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ON MARRERO'S J_m-HADAMARD MATRICES

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Abstract. In this paper, generalizing Marrero's construction, we introduce the concept of J_m-Hadamard matrices, and by allowing permutations, we construct other $2^m m! - 1$ J_m-Hadamard matrices from a given one of order mt; previous construction generated only other $2^m - 1$ ones. We also generalize Craigen's construction of products of two Hadamard matrices to those of several Hadamard matrices and a J_m-Hadamard matrix, yielding generalizations of Craigen's results. Furthermore, we introduce the J_m-class CJ_m for $m = 2$ or $4k$ and study the partially ordered set $\mathfrak{M}$ of J_m-classes CJ_m. Our main result shows that $CJ_8 \subsetneqq CJ_4 \subsetneqq CJ_2$.

1. INTRODUCTION

The remained unsolved Hadamard Conjecture asserts the existence of Hadamard matrices for all orders that are divisible by four. A step toward solving Hadamard conjecture is to construct other Hadamard matrices from a given one. Here we study two such constructions: Marrero's construction and Craigen's construction. In a previous paper [5], we generalized Marrero's construction of J_2-Hadamard matrices to J_m-Hadamard matrices, $m = 2$ or $m = 4k$, $k \in \mathbb{N}$. A Marrero's J_2-Hadamard matrix (see [2]) is a normalized Hadamard matrix of order $2t$ of the form

$$\begin{pmatrix} J & J & A \\ J & -J & B \end{pmatrix},$$

where $J \in \mathbb{M}_{t\times 1}(\{1\})$ and $A, B \in \mathbb{M}_{t\times(2t-2)}(\{\pm 1\})$. By changing A into $-A$ or B into $-B$, he yielded other $2^2 - 1$ Hadamard matrices from the given one. A J_m-Hadamard matrix is an Hadamard matrix of order mt of the form

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$$\left(\begin{array}{c|c} & A_1 \\ & A_2 \\ M \otimes J & \vdots \\ & A_m \end{array} \right),$$

where M is an Hadamard matrix of order m, $J \in \mathbb{M}_{t\times 1}(\{1\})$,$A_1, A_2, \cdots A_m \in \mathbb{M}_{t\times(mt-m)}(\{\pm 1\})$, and $\otimes$ is the Kronecker product (see [5], Definition 2.1). By changing A_i to $\pm A_i$, we constructed other $2^m - 1$ Hadamard matrices ([5], Theorem 2.2).

In this paper, our main results will be stated in Sections 2, 3, 4, and 5. In Section 2, by revisiting and simplifying the proof of our previous result in [5], it turns out that we can yield other $2^m m! - 1$ Hadamard matrices by allowing permutations on $\{1, 2, ..., m\}$ $\sigma \in S_m$ (Theorem 2.1 and Remark below). In fact, if we transform A_i mentioned above into $\pm A_{\sigma(i)}$ for $i = 1, 2, ..., m$, where σ is a permutation of the set $\{1, 2, ..., m\}$, then the new matrices are still J_m-Hadamard matrices (Theorem 2.1). Thus we can construct other $2^m m! - 1$ Hadamard matrices from a given J_m-Hadamard matrix.

In Section 3, we study the Kronecker product of several Hadamard matrices and a J_m-Hadamard matrix. For a given Hadamard matrix of order $4k$ and a $J_{4h}-$Hadamard matrix, the Kronecker product enables us to yield a $J_{16kh}-$Hadamard matrix (Proposition 3.3). Continuing this process, one easily gets a $J_{2^{2n+2}k_1k_2\cdots k_nh}-$ Hadamard matrix from given n Hadamard matrices of orders $4k_1, 4k_2, ..., 4k_n$, respectively, and a $J_{4h}-$Hadamard matrix. On the other hand, there is another technique due to Craigen to construct a $J_{2^lh}-$Hadamard matrix with smaller 2-exponent l from the given Hadamard matrices: In Proposition 3.4, we use Craigen's construction (see [1], Theorem 1) to generate a $J_{8kh}-$Hadamard matrix from a given Hadamard matrix of order $4k$ and a J_{4h}-Hadamard matrix. Moreover, our main result in this Section is Theorem 3.5 which generalizes Craigen's result (see [1], Theorem 3) to yield a $J_{2^{3l-3}mnp_1p_2\cdots p_{l-2}}$-Hadamard matrix of order $2^{3l-3}mntp_1p_2\cdots p_{l-2}$ from a given Hadamard matrix of order $2^l m$, a J_{2^ln}-Hadamard matrix of order $2^l nt$, and $l-2$ different pairs of $DW(4p_i)$ for $i = 1, 2, ..., l-2$. In particular, Craigen's Theorem 3 [1] is a Corollary of our Theorem 3.5 for $l = 2$ and $t = 1$, and Proposition 3.4 is a Corollary of our Theorem 3.5 for $l = 2$ and arbitrary t. Moreover, the product construction in Theorem 3.5 can be applied to a general case of several Hadamard matrices and a J_m-Hadamard matrix (Remark at the end of Section 3).

In Section 4, we introduce the concept of J_m-classes, $m = 2$ or $m = 4k, k \in \mathbb{N}$, denoted by CJ_m which contains the equivalent class of J_m-Hadamard matrices. By Marrero's approach, each Hadamard matrix belongs to CJ_2. For a given Hadamard matrix, it seems difficult to determine to which CJ_m it belongs. Nevertheless, we can decide to which CJ_m it doesn't belong. In Proposition 4.1 and Proposition 4.2 we show that an Hadamard matrix of order $12h$ or $20h$ doesn't belong to CJ_{4h}.

Our main result in Section 4 is Theorem 4.3 which asserts that $CJ_8 \subsetneqq CJ_4 \subsetneqq CJ_2$.

In the last Section 5, we study the poset $\mathfrak{M}$ of J_m-classes CJ_m : Firstly, $\mathfrak{M}$ is not totally ordered (or a chain). We end this note by leaving the question open whether for a given k, $CJ_{2^k} \subseteq CJ_{2^h}$ for some $1 \neq h < k$.

2. Further Results on J_m-Hadamard Matrices

In our previous paper [5], Theorem 2.2, for a given J_m-Hadamard matrix H as in Introduction, we show that all the matrix of the form $\hat{H} = \left(\begin{array}{c|c} & \pm A_1 \\ & \pm A_2 \\ M \otimes J & \vdots \\ & \pm A_m \end{array}\right)$ are J_m-Hadamard matrices, generalizing Marrero's result ([2], Proposition). In the following, we will prove a stronger result where permutations are allowed:

Theorem 2.1. *Let H be a J_m-Hadamard matrix of the form as above. Then*

$$\hat{H} = \left(\begin{array}{c|c} & B_1 \\ & B_2 \\ M \otimes J & \vdots \\ & B_m \end{array}\right)$$

is also a J_m-Hadamard matrix, where $B_i = A_{\sigma(i)}$ or $B_i = -A_{\sigma(i)}$ for $i = 1, 2, \ldots, m$ and $\sigma \in S_m$.

Proof. We have:

$$H = \left(\begin{array}{cc} M_1 & A_{11} \\ M_1 & A_{12} \\ \vdots & \vdots \\ M_1 & A_{1t} \\ \hline M_2 & A_{21} \\ M_2 & A_{22} \\ \vdots & \vdots \\ M_2 & A_{2t} \\ \hline \vdots & \vdots \\ \hline M_m & A_{m1} \\ M_m & A_{m2} \\ \vdots & \vdots \\ M_m & A_{mt} \end{array}\right), \text{ and } \hat{H} = \left(\begin{array}{cc} M_1 & B_{11} \\ M_1 & B_{12} \\ \vdots & \vdots \\ M_1 & B_{1t} \\ \hline M_2 & B_{21} \\ M_2 & B_{22} \\ \vdots & \vdots \\ M_2 & B_{2t} \\ \hline \vdots & \vdots \\ \hline M_m & B_{m1} \\ M_m & B_{m2} \\ \vdots & \vdots \\ M_m & B_{mt} \end{array}\right),$$

if we write $M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_m \end{pmatrix}$, $A_i = \begin{pmatrix} A_{i1} \\ A_{i2} \\ \vdots \\ A_{it} \end{pmatrix}$ and $B_i = \begin{pmatrix} B_{i1} \\ B_{i2} \\ \vdots \\ B_{it} \end{pmatrix}$, where M_i, A_{ik} and B_{ik} are the row vectors of M, A_i and B_i, respectively, for $i = 1, 2, ..., m$ and $k = 1, 2, ..., t$.

Since H is an Hadamard matrix, then for $i, j = 1, 2, ..., m$ and $k, l = 1, 2, ..., t$, we have

$$M_i M_j^T + A_{ik} A_{jl}^T = \begin{cases} mt, & \text{if } i = j \text{ and } k = l, \\ 0, & \text{otherwise.} \end{cases}$$

This implies

$$A_{ik} A_{jl}^T = \begin{cases} mt - m, & \text{if } i = j \text{ and } k = l, \\ -m, & \text{if } i = j \text{ and } k \neq l, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.1}$$

It suffices to prove that $M_i M_j^T + B_{ik} B_{jl}^T = \begin{cases} mt, & \text{if } i = j \text{ and } k = l, \\ 0, & \text{otherwise.} \end{cases}$

Case 1. For $i = j$ and $k = l$, i.e. $\sigma(i) = \sigma(j)$, $M_i M_j^T + B_{ik} B_{jl}^T = M_i M_i^T + B_{ik} B_{ik}^T = m + B_{ik} B_{ik}^T = M_{\sigma(i)} M_{\sigma(i)}^T + A_{\sigma(i)k} A_{\sigma(i)k}^T = mt$, by (2.1).

Case 2. For $i = j$ and $k \neq l$, $M_i M_j^T + B_{ik} B_{jl}^T = M_i M_i^T + B_{ik} B_{il}^T = M_i M_i^T + A_{\sigma(i)k} A_{\sigma(i)l}^T = m + (-m) = 0$, by (2.1).

Case 3. For $i \neq j$, i.e. $\sigma(i) \neq \sigma(j)$, then $M_i M_j^T + B_{ik} B_{jl}^T = M_i M_j^T \pm A_{\sigma(i)k} A_{\sigma(j)l}^T = 0 + 0 = 0$, by (2.1). This completes the proof. ■

Remark. It seems that one gets more Hadamard matrices from the J_m-Hadamard matrix above by also permuting rows inside each B_i, $i = 1, 2, ..., m$. However, by these permutations, one actually gets equivalent ones. Furthermore, it fails to produce Hadamard matrices if one permutes rows from different B_is.

By Theorem 2.1, we may produce $2^m m! - 1$ other Hadamard matrices from a given J_m-Hadamard matrix. In passing, we note the following further characterization of Hadamard matrices which will be useful in our discussion later on J_m-classes (our last Section 5).

Corollary 2.2. *Let H be a J_m-Hadamard matrix of the form as above. If M*

is a J_l-Hadamard matrix of the form

$$\left(\begin{array}{c|c} & C_1 \\ & C_2 \\ L \otimes J' & \vdots \\ & C_l \end{array}\right),$$

then

$$\hat{H} = \left(\left(\begin{array}{c|c} & \pm C_{\delta(1)} \\ & \pm C_{\delta(2)} \\ L \otimes J' & \vdots \\ & \pm C_{\delta(l)} \end{array}\right) \otimes J \left|\begin{array}{c} \pm A_{\sigma(1)} \\ \pm A_{\sigma(2)} \\ \vdots \\ \pm A_{\sigma(m)} \end{array}\right.\right)$$

is also a J_l-Hadamard matrix, where $\sigma \in S_m$ and $\delta \in S_l$. In particular, H itself is a J_l-Hadamard matrix.

Proof. Let $\hat{M} = \left(\begin{array}{c|c} & \pm C_{\delta(1)} \\ & \pm C_{\delta(2)} \\ L \otimes J' & \vdots \\ & \pm C_{\delta(l)} \end{array}\right)$. By Theorem 2.1, $\hat{M}$ is a J_l-Hadamard matrix of order m and trivially $\hat{H}$ is an Hadamard matrix. It remains to prove that $\hat{H}$ is evidently a J_l-Hadamard matrix.

To this end, just put $L \otimes (J' \otimes J) = L \otimes J''$, where $J' \in \mathbb{M}_{t' \times 1}(\{1\}), J \in \mathbb{M}_{t \times 1}(\{1\})$ and $J'' \in \mathbb{M}_{tt' \times 1}(\{1\})$, here $t' = \frac{m}{l}$, then clearly,

$$\hat{H} = \left(\left(\begin{array}{c|c} & \pm C_{\delta(1)} \otimes J \\ & \pm C_{\delta(2)} \otimes J \\ L \otimes J'' & \vdots \\ & \pm C_{\delta(l)} \otimes J \end{array}\right) \left|\begin{array}{c} \pm A_{\sigma(1)} \\ \pm A_{\sigma(2)} \\ \vdots \\ \pm A_{\sigma(m)} \end{array}\right.\right)$$

is a J_l-Hadamard matrix and the proof follows. ■

3. A Generalization of Craigen's Result to J_m-Hadamard Matrices

In this section, our main purpose is to generalize a Craigen's result (Theorem 3.1 below) to J_m-Hadamard matrix (Theorem 3.5). To this end, we first review some pertinent definitions and results.

A pair (S, P), where $S, P \in \mathbb{M}_{4h \times 4h}(\{\pm 1\})$, is an orthogonal pair, notation: (S, P) is an $OP(4h)$, if it satisfies

$$SS^T + PP^T = 8hI_{4h} \text{ and } SP^T = PS^T = O_{4h}.$$

In [1], using certain product construction, Craigen proved the following result (see Theorem 3, [1]) on how to construct an Hadamard matrix from given two Hadamard matrices.

Theorem 3.1. [(Craigen)] *If there are Hadamard matrices of orders* $4h$ *and* $4k$, *then there is an* $OP(4hk)$ (S, P). *Moreover,* $\begin{pmatrix} S & P \\ P & S \end{pmatrix}$ *is an Hadamard matrix of order* $8hk$.

A matrix $M \in \mathbb{M}_{4m\times 4m}(\{0, \pm 1\})$ is a weighing matrix of order $4m$ with weight $2m$ if $MM^T = 2mI_{4m}$. Two weighing matrices of order $4m$, namely $W = (w_{ij})$ and $U = (u_{ij})$, are disjoint if $w_{ij}u_{ij} = 0$. For convenience, we say that (W, U) is a pair of $DW(4m)$ if W and U are two disjoint weighing matrices of order $4m$ with weight $2m$. Seberry and Zhang [4] proved the following result.

Lemma 3.2. [(Seberry and Zhang)] *If there are two Hadamard matrices of orders* $4m$ *and* $4n$, *there exists a pair of* $DW(4mn)$.

We start with the Kronecker product of an Hadamard matrix K of order $4k$, and a $J_{4h}-$Hadamard matrix $H = (M \otimes J\,|A)$ of order $4ht$. In our previous paper [5], Theorem 2.5, using combinatorial arguments, we showed that $K \otimes H$ is equivalent to a $J_{16kh}-$Hadamard matrix $(K \otimes M \otimes J\,|K \otimes A)$. In the following, using only matrix multiplications and Kronecker product (see e.g. Craigen's paper [1], p. 57), we reprove the result as follows.

Proposition 3.3. *Let* K *be an Hadamard matrix of order* $m = 2$ *or* $4k$. *If* $H = (M \otimes J\,|A)$ *is a* J_n-*Hadamard matrix of order* nt, $n = 2$ *or* $4h$, *then* $K \otimes H \sim (K \otimes M \otimes J\,|K \otimes A)$ *and* $(K \otimes M \otimes J\,|K \otimes A)$ *is a* $J_{mn}-$*Hadamard matrix of order* mnt.

Proof. Let $\tilde{H} = (K \otimes M \otimes J\,|K \otimes A)$. Then

$$\begin{aligned}\tilde{H}\tilde{H}^T &= KK^T \otimes MM^T \otimes JJ^T + KK^T \otimes AA^T \\ &= KK^T \otimes (MM^T \otimes JJ^T + KK^T \otimes AA^T) \\ &= 4kI_{4k} \otimes 4htI_{4ht} = 16khtI_{16kht}.\end{aligned}$$

Since $K \otimes M$ is an Hadamard matrix of order $16kh$, hence $\tilde{H}$ is a J_{16kh}-Hadamard matrix. ■

With the supposedly existing Hadamard matrices K and H as in Proposition 3.3, using successively Sylvester's constructions, we yield a $J_{2^{l+4}kh}-$Hadamard matrix for $l \geq 0$. Now, using Craigen's technique, we shall obtain a $J_{2^{l+4}kh}-$Hadamard matrix with $l = -1$. In fact, we have the following result which is a generalization of Craigen's Theorem 1 in [1].

Proposition 3.4. *If there exist an Hadamard matrix K of order $4k$ and a $J_{4h}-$Hadamard matrix H of order $4ht$, then there is a $J_{8kh}-$Hadamard matrix of order $8kht$.*

Proof. Write $K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ and $H = \left(\begin{pmatrix} H_1 & H_2 \end{pmatrix} \otimes J \,\middle|\, A_1 \quad A_2 \right)$, where $K_i \in \mathbb{M}_{2k\times 4k}(\{\pm 1\})$, $H_i \in \mathbb{M}_{4h\times 2h}(\{\pm 1\})$, $A_i \in \mathbb{M}_{4ht\times(2ht-2h)}(\{\pm 1\})$ for $i = 1, 2$, and $J \in \mathbb{M}_{t\times 1}(\{1\})$. Since K and H both are Hadamard matrices, we have

$$K_1K_1^T = K_2K_2^T = 4kI_{2k},\ K_1K_2^T = K_2K_1^T = O_{2k},$$

$$(H_1H_1^T + H_2H_2^T) \otimes JJ^T + A_1A_1^T + A_2A_2^T = 4htI_{4ht}.$$

As in Craigen's constructions, put

$$S = \frac{1}{2}(K_1 + K_2) \otimes H_1 + \frac{1}{2}(K_1 - K_2) \otimes H_2,$$

$$P = \frac{1}{2}(K_1 + K_2) \otimes A_1 + \frac{1}{2}(K_1 - K_2) \otimes A_2.$$

Let $\hat{H} = \left(S \ \otimes J \,\middle|\, P \right)$. Since by direct calculations,

$$\hat{H}\hat{H}^T = SS^T \otimes JJ^T + PP^T = 8khtI_{8kht},$$

we conclude that $\hat{H}$ is a $J_{8kh}-$Hadamard matrix. ■

Now we are in a position to prove our main result and apply it to the Kronecker product of several Hadamard matrices and a J_m-Hadamard matrix.

Theorem 3.5. *If there are an Hadamard matrix H of order $2^l m$, a $J_{2^l n}-$Hadamard matrix K of order $2^l nt$, and $l-2$ different pairs of $DW(4p_i)$ for $i = 1, 2, ..., l-2$, then there is a $J_{2^{3l-3}mnp_1p_2\cdots p_{l-2}}-$Hadamard matrix of order $2^{3l-3}mntp_1p_2...p_{l-2}$.*

Proof. Let $H = \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_{2^l} \end{pmatrix}$ and $K = \left(\begin{pmatrix} K_1\ K_2 \cdots K_{2^l} \end{pmatrix} \otimes J \,\middle|\, A_1\ A_2 \cdots A_{2^l} \right)$, where $H_i \in \mathbb{M}_{m\times 2^l m}(\{\pm 1\})$, $K_i \in \mathbb{M}_{2^l n\times n}(\{\pm 1\})$, $J \in \mathbb{M}_{t\times 1}(\{1\})$, and $A_i \in \mathbb{M}_{2^l nt\times n(t-1)}(\{\pm 1\})$ for $i = 1, 2, \ldots, 2^l$. Then

$$H_i H_j^T = \begin{cases} 2^l m I_m & \text{, if } i = j, \\ O_m & \text{, otherwise,} \end{cases} \tag{3.1}$$

$$\sum_{i=1}^{2^l} K_i K_i^T = 2^l n I_{2^l n}. \tag{3.2}$$

and

$$KK^T = \sum_{i=1}^{2^l} K_i K_i^T \otimes JJ^T + \sum_{i=1}^{2^l} A_i A_i^T = 2^l nt I_{2^l nt} \tag{3.3}$$

Let (X_i, Y_i) be $l-2$ different pairs of $DW(4p_i)$ for $i = 1, 2, ..., l-2$, and set $\mathbb{F} = \{Z_1 \otimes Z_2 \otimes \cdots \otimes Z_{l-2} \mid Z_i = X_i \text{ or } Y_i \text{ for } i = 1, 2, ..., l-2\}$. Clearly, the cardinal number of $\mathbb{F}$ is 2^{l-2}, and we have for $F_i \in \mathbb{F}$, $i = 1, 2, ..., 2^{l-2}$:

$$\begin{aligned} F_i F_i^T &= (Z_1 \otimes Z_2 \otimes \cdots \otimes Z_{l-2})(Z_1 \otimes Z_2 \otimes \cdots \otimes Z_{l-2})^T \\ &= Z_1 Z_1^T \otimes Z_2 Z_2^T \otimes \cdots \otimes Z_{l-2} Z_{l-2}^T \\ &= 2p_1 I_{4p_1} \otimes 2p_2 I_{4p_2} \otimes \cdots \otimes 2p_{l-2} I_{4p_{l-2}}, \\ &= 2^{l-2} p_1 p_2 \cdots p_{l-2} I_{4^{l-2} p_1 p_2 \cdots p_{l-2}}. \end{aligned}$$

We define

$$\begin{aligned} 2S &= \sum_{i=1}^{2^{l-2}} F_i \otimes \{(H_{2i-1} + H_{2i}) \otimes K_{2i-1} + (H_{2i-1} - H_{2i}) \otimes K_{2i}\}, \\ 2P &= \sum_{i=1}^{2^{l-2}} F_i \otimes \{(H_{2^{l-1}+2i-1} + H_{2^{l-1}+2i}) \otimes K_{2^{l-1}+2i-1} \\ &\quad + (H_{2^{l-2}+2i-1} - H_{2^{l-1}+2i}) \otimes K_{2^{l-1}+2i}\}, \end{aligned}$$

$$\begin{aligned} 2V &= \sum_{i=1}^{2^{l-2}} F_i \otimes \{(H_{2i-1} + H_{2i}) \otimes A_{2i-1} + (H_{2i-1} - H_{2i}) \otimes A_{2i}\}, \\ 2U &= \sum_{i=1}^{2^{l-2}} F_i \otimes \{(H_{2^{l-1}+2i-1} + H_{2^{l-1}+2i}) \otimes A_{2^{l-1}+2i-1} \\ &\quad + (H_{2^{l-2}+2i-1} - H_{2^{l-1}+2i}) \otimes A_{2^{l-1}+2i}\}, \end{aligned}$$

and

$$W = \left(\begin{pmatrix} S & P \\ P & S \end{pmatrix} \otimes J \,\middle|\, \begin{pmatrix} V & U \\ U & V \end{pmatrix} \right).$$

It suffices to show that $\begin{pmatrix} S & P \\ P & S \end{pmatrix}$ is an Hadamard matrix of order $2^{3l-3}mnp_1 p_2 \cdots p_{l-2}$ and W is an Hadamard matrix of order $2^{3l-3}mntp_1p_2\cdots p_{l-2}$. The following algebraic calculation shows that (S, P) is an $OP(2^{3l-4}mnp_1p_2\cdots p_{l-2})$. In fact, first we calculate SS^T using Equation (3.1):

$$SS^T = \tfrac{1}{4}\textstyle\sum_{i=1}^{2^{l-2}} F_iF_i^T \otimes \{(H_{2i-1}H_{2i-1}^T + H_{2i}H_{2i}^T) \otimes K_{2i-1}K_{2i-1}^T$$
$$+(H_{2i-1}H_{2i-1}^T + H_{2i}H_{2i}^T) \otimes K_{2i}K_{2i}^T\},$$

by Equation (3.1),

$$= \frac{1}{4}\sum_{i=1}^{2^{l-2}} 2^{l-2}p_1p_2\cdots p_{l-2}I_{4^{l-2}p_1p_2\cdots p_{l-2}} \otimes 2\cdot 2^l mI_m \otimes \{K_{2i-1}K_{2i-1}^T + K_{2i}K_{2i}^T\}$$
$$= \sum_{i=1}^{2^{l-2}} 2^{2l-3}mp_1p_2\cdots p_{l-2}I_{2^{2l-4}mp_1p_2\cdots p_{l-2}} \otimes \{K_{2i-1}K_{2i-1}^T + K_{2i}K_{2i}^T\}.$$

Analogously,

$$PP^T = \sum_{i=1}^{2^{l-2}} 2^{2l-3}mp_1p_2\cdots p_{l-2}I_{2^{2l-4}mp_1p_2\cdots p_{l-2}}$$
$$\otimes\{K_{2^{l-1}+2i-1}K_{2^{l-1}+2i-1}^T + K_{2^{l-1}+2i}K_{2^{l-1}+2i}^T\}.$$

Using Equation (3.2), we get:

$$SS^T + PP^T = \sum_{i=1}^{2^l} 2^{2l-3}mp_1p_2\cdots p_{l-2}I_{2^{2l-4}mp_1p_2\cdots p_{l-2}} \otimes K_iK_i^T$$
$$= 2^{3l-3}mnp_1p_2\cdots p_{l-2}I_{2^{3l-4}mnp_1p_2\cdots p_{l-2}}.$$

A direct calculation, using Equation (3.1), proves that $SP^T = PS^T = O_{2^{3l-4}mnp_1p_2\cdots p_{l-2}}$. This shows (S, P) is an $OP(2^{3l-4}mnp_1p_2...p_{l-2})$ and this orthogonal pair (S, P) produces an Hadamard matrix $\begin{pmatrix} S & P \\ P & S \end{pmatrix}$ of order $2^{3l-3}mnp_1p_2\cdots p_{l-2}$, by Craigen's Theorem 3.1.

Similarly,

$$VV^T + UU^T = \sum_{i=1}^{2^l} 2^{2l-3}mp_1p_2\cdots p_{l-2}I_{2^{2l-4}mp_1p_2\cdots p_{l-2}} \otimes A_iA_i^T, \text{ and}$$

$VU^T = UV^T = O_{2^{3l-4}mntp_1p_2\cdots p_{l-2}}$.
Hence we obtain, by Equation (3.3),

$$(SS^T + PP^T) \otimes JJ^T + (VV^T + UU^T) = 2^{3l-3}mntp_1p_2\cdots p_{l-2}I_{2^{3l-4}mntp_1p_2\cdots p_{l-2}}. \tag{3.4}$$

Clearly,

$$(SP^T + PS^T) \otimes JJ^T + (VU^T + UV^T) = O_{2^{3l-4}mntp_1p_2\cdots p_{l-2}}. \tag{3.5}$$

Finally, by definition of W and by Equations (3.4), (3.5), we show easily that

$$WW^T = 2^{3l-3}mntp_1p_2\cdots p_{l-2}I_{2^{3l-3}mntp_1p_2\cdots p_{l-2}}.$$

This completes the proof of the Theorem. ■

Theorem 3.1 is a Corollary of Theorem 3.5 whenever we choose $l = 2$ and $t = 1$. On the other hand, if we take $l = 2$ and arbitrary t, then we obtain Proposition 3.4. To illustrate how Theorem 3.5 yields better result, we choose a special example of four Hadamard matrices of orders $4m_1, 4m_2, 4m_3, 4m_4$, and a $J_{2^3m_5}-$Hadamard matrix: Repeated Craigen's construction yields a $J_{2^7m_1m_2m_3m_4m_5}-$Hadamard matrix. However, by Theorem 3.5, we can improve the 2-exponent 7 to 6.

Remark. The product construction in Theorem 3.5 can be applied to several Hadamard matrices and a J_m-Hadamard matrix as follows: Firstly, use Seberry-Zhang's construction to generate $l-2$ different pairs of $DW(4p_i), i = 1, 2, ..., l-2$, secondly, proceed with Craigen's construction to generate an Hadamard matrix of order $2^l m'$, and next proceed with the product construction in Proposition 3.4 to generate a $J_{2^l n}-$Hadamard matrix, then, using Theorem 3.5, we get a $J_{2^{3l-3}m'np_1p_2\cdots p_{l-2}}-$Hadamard matrix.

4. Hadamard Matrices Belonging to J_m-Classes CJ_m

In this section, we are interested in the problem to which J_m-Hadamard matrix does a given Hadamard matrix belong? For convenience, we define such family as follows: The family of all Hadamard matrices equivalent to some J_m-Hadamard matrix is called a J_m-class and denoted by CJ_m.

By Marrero's construction, each Hadamard matrix belongs to CJ_2. For a given Hadamard matrix, it seems difficult to determine to which CJ_m it belongs. Nevertheless, for some particular Hadamard matrices, we can decide to which CJ_m each of them doesn't belong. The following two results supply us criteria for this purpose which are generalizations of Example 3.1 and Example 3.2 in [5], respectively.

Proposition 4.1. *If H is an Hadamard matrix of order $12h$, then H doesn't belong to CJ_{4h}.*

Proof. If H were equivalent to a J_{4h}-Hadamard matrix, then $H \sim \left(\begin{array}{c|c} M \otimes J & \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_{4h} \end{matrix} \end{array}\right)$, where M is an Hadamard matrix of order $4h$, $J \in \mathbb{M}_{3\times 1}(\{1\})$ and $A_i \in \mathbb{M}_{3\times 8h}(\{\pm 1\})$ for $i = 1, 2, ..., 4h$. By multiplying rows or columns of $\left(\begin{array}{c|c} M \otimes J & \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_{4h} \end{matrix} \end{array}\right)$, M can be normalized. Hence H must be equivalent to the J_{4h}-Hadamard matrix of the form:

$$H \sim \tilde{H} = \begin{pmatrix} \overbrace{J \quad J \quad \cdots \quad J}^{4h} & A_1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots & \vdots \end{pmatrix} = \left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \end{matrix}}^{4h} & \begin{matrix} A_1 \\ \\ \vdots \end{matrix} \end{array}\right).$$

By eventually multiplying columns of $\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{4h} \end{pmatrix}$ by -1, $\tilde{H}$ can be normalized. However, $\tilde{H}$ is not an Hadamard matrix, since there are at least $4h$ 1s at the same positions between the second row and the third row contradicting to the fact that there are exactly $\frac{12h}{4}$ 1s at the same positions in both rows except the first one (see [3], Theorem 10.9, p. 429). Thus $\tilde{H}$ is not a J_{4h}-Hadamard matrix. ■

Proposition 4.2. *If H is an Hadamard matrix of order $20h$, then H doesn't belong to CJ_{4h}.*

Proof. Suppose that H is a normalized J_{20h}-Hadamard matrix of the form as in Proposition 4.1 with $J \in \mathbb{M}_{5\times 1}(\{1\})$ and $A_i \in \mathbb{M}_{5\times 16h}(\{\pm 1\})$ for $i = 1, 2, ..., 4h$. We will use the same argument as above to derive a contradiction by counting the number of 1s in the second, the third, the fourth and the fifth row. As before, we know that there are exactly $10h$ 1s at each row and $\frac{20h}{4}$ 1s at the same positions between any two different rows except the first one. By arranging the 1s as forward as possible, so H, with the first five rows written down, is of the following form:

$$H = \begin{pmatrix} \overbrace{J \;\; J \;\; \cdots \;\; J}^{4h} & A_1 \\ \vdots \;\; \vdots \;\; \vdots \;\; \vdots & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \overbrace{1\ 1\ \cdots\ 1}^{4h} & \overbrace{1\ 1\ \cdots\ 1}^{h} & \overbrace{1\ \ 1\ \ \cdots\ \ 1}^{5h} & \overbrace{1\ \ 1\ \ \cdots\ \ 1}^{10h} \\ 1\ 1\ \cdots\ 1 & 1\ 1\ \cdots\ 1 & 1\ \ 1\ \ \cdots\ \ 1 & -1\ -1\ \cdots\ -1 \\ 1\ 1\ \cdots\ 1 & 1\ 1\ \cdots\ 1 & -1\ -1\ \cdots\ -1 & \\ 1\ 1\ \cdots\ 1 & & & \\ 1\ 1\ \cdots\ 1 & & & \end{pmatrix}.$$

Looking at the $(10h+1)^{th}$ column up to the $(20h)^{th}$ column , to fill in the $10h$ 1s in the third row, we need $5h$ positions in last $10h$ columns. With the same argument, to fill in the $10h$ 1s in the fourth row, we need at least $4h$ positions in the last $10h$ columns differ from the positions already taken in the third row. Finally, in the fifth row, we need at least $3h$ positions in the last $10h$ columns differ from the positions already taken in the third and the fourth rows. This means that we need in total at least $5h + 4h + 3h = 12h$ positions to fill in the 1s in the last ten columns which is impossible. Therefore, we conclude that every Hadamard matrix of order $20h$ is not equivalent to a J_{4h}-Hadamard matrix. ■

A natural question is whether $CJ_{2^{k+1}} \subseteq CJ_{2^k}$. Our initial contribution to this question, using Proposition 3.3 and Proposition 4.1, is to show the following result; this works in the special case of Hadamard matrices of order 8 which is known to be unique up to equivalence.

Theorem 4.3. $CJ_8 \subsetneqq CJ_4 \subsetneqq CJ_2$.

Proof. By Marrero's construction and Example 3.1 in [5], we obtain $CJ_4 \subsetneqq CJ_2$. It remains to show that $CJ_8 \subsetneqq CJ_4$.

By the uniqueness of Hadamard matrices, every J_8-Hadamard matrix of order 8t is equivalent to the following normalized Hadamard matrix (see e.g. [6])

$$\left(\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \otimes J \;\middle|\; \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \end{matrix} \right)$$

$$= \left(\left(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| \begin{matrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{matrix} \right) \otimes J \middle| \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \end{matrix} \right)$$

$$= \left(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} J \\ J \end{pmatrix} \middle| \begin{matrix} J & J & J & J & A_1 \\ -J & -J & -J & -J & A_2 \\ J & J & -J & -J & A_3 \\ -J & -J & J & J & A_4 \\ J & -J & J & -J & A_5 \\ -J & J & -J & J & A_6 \\ J & -J & -J & J & A_7 \\ -J & J & J & -J & A_8 \end{matrix} \right),$$

where $J \in \mathbb{M}_{t\times 1}(\{1\})$ and $A_i \in \mathbb{M}_{t\times(8t-8)}(\{\pm 1\})$ for $i = 1, 2, .., 8$. This yields $CJ_8 \subseteq CJ_4$. Next, let H be an Hadamard matrix of order 12 of the form $\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes J \,|A \right)$, where $J \in \mathbb{M}_{6\times 1}(\{1\})$ and $A \in \mathbb{M}_{12\times 10}(\{\pm 1\})$.
Set $\hat{H} = \left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes J \middle| \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes A \right)$.
By Proposition 3.3, $\hat{H} \in CJ_4$. Since $\hat{H}$ is an Hadamard matrix of order 24, by Proposition 4.1, $\hat{H}$ doesn't belong to CJ_8, and this gives $CJ_8 \subsetneqq CJ_4$. ■

5. The Partially Ordered Set $\mathfrak{M}$ of J_m−Classes CJ_m

In this section, we consider the poset $\mathfrak{M} = \{CJ_m | m = 2 \text{ or } m \in 4k,\ k \in N\}$ which is easily seen not to be a chain. To see this, we show the following non-inclusions:

$$CJ_8 \nsubseteq CJ_{12} \nsubseteq CJ_8.$$

In fact, let H be an Hadamard matrix of order 12. Then the Sylvester-Hadamard matrix $\begin{pmatrix} H & H \\ H & -H \end{pmatrix} \sim \left(H \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid H \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \in CJ_{12}$ which has order 24, hence by Proposition 4.2 it doesn't belong to CJ_8. This shows the non-inclusion $CJ_{12} \nsubseteq CJ_8$. To prove the other non-inclusion, we construct a Sylvester-Hadamard matrix of order 16 which belongs to CJ_8 and doesn't belong to CJ_{12} because 16 is not a multiple of 12.

As a consequence of our Corollary 2.2, a J_m-Hadamard matrix H is a J_l-Hadamard matrix for some $l \mid m$, where l depends on m and H. In particular, if m is a 2 power, then l is also. So we are led to consider the smaller poset $\mathfrak{M}_2 = \{CJ_{2^k} | k \in N\}$ and pose the following question: For a given k, does there exist a $1 \neq h < k$ such that $CJ_{2^k} \subseteq CJ_{2^h}$? Another related question is whether $\mathfrak{M}_2$ is a lattice: Given $h, k \in \mathbb{N}$, whether CJ_{2^h} and CJ_{2^k} has a least upper bound $CJ_{2^h} \vee CJ_{2^k}$ and a greatest lower bound $CJ_{2^h} \wedge CJ_{2^k}$?

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