}$ in (2) is still the best possible. Otherwise, suppose $||T|| > 0$, there exists a positive number K, with $K < ||T||$,

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such that (2) is still valid if one replaces $||T||$ by K. In particular, for $a = Tb(\neq \theta)$, by Cauchy-Schwarz's inequality (see [3]), one has

$$||Tb||^4 = (Tb, Tb)^2 \leq \frac{K^2}{2}(||Tb||^2||b||^2 + (Tb, b)^2).$$
$$\leq \frac{K^2}{2}(||Tb||^2||b||^2 + ||Tb||^2||b||^2) = (K||Tb||||b||)^2,$$

and then $||Tb|| \leq K||b||$. This contradicts the fact that $||T||$ is the norm of T.

Recently, Yang [9] considered the norm of a bounded self-adjoint operator $T : l^2 \to l^2$ and its applications to the Hilbert-type inequalities. In this paper, the norm of a bounded self-adjoint integral operator $T : L^2(0, \infty) \to L^2(0, \infty)$ is obtained. As applications, a new bilinear integral inequality with a best constant factor is given, and as its particular cases, some new Hilbert-type integral inequalities are established.

We need the formula of the Beta function $B(u, v)$ as (cf. Wang et al. [4]):

$$B(u,v) = \int_0^\infty \frac{t^{u-1}dt}{(1+t)^{u+v}} = \int_0^1 (1-t)^{u-1}t^{v-1}dt = B(v,u) \quad (u, v > 0). \tag{3}$$

2. Main Results

Lemma 1. *Let the function $k(x, y)$ be non-negative measurable and -1-homogeneous in $(0,\infty) \times (0,\infty)$, satisfying $k(x, y) = k(y, x)$, for $x, y \in (0,\infty)$. If $k(u, 1)(u \in (0, 1))$ is a positive continuous function, and there exist constants $0 \leq \alpha < \frac{1}{2}$, $\beta < 1$ and $C_1, C_2 \geq 0$, such that $\lim_{u\to 0^+} u^\alpha k(u, 1) = C_1$ and $\lim_{u\to 1^-}(1-u)^\beta k(u, 1) = C_2$, then for $\varepsilon \in [0, \min\{\frac{1}{2}, 1-2\alpha\})$, the integral $\int_0^\infty k(u, 1)u^{-\frac{1+\varepsilon}{2}}du$ is a constant dependent on ε, and*

$$k(\varepsilon) := \int_0^\infty k(u,1)u^{-\frac{1+\varepsilon}{2}}du = k(0) + o(1) \quad (\varepsilon \to 0^+). \tag{4}$$

Proof. One finds that $\lim_{u\to 0^+} u^\alpha(1-u)^\beta k(u, 1) = C_1$ and $\lim_{u\to 1^-} u^\alpha(1-u)^\beta k(u, 1) = C_2$. Since $k(u, 1)$ is continuous in $(0, 1)$, there exists a constant $L > 0$ such that $u^\alpha(1-u)^\beta k(u, 1) \leq L(u \in [0, 1])$. Setting $u = 1/v$ in the following second integral, since $k(\frac{1}{v}, 1) = vk(v, 1)$, one finds from (3) that

$$\begin{aligned} 0 < k(\varepsilon) &= \int_0^1 k(u,1)u^{-\frac{1+\varepsilon}{2}}du + \int_1^\infty k(u,1)u^{-\frac{1+\varepsilon}{2}}du \\ &= \int_0^1 k(u,1)u^{-\frac{1+\varepsilon}{2}}du + \int_0^1 k(v,1)v^{-\frac{1-\varepsilon}{2}}dv \end{aligned}$$

$$= \int_0^1 [u^\alpha(1-u)^\beta k(u,1)](1-u)^{-\beta}u^{-\alpha}(u^{-\frac{1+\varepsilon}{2}} + u^{-\frac{1-\varepsilon}{2}})du$$
$$\le L\int_0^1 (1-u)^{(1-\beta)-1}[u^{(\frac{1-\varepsilon}{2}-\alpha)-1} + u^{(\frac{1+\varepsilon}{2}-\alpha)-1}]du$$
$$= L[B(1-\beta, \frac{1-\varepsilon}{2}-\alpha) + B(1-\beta, \frac{1+\varepsilon}{2}-\alpha)].$$

Hence the integral $\int_0^\infty k(u,1)u^{-\frac{1+\varepsilon}{2}}du$ is a constant dependent on ε. Since by (3), one obtains

$$|k(\varepsilon)-k(0)| = |\int_0^1 k(u,1)(u^{-\frac{1+\varepsilon}{2}} + u^{-\frac{1-\varepsilon}{2}} - 2u^{-\frac{1}{2}})du|$$
$$\le \int_0^1 u^\alpha(1-u)^\beta k(u,1)(1-u)^{-\beta}|u^{-\frac{1+\varepsilon}{2}-\alpha} + u^{-\frac{1-\varepsilon}{2}-\alpha} - 2u^{-\frac{1}{2}-\alpha}|du$$
$$\le L\int_0^1 (1-u)^{-\beta}|(u^{-\frac{1+\varepsilon}{2}-\alpha} - u^{-\frac{1}{2}-\alpha}) + (u^{-\frac{1-\varepsilon}{2}-\alpha} - u^{-\frac{1}{2}-\alpha})|du$$
$$\le L\int_0^1 (1-u)^{-\beta}(|u^{-\frac{1+\varepsilon}{2}-\alpha} - u^{-\frac{1}{2}-\alpha}| + |u^{-\frac{1}{2}-\alpha} - u^{-\frac{1-\varepsilon}{2}-\alpha}|)du$$
$$= L|\int_0^1 (1-u)^{-\beta}(u^{-\frac{1+\varepsilon}{2}-\alpha} - u^{-\frac{1}{2}-\alpha} + u^{-\frac{1}{2}-\alpha} - u^{-\frac{1-\varepsilon}{2}-\alpha})du|$$
$$= L|\int_0^1 (1-u)^{(1-\beta)-1}[u^{(\frac{1-\varepsilon}{2}-\alpha)-1} - u^{(\frac{1+\varepsilon}{2}-\alpha)-1}]du|$$
$$= L|B(1-\beta, \frac{1-\varepsilon}{2}-\alpha) - B(1-\beta, \frac{1+\varepsilon}{2}-\alpha)|,$$

then $k(\varepsilon) = k(0) + o(1)\ \ (\varepsilon \to 0^+)$. The lemma is proved.

Note 1. In applying Lemma 1, if $k(u,1)$ is continuous in $[0,1)$, then one can set $\alpha = 0$ and only considers $\lim_{u\to 1^-}(1-u)^\beta k(u,1)$; if $k(u,1)$ is continuous in $(0,1]$, then one can set $\beta = 0$ and only considers $\lim_{u\to 0^+} u^\alpha k(u,1)$; if $k(u,1)$ is continuous in $[0,1]$, then one can set $\alpha = \beta = 0$ and does not consider the above two types of limit.

Theorem 1. *Suppose that $k(x,y)$ satisfies the conditions of Lemma 1. If $L^2(0,\infty)$ is a real space and the integral operator $T : L^2(0,\infty) \to L^2(0,\infty)$ is defined by: for all $f \in L^2(0,\infty)$ and $y \in (0,\infty)$,*

$$(Tf)(y) := \int_0^\infty k(x,y)f(x)dx,$$

then, T is a bounded self-adjoint operator and

$$||T|| = k := k(0) = \int_0^\infty k(u,1)u^{-\frac{1}{2}}du = 2\int_0^1 k(u,1)u^{-\frac{1}{2}}du > 0. \tag{5}$$

Proof. Setting $u = x/y$, one finds $\int_0^\infty k(y,x)(\frac{y}{x})^{\frac{1}{2}}dx = \int_0^\infty k(u,1)u^{-\frac{1}{2}}du = k$. By Cauchy's inequality with weight (see[2]), one obtains that: for all $f \in L^2(0,\infty)$,

$$\left(\int_0^\infty k(x,y)f(x)dx\right)^2 = \left\{\int_0^\infty k(x,y)\left[\left(\frac{y}{x}\right)^{\frac{1}{4}}\right]\left[\left(\frac{x}{y}\right)^{\frac{1}{4}}f(x)\right]dx\right\}^2$$
$$\leq \left[\int_0^\infty k(y,x)\left(\frac{y}{x}\right)^{\frac{1}{2}}dx\right]\int_0^\infty k(x,y)\left(\frac{x}{y}\right)^{\frac{1}{2}}f^2(x)dx$$
$$= k\int_0^\infty k(x,y)\left(\frac{x}{y}\right)^{\frac{1}{2}}f^2(x)dx.$$

Since $||f|| = \{\int_0^\infty f^2(x)dx\}^{1/2}$, in view of the above result, one finds that

$$||Tf||^2 = \int_0^\infty\left(\int_0^\infty k(x,y)f(x)dx\right)^2dy$$
$$\leq k\int_0^\infty\int_0^\infty k(x,y)\left(\frac{x}{y}\right)^{\frac{1}{2}}f^2(x)dxdy \tag{6}$$
$$= k\int_0^\infty\left[\int_0^\infty k(x,y)\left(\frac{x}{y}\right)^{\frac{1}{2}}dy\right]f^2(x)dx = k^2||f||^2,$$

and then $||Tf|| \leq k||f||$. It follows that $Tf \in L^2(0,\infty)$ with $||T|| \leq k$.

Since $k > 0$, if $||T|| < k$, then, there exists $0 < k_1 < k$, such that $||Tf|| < k_1||f||($ for $||f|| > 0)$. It follows

$$\int_0^\infty\left(\int_0^\infty k(x,y)f(x)dx\right)^2dy < k_1^2\int_0^\infty f^2(x)dx. \tag{7}$$

Since $\alpha < \frac{1}{2}$, there exists a constant $\gamma > 0$, such that $\alpha + \gamma < \frac{1}{2}$. For $0 < \varepsilon < \min\{\frac{1}{2}, 1 - 2(\alpha+\gamma)\}$, setting f_ε as: $f_\varepsilon(x) = 0, x \in (0,1)$; $f_\varepsilon(x) = x^{-(1+\varepsilon)/2}$, $x \in [1,\infty)$, one obtains

$$I := \int_0^\infty\left(\int_0^\infty k(x,y)f_\varepsilon(x)dx\right)^2dy \geq \int_1^\infty\left(\int_1^\infty k(x,y)x^{-\frac{1+\varepsilon}{2}}dx\right)^2dy$$
$$= \int_1^\infty\frac{1}{y^{1+\varepsilon}}\left(\int_{y^{-1}}^\infty k(u,1)u^{-\frac{1+\varepsilon}{2}}du\right)^2dy$$
$$= \int_1^\infty\frac{1}{y^{1+\varepsilon}}\left(k(\varepsilon) - \int_0^{y^{-1}}k(u,1)u^{-\frac{1+\varepsilon}{2}}du\right)^2dy$$

$$\geq \int_1^\infty \frac{1}{y^{1+\varepsilon}}(k^2(\varepsilon)-2k(\varepsilon)\int_0^{y^{-1}} k(u,1)u^{-\frac{1+\varepsilon}{2}}du)dy$$
$$= \frac{k^2(\varepsilon)}{\varepsilon} - 2k(\varepsilon)\int_1^\infty \frac{1}{y^{1+\varepsilon}}\left[\int_0^{y^{-1}}[u^\alpha(1-u)^\beta k(u,1)]u^\gamma(1-u)^{-\beta}u^{-\frac{1+\varepsilon}{2}-\alpha-\gamma}du\right]dy$$
$$\geq \frac{k^2(\varepsilon)}{\varepsilon} - 2k(\varepsilon)L\int_1^\infty \frac{1}{y}\left[\int_0^{y^{-1}} u^\gamma(1-u)^{-\beta}u^{-\frac{1+\varepsilon}{2}-\alpha-\gamma}du\right]dy$$
$$\geq \frac{k^2(\varepsilon)}{\varepsilon} - 2k(\varepsilon)L\int_1^\infty \frac{1}{y}\left[y^{-\gamma}\int_0^1 (1-u)^{(1-\beta)-1}u^{(\frac{1-\varepsilon}{2}-\alpha-\gamma)-1}du\right]dy$$
$$= \frac{k^2(\varepsilon)}{\varepsilon} - 2k(\varepsilon)\frac{L}{\gamma}B(1-\beta,\frac{1-\varepsilon}{2}-\alpha-\gamma).$$

Hence by (7), one finds

$$k^2(\varepsilon) - 2\varepsilon k(\varepsilon)\frac{L}{\gamma}B\left(1-\beta,\frac{1-\varepsilon}{2}-\alpha-\gamma\right) \leq \varepsilon I < \varepsilon k_1^2\int_0^\infty f_\varepsilon^2(x)dx = k_1^2, \tag{8}$$

and $k = k(0) \leq k_1(\varepsilon \to 0^+)$. This contradiction shows that $||T|| \geq k$, and hence $||T|| = k$.

By Fubini's theorem, one has

$$(Tf,g) = \int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy = (f,Tg).$$

It follows that $T = T^*$, and T is a bounded self-adjoint operator (see [3]).

Note 2. By (6), one has a inequality with the best constant factor $k^2 = ||T||^2$ as follows:

$$\int_0^\infty\left(\int_0^\infty k(x,y)f(x)dx\right)^2 dy \leq k^2||f||^2.$$

By (1) and (5), one has

Theorem 2. *If $L^2(0,\infty)$ is a real space, $f,g \in L^2(0,\infty)$, the operator T and the function $k(x,y)$ are indicated as in Theorem 1, then*

$$\left|\int_0^\infty\int_0^\infty k(x,y)\ f(x)g(y)dxdy\right| = |(Tf,g)| \leq k||f||||g||, \tag{9}$$

where the constant factor $k(=\int_0^\infty k(u,1)u^{-\frac{1}{2}}du = 2\int_0^1 k(u,1)u^{-\frac{1}{2}}du)$ is the best possible.

Note 3. It is obvious that Theorems 1 and Theorem 2 still hold when $L^2(0,\infty)$ is replaced by $L^2(a,b)$ in some certain conditions.

3. Applications to Bilinear Integral Inequalities

(a) Let $k(x,y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}(xy)^{\frac{\lambda-1}{2}}$ $(\lambda > 0)$. Setting $k(1,1) = \frac{1}{\lambda}$, one finds that $k(u,1) = \frac{\ln u}{u^\lambda - 1}u^{\frac{\lambda-1}{2}}$ $(u \in (0,1])$ is continuous, and $\lim_{u\to 0^+} u^\alpha k(u,1) = 0(\alpha > \max\{\frac{1-\lambda}{2}, 0\})$. Since $\int_0^\infty \frac{\ln u}{u-1}u^{-\frac{1}{2}}du = \pi^2$ (cf. [1]), setting $v = u^\lambda$, one obtains from (5) that

$$k = \int_0^\infty \frac{\ln u}{u^\lambda - 1}u^{\frac{\lambda-1}{2}-\frac{1}{2}}du = \frac{1}{\lambda^2}\int_0^\infty \frac{\ln v}{v-1}v^{-\frac{1}{2}}dv = (\frac{\pi}{\lambda})^2.$$

Hence by (9), one has

Corollary 1. *If $L^2(0,\infty)$ is a real space, $f, g \in L^2(0,\infty)$, then for $\lambda > 0$,*

$$\left|\int_0^\infty \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}\ln(\frac{x}{y})}{x^\lambda - y^\lambda} f(x)g(y)dxdy\right| \le (\frac{\pi}{\lambda})^2 ||f||||g||, \tag{10}$$

where the constant factor $(\frac{\pi}{\lambda})^2$ is the best possible.

(b) Let $k(x,y) = \frac{|x-y|^{\lambda-1}}{(\max\{x,y\})^\lambda}$ $(\lambda > 0)$. One obtains that $k(u,1) = \frac{(1-u)^{\lambda-1}}{(\max\{u,1\})^\lambda} = (1-u)^{\lambda-1}(u \in [0,1))$ is continuous, and $\lim_{u\to 1^-}(1-u)^\beta k(u,1) = 1(\beta = 1-\lambda < 1)$. Then one obtains from (5) and (3) that

$$k = 2\int_0^1 (1-u)^{\lambda-1}u^{\frac{1}{2}-1}du = 2B(\lambda, \frac{1}{2}).$$

Hence by (9), one has

Corollary 2. *If $L^2(0,\infty)$ is a real space, $f, g \in L^2(0,\infty)$, then for $\lambda > 0$,*

$$\left|\int_0^\infty \int_0^\infty \frac{|x-y|^{\lambda-1}}{(\max\{x,y\})^\lambda} f(x)g(y)dxdy\right| \le 2B(\lambda, \frac{1}{2})||f||||g||, \tag{11}$$

where the constant factor $2B(\lambda, \frac{1}{2})$ is the best possible.

(c) Let $k(x,y) = \frac{|x^{\lambda-1} - y^{\lambda-1}|}{(\max\{x,y\})^\lambda}$ $(\lambda > \frac{1}{2}, \lambda \ne 1)$. One obtain that $k(u,1) = \frac{|u^{\lambda-1}-1|}{(\max\{u,1\})^\lambda} = |u^{\lambda-1} - 1|$ $(u \in (0,1])$ is continuous, and $\lim_{u\to 0^+} u^\alpha k(u,1) = 0(\alpha > \max\{1-\lambda, 0\})$. By (5), one obtains that

(i) if $\frac{1}{2} < \lambda < 1$, then

$$k = 2\int_0^1 (u^{\lambda-1} - 1)u^{-\frac{1}{2}}du = \frac{8(1-\lambda)}{2\lambda - 1};$$

(ii) if $\lambda > 1$, then

$$k = 2\int_0^1 (1 - u^{\lambda-1})u^{-\frac{1}{2}}du = \frac{8(\lambda-1)}{2\lambda - 1}.$$

By (9), it follows that

Corollary 3. *If $L^2(0,\infty)$ is a real space, $f, g \in L^2(0,\infty)$, then for $\lambda > \frac{1}{2}$ $(\lambda \neq 1)$,*

(12) $$\left|\int_0^\infty \int_0^\infty \frac{|x^{\lambda-1} - y^{\lambda-1}|}{(\max\{x,y\})^\lambda} f(x)g(y)dxdy| \le \frac{8|\lambda-1|}{2\lambda-1}||f||||g|\right|,$$

where the constant factor $\frac{8|\lambda-1|}{2\lambda-1}$ is the best possible. In particular, for $\lambda = 2$, one has

(13) $$\left|\int_0^\infty \int_0^\infty \frac{|x-y|}{(\max\{x,y\})^2} f(x)g(y)dxdy| \le \frac{8}{3}||f||||g|\right|.$$

(d) Let $k(x,y) = \frac{(\min\{(x/y),(y/x)\})^{\lambda/2}}{\max\{x,y\}}$ $(\lambda \ge 0)$. One obtains that $k(u,1) = \frac{(\min\{u,1/u\})^{\lambda/2}}{\max\{u,1\}} = u^{\lambda/2}(u \in (0,1])$ is continuous, and $\lim_{u\to 0^+} u^\alpha k(u,1) = 0(0 < \alpha < \frac{1}{2})$. By (5), one obtains that

$$k = 2\int_0^1 \frac{(\min\{u,1/u\})^{\lambda/2}}{\max\{u,1\}} u^{-\frac{1}{2}}du = 2\int_0^1 u^{\frac{\lambda-1}{2}}du = \frac{4}{1+\lambda}.$$

By (9), it follows that

Corollary 4. *If $L^2(0,\infty)$ is a real space, $f, g \in L^2(0,\infty)$, then for $\lambda \ge 0$,*

(14) $$\left|\int_0^\infty \int_0^\infty \frac{(\min\{(\frac{x}{y}),(\frac{y}{x})\})^{\lambda/2}}{\max\{x,y\}} f(x)g(y)dxdy| \le \frac{4}{1+\lambda}||f||||g|\right|,$$

where the constant factor $4/(1+\lambda)$ is the best possible.

(e) Let $k(x,y) = \frac{|x-y|^{\lambda-1}}{(\min\{x,y\})^{\lambda}}$ $(0 < \lambda < \frac{1}{2})$. One obtains that $k(u,1) = \frac{|u-1|^{\lambda-1}}{(\min\{u,1\})^{\lambda}} = (1-u)^{\lambda-1}u^{-\lambda}$ $(u \in (0,1))$ is continuous, and $\lim_{u\to 0^+} u^{\alpha}k(u,1) = 1(\alpha = \lambda)$; $\lim_{u\to 1^-}(1-u)^{\beta}k(u,1) = 1(\beta = 1-\lambda)$. By (5), one obtains that

$$k = 2\int_0^1 (1-u)^{\lambda-1}u^{(\frac{1}{2}-\lambda)-1}du = 2B(\lambda, \frac{1}{2}-\lambda).$$

By (9), it follows that

Corollary 5. *If $L^2(0,\infty)$ is a real space, $f, g \in L^2(0,\infty)$, then for $0 < \lambda < \frac{1}{2}$,*

$$\left|\int_0^{\infty}\int_0^{\infty} \frac{|x-y|^{\lambda-1}}{(\min\{x,y\})^{\lambda}} f(x)g(y)dxdy\right| \leq 2B(\lambda, \frac{1}{2}-\lambda)||f||||g||, \tag{15}$$

where the constant factor $2B(\lambda, \frac{1}{2}-\lambda)$ is the best possible.

(f) Let $k(x,y) = \frac{(xy)^{(\lambda-1)/2}}{|x-y|^{\lambda}}(0 < \lambda < 1)$. One obtains that $k(u,1) = \frac{u^{(\lambda-1)/2}}{(1-u)^{\lambda}}(u \in (0,1))$ is continuous and $\lim_{u\to 0^+} u^{\alpha}k(u,1) = 1(\alpha = (1-\lambda)/2)$; $\lim_{u\to 1^-}(1-u)^{\beta}k(u,1) = 1(\beta = \lambda)$. By (5), one obtains that

$$k = 2\int_0^1 (1-u)^{(1-\lambda)-1}u^{\frac{\lambda}{2}-1}du = 2B\left(1-\lambda, \frac{\lambda}{2}\right).$$

By (9), it follows that

Corollary 6. *If $L^2(0,\infty)$ is a real space, $f, g \in L^2(0,\infty)$, then for $0 < \lambda < 1$,*

$$\left|\int_0^{\infty}\int_0^{\infty} \frac{(xy)^{(\lambda-1)/2}}{|x-y|^{\lambda}} f(x)g(y)dxdy\right| \leq 2B(1-\lambda, \frac{\lambda}{2})||f||||g||, \tag{16}$$

where the constant factor $2B(1-\lambda, \frac{\lambda}{2})$ is the best possible (cf. [7]).

(g) Let $k(x,y) = \frac{|\ln(x/y)|(xy)^{(\lambda-1)/2}}{(\max\{x,y\})^{\lambda}}$ $(\lambda > 0)$. One obtains that $k(u,1) = \frac{|\ln u|u^{(\lambda-1)/2}}{(\max\{u,1\})^{\lambda}} = (-\ln u)u^{(\lambda-1)/2}$ $(u \in (0,1])$ is continuous, and $\lim_{u\to 0^+} u^{\alpha}k(u,1) = 0(\max\{\frac{1-\lambda}{2}, 0\} < \alpha < \frac{1}{2})$. By (5), one obtains that

$$k = 2\int_0^1 (-\ln u)u^{(\lambda-1)/2}u^{-\frac{1}{2}}du = \frac{4}{\lambda}\int_0^1 (-\ln u)du^{\frac{\lambda}{2}} = \frac{8}{\lambda^2}.$$

By (9), it follows that

Corollary 7. *If $L^2(0,\infty)$ is a real space, $f, g \in L^2(0,\infty)$, then for $\lambda > 0$,*

$$\left|\int_0^\infty \int_0^\infty \frac{|\ln(x/y)|(xy)^{(\lambda-1)/2}}{(\max\{x,y\})^\lambda} f(x)g(y)dxdy\right| \leq \frac{8}{\lambda^2}||f||||g||, \tag{17}$$

where the constant factor $\frac{8}{\lambda^2}$ is the best possible.

Remarks.

(i) For $\lambda = 2$, inequality (11) also reduces to (13). Hence inequalities (11) and (12) are extensions of (13).

(ii) For $\lambda = 1$ in (11) and $\lambda = 0$ in (14) , both of them reduce to the following base Hilbert-type inequality (see [1]):

$$\left|\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy\right| \leq 4\left\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right\}^{\frac{1}{2}}. \tag{18}$$

Hence inequalities (11) and (14) are extensions of (18). Another extension of (18) was given in [5].

(iii) For $\lambda = 1$ in (10), one has the following base Hilbert- type inequality (see [1]):

$$\left|\int_0^\infty \int_0^\infty \frac{\ln(\frac{x}{y})}{x-y} f(x)g(y)dxdy\right| \leq \pi^2 \left\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right\}^{\frac{1}{2}}. \tag{19}$$

Hence inequality (10) is an extension of (19). One also has another extension of (19) (see [6]).

(iv) F $\lambda = 1$ in (17), one has the following new base Hilbert- type inequality (see [8]):

$$\left|\int_0^\infty \int_0^\infty \frac{|\ln(\frac{x}{y})||f(x)g(y)}{\max\{x,y\}} dxdy\right| \leq 8\left\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right\}^{\frac{1}{2}}. \tag{20}$$

(v) Inequality (9) is a new bilinear integral inequality with a best constant factor . By using (9), one can establish many new Hilbert's type integral inequalities with the best constant factors such as (10-12, 14-16) and (17).

Open Problem. Is the operator T defined by Theorem 1 semi-positive definite and is it suitable to use (2)?

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