TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 2, pp. 293-300, April 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

ON A CLASS OF n-STARLIKE FUNCTIONS

Mugur Acu

Abstract. In this paper we define a general class of n-starlike functions with respect to a convex domain D contained in the right half plane by using a generalized Sălăgean operator introduced by F. M. Al-Oboudi in [2] and we give some properties of this class.

1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f \text{ is univalent in } U\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

Let D^n be the Sălăgean differential operator (see [9]) defined as:

$$D^n : A \to A\ , \quad n \in \mathbb{N} \text{ and } D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z)\ , \quad D^n f(z) = D(D^{n-1} f(z)).$$

Remark 1.1. If $f \in S, f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$, $j = 2, 3, ...,$ $z \in U$ then $D^n f(z) = z + \sum\limits_{j=2}^{\infty} j^n a_j z^j$.

The aim of this paper is to define a general class of n-starlike functions with respect to a convex domain D contained in the right half plane by using a generalized Sălăgean operator introduced by F. M. Al-Oboudi in [2] and to obtain some properties of this class.

Received August 29, 2005, accepted June 19, 2006.
Communicated by Der-Chen Chang.
2000 *Mathematics Subject Classification*: 30C45.
Key words and phrases: Starlike functions, Libera-Pascu integral operator, Briot-Bouquet differential subordination, Generalized Sălăgean operator.

2. Preliminary Results

We recall here the definition of the well-known class of starlike functions

$$S^* = \left\{ f \in A : Re \frac{zf'(z)}{f(z)} > 0 \ , \ z \in U \right\} .$$

Remark 2.1. By using the subordination relation, we may define the class S^* thus if $f(z) = z + a_2 z^2 + \ldots, \ z \in U$, then $f \in S^*$ if and only if $\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}$, $z \in U$, where by "$\prec$" we denote the subordination relation.

Let consider the Libera-Pascu integral operator $L_a : A \to A$ defined as:

$$(1) \qquad f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt \ , \quad a \in \mathbb{C} \ , \quad Re \ a \geq 0.$$

In the case $a = 1, 2, 3, \ldots$ this operator was introduced by S. D. Bernardi and it was studied by many authors in different general cases.

Definition 2.1. ([2]) Let $n \in \mathbb{N}$ and $\lambda \geq 0$. We denote with D^n_λ the operator defined by

$$D^n_\lambda : A \to A \, ,$$

$$D^0_\lambda f(z) = f(z) \ , \ D^1_\lambda f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z),$$

$$D^n_\lambda f(z) = D_\lambda \left(D^{n-1}_\lambda f(z) \right) \, .$$

Remark 2.2. ([2]) We observe that D^n_λ is a linear operator and for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we have

$$D^n_\lambda f(z) = z + \sum_{j=2}^{\infty} \left(1 + (j-1)\lambda\right)^n a_j z^j \, .$$

Also, it is easy to observe that if we consider $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator.

The next theorem is result of the so called "admissible functions method" introduced by P. T. Mocanu and S. S. Miller (see [6-8]).

Theorem 2.1. *Let h convex in U and $Re[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H(U)$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \textit{then} \ \ p(z) \prec h(z).$$

2. Main Results

Definition 3.1. Let $q(z) \in \mathcal{H}_u(U)$, with $q(0) = 1$ and $q(U) = D$, where D is a convex domain contained in the right half plane, $n \in \mathbb{N}$ and $\lambda \geq 0$. We say that a function $f(z) \in A$ is in the class $SL_n^*(q)$ if $\dfrac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \prec q(z)\,,\, z \in U.$

Remark 3.1. Geometric interpretation: $f(z) \in SL_n^*(q)$ if and only if $\dfrac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)}$ take all values in the convex domain D contained in the right half-plane.

Remark 3.2. It is easy to observe that if we choose different function $q(z)$ we obtain variously classes of starlike functions, such as (for example), for $\lambda = 1$ and $n = 0$, the class of starlike functions, the class of starlike functions of order γ (see [4]), the class of starlike functions with respect to a hyperbola (see [10]), and, for $\lambda = 1$, the class of n-starlike functions (see [9]), the class of n-starlike functions with respect to a hyperbola (see [1]), the class of n-uniformly starlike functions of order γ and type α(see [5]).

Remark 3.3. For $q_1(z) \prec q_2(z)$ we have $SL_n^*(q_1) \subset SL_n^*(q_2)$. From the above we obtain $SL_n^*(q) \subset SL_n^*\left(\dfrac{1+z}{1-z}\right)$.

Theorem 3.1. *Let $n \in \mathbb{N}$ and $\lambda \geq 0$. We have*

$$SL_{n+1}^*(q) \subset SL_n^*(q)\,.$$

Proof. Let $f(z) \in SL_{n+1}^*(q)$.
With notation

$$p(z) = \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)}\,,\, p(0) = 1\,,$$

we obtain

$$\frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} = \frac{D_\lambda^{n+2} f(z)}{D_\lambda^n f(z)} \cdot \frac{D_\lambda^n f(z)}{D_\lambda^{n+1} f(z)} = \frac{1}{p(z)} \cdot \frac{D_\lambda^{n+2} f(z)}{D_\lambda^n f(z)} \tag{2}$$

Also, we have

$$\frac{D_\lambda^{n+2} f(z)}{D_\lambda^n f(z)} = \frac{z + \sum\limits_{j=2}^{\infty} (1 + (j-1)\lambda)^{n+2}\, a_j z^j}{z + \sum\limits_{j=2}^{\infty} (1 + (j-1)\lambda)^{n}\, a_j z^j}$$

and

$$zp'(z) = \frac{z\left(D_\lambda^{n+1}f(z)\right)'}{D_\lambda^n f(z)} - \frac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)} \cdot \frac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)}$$

$$= \frac{z\left(1+\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{n+1} j a_j z^{j-1}\right)}{D_\lambda^n f(z)}$$

$$-p(z) \cdot \frac{z\left(1+\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{n} j a_j z^{j-1}\right)}{D_\lambda^n f(z)}$$

or

$$zp'(z) = \frac{z+\sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{n+1} a_j z^j}{D_\lambda^n f(z)} \tag{3}$$

$$-p(z) \cdot \frac{z+\sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{n} a_j z^j}{D_\lambda^n f(z)}.$$

We have

$$\begin{aligned}
& z+\sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{n+1} a_j z^j \\
= & \; z+\sum_{j=2}^{\infty} ((j-1)+1)\,(1+(j-1)\lambda)^{n+1} a_j z^j \\
= & \; z+\sum_{j=2}^{\infty} (1+(j-1)\lambda)^{n+1} a_j z^j + \sum_{j=2}^{\infty}(j-1)\,(1+(j-1)\lambda)^{n+1} a_j z^j \\
= & \; z+D_\lambda^{n+1}f(z) - z + \sum_{j=2}^{\infty}(j-1)\,(1+(j-1)\lambda)^{n+1} a_j z^j \\
= & \; D_\lambda^{n+1}f(z) + \frac{1}{\lambda}\sum_{j=2}^{\infty} ((j-1)\lambda)\,(1+(j-1)\lambda)^{n+1} a_j z^j \\
= & \; D_\lambda^{n+1}f(z) + \frac{1}{\lambda}\sum_{j=2}^{\infty} (1+(j-1)\lambda-1)\,(1+(j-1)\lambda)^{n+1} a_j z^j \\
= & \; D_\lambda^{n+1}f(z) - \frac{1}{\lambda}\sum_{j=2}^{\infty} (1+(j-1)\lambda)^{n+1} a_j z^j + \frac{1}{\lambda}\sum_{j=2}^{\infty} (1+(j-1)\lambda)^{n+2} a_j z^j
\end{aligned}$$

$$= D_\lambda^{n+1}f(z) - \frac{1}{\lambda}\left(D_\lambda^{n+1}f(z) - z\right) + \frac{1}{\lambda}\left(D_\lambda^{n+2}f(z) - z\right)$$
$$= D_\lambda^{n+1}f(z) - \frac{1}{\lambda}D_\lambda^{n+1}f(z) + \frac{z}{\lambda} + \frac{1}{\lambda}D_\lambda^{n+2}f(z) - \frac{z}{\lambda}$$
$$= \frac{\lambda-1}{\lambda}D_\lambda^{n+1}f(z) + \frac{1}{\lambda}D_\lambda^{n+2}f(z)$$
$$= \frac{1}{\lambda}\left((\lambda-1)D_\lambda^{n+1}f(z) + D_\lambda^{n+2}f(z)\right) .$$

Similarly we have

$$z + \sum_{j=2}^{\infty} j\left(1+(j-1)\lambda\right)^n a_j z^j = \frac{1}{\lambda}\left((\lambda-1)D_\lambda^{n}f(z) + D_\lambda^{n+1}f(z)\right) .$$

From (3) we obtain

$$zp'(z) = \frac{1}{\lambda}\left(\frac{(\lambda-1)D_\lambda^{n+1}f(z) + D_\lambda^{n+2}f(z)}{D_\lambda^n f(z)} - p(z)\frac{(\lambda-1)D_\lambda^{n}f(z) + D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)}\right)$$
$$= \frac{1}{\lambda}\left((\lambda-1)p(z) + \frac{D_\lambda^{n+2}f(z)}{D_\lambda^n f(z)} - p(z)\left((\lambda-1)+p(z)\right)\right)$$
$$= \frac{1}{\lambda}\left(\frac{D_\lambda^{n+2}f(z)}{D_\lambda^n f(z)} - p(z)^2\right)$$

Thus

$$\lambda zp'(z) = \frac{D_\lambda^{n+2}f(z)}{D_\lambda^n f(z)} - p(z)^2$$

or

$$\frac{D_\lambda^{n+2}f(z)}{D_\lambda^n f(z)} = p(z)^2 + \lambda zp'(z) .$$

From (2) we obtain

$$\frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n+1} f(z)} = \frac{1}{p(z)}\left(p(z)^2 + \lambda zp'(z)\right) = p(z) + \lambda\frac{zp'(z)}{p(z)} ,$$

where $\lambda \geq 0$.

From $f(z) \in SL^*_{n+1}(q)$ we have

$$p(z) + \lambda\frac{zp'(z)}{p(z)} \prec q(z) ,$$

with $p(0) = q(0) = 1$ and $\lambda \geq 0$. In this conditions from Theorem 2.1, we obtain

$$p(z) \prec q(z)$$

or

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \prec q(z)\,.$$

This means $f(z) \in SL_n^*(q)$.

Corollary 3.1. *For every $n \in \mathbb{N}^*$ we have $SL_n^*(q) \subset SL_0^*(q) \subset S^*$.*

Theorem 3.2. *Let $n \in \mathbb{N}$ and $\lambda \geq 1$. If $F(z) \in SL_n^*(q)$ then $f(z) = L_a F(z) \in SL_n^*(q)$, where L_a is the Libera-Pascu integral operator defined by* (1).

Proof. From (1) we have

$$(1+a)F(z) = af(z) + zf'(z)$$

and, by using the linear operator D_λ^{n+1}, we obtain

$$\begin{aligned}(1+a)D_\lambda^{n+1}F(z) &= aD_\lambda^{n+1}f(z) + D_\lambda^{n+1}\left(z + \sum_{j=2}^{\infty} ja_j z^j\right)\\ &= aD_\lambda^{n+1}f(z) + z + \sum_{j=2}^{\infty}\left(1+(j-1)\lambda\right)^{n+1} ja_j z^j\end{aligned}$$

We have (see the proof of the above theorem)

$$z + \sum_{j=2}^{\infty} j\left(1+(j-1)\lambda\right)^{n+1} a_j z^j = \frac{1}{\lambda}\left((\lambda-1)D_\lambda^{n+1}f(z) + D_\lambda^{n+2}f(z)\right)$$

Thus

$$\begin{aligned}(1+a)D_\lambda^{n+1}F(z) &= aD_\lambda^{n+1}f(z) + \frac{1}{\lambda}\left((\lambda-1)D_\lambda^{n+1}f(z) + D_\lambda^{n+2}f(z)\right)\\ &= \left(a + \frac{\lambda-1}{\lambda}\right)D_\lambda^{n+1}f(z) + \frac{1}{\lambda}D_\lambda^{n+2}f(z)\end{aligned}$$

or

$$\lambda(1+a)D_\lambda^{n+1}F(z) = \left((a+1)\lambda - 1\right)D_\lambda^{n+1}f(z) + D_\lambda^{n+2}f(z)\,.$$

Similarly, we obtain

$$\lambda(1+a)D_\lambda^{n}F(z) = \left((a+1)\lambda - 1\right)D_\lambda^{n}f(z) + D_\lambda^{n+1}f(z)\,.$$

Then

$$\frac{D_\lambda^{n+1}F(z)}{D_\lambda^n F(z)} = \frac{\dfrac{D_\lambda^{n+2}f(z)}{D_\lambda^{n+1}f(z)}\cdot\dfrac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)} + \left((a+1)\lambda-1\right)\cdot\dfrac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)}}{\dfrac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)} + \left((a+1)\lambda-1\right)}\,.$$

With notation

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} = p(z)\,,\; p(0) = 1\,,$$

we obtain

$$\text{(4)} \qquad \frac{D_\lambda^{n+1} F(z)}{D_\lambda^n F(z)} = \frac{\dfrac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} \cdot p(z) + ((a+1)\lambda - 1) \cdot p(z)}{p(z) + ((a+1)\lambda - 1)}$$

We have (see the proof of the above theorem)

$$\lambda z p'(z) = \frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} \cdot \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - p(z)^2 = \\ = \frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} \cdot p(z) - p(z)^2\,.$$

Thus

$$\frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} = \frac{1}{p(z)} \cdot \left(p(z)^2 + \lambda z p'(z)\right).$$

Then, from (4), we obtain

$$\frac{D_\lambda^{n+1} F(z)}{D_\lambda^n F(z)} = \frac{p(z)^2 + \lambda z p'(z) + ((a+1)\lambda - 1)\, p(z)}{p(z) + ((a+1)\lambda - 1)} \\ = p(z) + \lambda \frac{z p'(z)}{p(z) + ((a+1)\lambda - 1)}\,,$$

where $a \in \mathbb{C}$, $Re\, a \geq 0$ and $\lambda \geq 1$. From $F(z) \in SL_n^*(q)$ we have

$$p(z) + \frac{z p'(z)}{\dfrac{1}{\lambda}\left(p(z) + ((a+1)\lambda - 1)\right)} \prec q(z)\,,$$

where $a \in \mathbb{C}$, $Re\, a \geq 0$, $\lambda \geq 1$, and from her construction, we have $Re\, q(z) > 0$. In this conditions from Theorem 2.1, we obtain

$$p(z) \prec q(z)$$

or

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \prec q(z)\,.$$

This means $f(z) = L_a F(z) \in SL_n^*(q)$. For $\lambda = 1$ we obtain

Corollary 3.2. *If $F(z) \in S_n^*(q)$ then $f(z) = L_a F(z) \in S_n^*(q)$, where L_a is the Libera-Pascu integral operator and by $S_n^*(q)$ we denote the class of n-starlike functions subordinate to the function $q(z)$ (see* [3]*).*

ACKNOWLEDGMENT

REFERENCES

1. M. Acu, On a subclass of n-starlike functions associated with some hyperbola, *General Mathematics*, **13(1)**, (2005), 91-98.
2. F. M. Al-Oboudi, On univalent funtions defined by a generalized Sălăgean operator, *Ind. J. Math. Math. Sci.*, 2004, no. 25-28, 1429-1436.
3. D. Blezu, On the close to convex functions with respect to a convex set II, *Mathematica*, **Tome 3(1)**, (1989), 15-23.
4. P. Duren, *Univalent functions*, Springer Verlag, Berlin Heildelberg, 1984.
5. I. Magdaş, *A new subclass of uniformly convex functions with negative coefficients*, Doctoral Thesis, "Babeş-Bolyai" University, Cluj-Napoca, 1999.
6. S. S. Miller and P. T. Mocanu, Differential subordination and univalent functions, *Mich. Math.*, **28** (1981), 157-171.
7. S. S. Miller and P. T. Mocanu, Univalent solution of Briot-Bouquet differential equation, *J. Differential Equations*, **56** (1985), 297-308.
8. S. S. Miller and P. T. Mocanu, On some classes of first-order differential subordination, *Mich. Math.*, **32** (1985), 185-195.
9. Gr. Sălăgean, *On some classes of univalent functions*, Seminar of geometric function theory, Cluj-Napoca, 1983.
10. J. Stankiewicz, A. Wisniowska, Starlike functions associated with some hyperbola, Folia Scientarum Universitatis Tehnicae Resoviensis 147, *Matematyka*, **19** (1996), 117-126.

Mugur Acu
University "Lucian Blaga" of Sibiu
Department of Mathematics,
Str. Dr. I. Raţiu, No. 5-7,
550012-Sibiu,
Romania
E-mail: acu_mugur@yahoo.com