TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 2, pp. 269-279, April 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

# A MULTIDIMENSIONAL GENERALIZATION OF HARDY-HILBERT'S INTEGRAL INEQUALITY

Hong Yong

**Abstract.** In this paper, by introducing norm $\|x\|_\alpha (x \in R^n)$, we give a multidimensional Hardy-Hilbert's integral inequality with two parameters $\alpha$, $\lambda$ and best constant factor.

## 1. Introduction and Main Results

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \geq 0$, $g \geq 0$, $0 < \int_0^\infty f^p(x)dx < +\infty$, $0 < \int_0^\infty g^q(x)dx < +\infty$, then the well known Hardy-Hilbert's inequality be given by (see [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(x)dx \right)^{\frac{1}{q}}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible. it's equivalent form is:

$$\int_0^\infty \left( \int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy < \left[ \frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \int_0^\infty f^p(x)dx, \tag{2}$$

where the constant factors of $[\frac{\pi}{\sin(\frac{\pi}{p})}]^p$ is also the best possible.

In recent years, many results (see [2-5])have been obtained in research of Hardy-Hilbert't inequality.By introducing a parameter $\lambda$, Yang [6] give generalizations of (1) and (2) as:

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy &< \frac{\pi}{\lambda \sin(\frac{\pi}{p})} \left( \int_0^\infty x^{(1-\lambda)(p-1)} f^p(x)dx \right)^{\frac{1}{p}} \\ &\quad \times \left( \int_0^\infty x^{(1-\lambda)(q-1)} g^q(x)dx \right)^{\frac{1}{q}}, \end{aligned} \tag{3}$$

Received August 3, 2005, accepted October 24, 2006.
Communicated by Sen-Yen Shaw.
2000 *Mathematics Subject Classification*: 26D15.
*Key words and phrases*: Multidimensional Hardy-Hilbert's integral inequality, $\Gamma$-function, $\beta$-function, the best constant factor.

(4) $$\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{f(x)}{x^\lambda+y^\lambda}dx\right)^p dy < \left[\frac{\pi}{\lambda\sin(\frac{\pi}{p})}\right]^p \int_0^\infty x^{(1-\lambda)(p-1)}f^p(x)dx,$$

where the constant factors $\frac{\pi}{\lambda\sin(\frac{\pi}{p})}$ and $[\frac{\pi}{\lambda\sin(\frac{\pi}{p})}]^p$ are all the best possible.

It is significant and important to generalize Hardy-Hilbert's integral inequalities into multidimensional forms. Recently, Hong [7] and Yang [8] give some multidimensional results. In the year 2003, Kuang and Debnath L. obtains (see [9]): If $a < \infty$ or $a = \infty$, $\Sigma_{k=1}^n \frac{1}{p_k} = 1 (p_k > 1)$, $\alpha_k(x)(k = 1, 2, \cdots, n)$ are measurable function in $(0, a)$, $f_k(x)(k = 1, 2, \cdots, n)$ are nonnegative measurable function in $(0, a)$, $\lambda \geq 1$ and

$$F_k(u) = e^u \int_0^a f_k(x)\frac{u^{(\alpha_k(x))^k-\frac{1}{2}}}{\Gamma\left((\alpha_k(x))^k+\frac{1}{2}\right)}dx, \quad k = 1, 2, \cdots, n,$$

then

(5) $$\begin{aligned}&\int_0^\infty \cdots \int_0^\infty \frac{\prod_{k=1}^n f_k(x_k)}{(\sum_{k=1}^n \alpha_k(x_k))^\lambda}dx_1\cdots dx_n \\ &\leq \prod_{k=1}^n \Gamma\left(1-\frac{1}{p_k}\right)\left(\int_0^a F_k^{p_k}(u)du\right)^{\frac{1}{p_k}},\end{aligned}$$

where the $\Gamma(\cdot)$ is $\Gamma$-function.

In this paper, by introduce norm $\|x\|_\alpha (x \in R^n)$ and parameters $\alpha$ and $\lambda$, we give multidimensional Hardy-Hilbert's integral inequalities with the best constant factors that correspond to (3) and (4). Let $R_n^+ = \{x = (x_1, \cdots, x_n) : x_1, \ldots, x_n > 0\}$, $\|x\|_\alpha = (x_1^\alpha + \cdots + x_n^\alpha)^{\frac{1}{\alpha}}, (\alpha > 0)$, our main theorem is:

**Theorem 1.1.** *If* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $n \in Z_+$, $\alpha > 0$, $\lambda > 0$, $f \geq 0$, $g \geq 0$, *and*

(6) $$0 < \int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx < \infty, 0 < \int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(q-1)} g^q(x)dx < \infty,$$

*then*

(7) $$\begin{aligned}&\int_{R_+^n}\int_{R_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha+\|y\|_\alpha)^\lambda}dxdy < \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right) \\ &\times\left(\int_{R_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{R_+^n}\|x\|_\alpha^{(n-\lambda)(q-1)}g^q(x)dx\right)^{\frac{1}{q}};\end{aligned}$$

(8)
$$\int_{R_+^n} \|y\|_\alpha^{\lambda-n} \left( \int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} dx \right)^p dy < \left[ \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \right]^p \int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x) dx,$$

*where the constant factors* $\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\frac{\lambda}{p}, \frac{\lambda}{q})$ *and* $[\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\frac{\lambda}{p}, \frac{\lambda}{q})]^p$ *are all the best possible.In particular*

(1) *for* $n = 1$, *we have*

(9)
$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \times \left( \int_{R_+^n} x^{(1-\lambda)(p-1)} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_{R_+^n} x^{(1-\lambda)(q-1)} g^q(x) dx \right)^{\frac{1}{q}};$$

(10)
$$\int_{R_+^n} \left( \int_{R_+^n} \frac{f(x)}{(x+y)^\lambda} dx \right)^p dy < B^p\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \int_0^\infty x^{(1-\lambda)(q-1)} f^p(x) dx,$$

*where the constant factors* $B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ *and* $B^p\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ *are all the best possible.*

(2) *for* $\alpha = 1$, *we have*

(11)
$$\int_{R_+^n} \int_{R_+^n} \frac{f(x)g(y)}{[\sum_{i=1}^n (x_i + y_i)]^\lambda} dx dy < \frac{1}{(n-1)!} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \times \left[ \int_{R_+^n} \left( \sum_{i=1}^n x_i \right)^{(n-\lambda)(p-1)} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{R_+^n} \left( \sum_{i=1}^n x_i \right)^{(n-\lambda)(q-1)} g^q(x) dx \right]^{\frac{1}{q}};$$

(12)
$$\int_{R_+^n} \left( \sum_{i=1}^n y_i \right)^{\lambda-n} \left( \int_{R_+^n} \frac{f(x)}{[\sum_{i=1}^n (x_i + y_i)]^\lambda} dx \right)^p dy < \left[ \frac{1}{(n-1)!} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \right]^p \int_{R_+^n} \left( \sum_{i=1}^n x_i \right)^{(n-\lambda)(p-1)} f^p(x) dx,$$

*where the constant factors* $\frac{1}{(n-1)!}B(\frac{\lambda}{p},\frac{\lambda}{q})$ *and* $[\frac{1}{(n-1)!}B(\frac{\lambda}{p},\frac{\lambda}{q})]^p$ *are all the best possible.*

(3) *for* $\lambda = n$, *we have*

(13)
$$\int_{R_+^n}\int_{R_+^n}\frac{f(x)g(y)}{(\|x\|_\alpha+\|y\|_\alpha)^n}dxdy<\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B\left(\frac{n}{p},\frac{n}{q}\right)\times\left(\int_{R_+^n}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{R_+^n}g^q(x)dx\right)^{\frac{1}{q}};$$

(14)
$$\int_{R_+^n}\left(\int_{R_+^n}\frac{f(x)}{(\|x\|_\alpha+\|y\|_\alpha)^n}dx\right)^p dy<\left[\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B\left(\frac{n}{p},\frac{n}{q}\right)\right]^p\int_{R_+^n}f^p(x)dx,$$

*where the constant factors* $\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\frac{n}{p},\frac{n}{q})$ *and* $[\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{n})}B(\frac{n}{p},\frac{n}{q})]^p$ *are all the best possible.*

(4) *for* $n=\lambda=p=q=2$, *we have*

(15)
$$\int_{R_+^2}\int_{R_+^2}\frac{f(x)g(y)}{(\|x\|_2+\|y\|_2)^2}dxdy<\frac{\pi}{2}\left(\int_{R_+^2}f^2(x)dx\right)^{\frac{1}{2}}\left(\int_{R_+^2}g^2(x)dx\right)^{\frac{1}{2}};$$

(16)
$$\int_{R_+^2}\left(\int_{R_+^2}\frac{f(x)}{(\|x\|_2+\|y\|_2)^2}dx\right)^2 dy<\frac{\pi^2}{4}\int_{R_+^2}f^2(x)dx,$$

*where the constant factors* $\frac{\pi}{2}$ *and* $\frac{\pi^2}{4}$ *are all the best possible.*

## 2. Some Lemmas

**Lemma 2.1.** (see [10]) *If* $p_i>0$, $a_i>0$, $\alpha_i>0$, $i=1,2,\cdots,n$, $\Psi(u)$ *is a measurable function, then*

(17)
$$\int\cdots\int_{x_1,\cdots,x_n>0;(\frac{x_1}{a_1})^{\alpha_1}+\cdots+(\frac{x_n}{a_n})^{\alpha_n}\le 1}\Psi\left((\frac{x_1}{a_1})^{\alpha_1}+\cdots+(\frac{x_n}{a_n})^{\alpha_n}\right)\times x_1^{p_1-1}\cdots x_n^{p_n-1}dx_1\cdots dx_n=\frac{a_1^{p_1}\cdots a_n^{p_n}\Gamma(\frac{p_1}{\alpha_1})\cdots\Gamma(\frac{p_n}{\alpha_n})}{\alpha_1\cdots\alpha_n\Gamma(\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n})}\int_0^1\Psi(u)u^{\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}-1}du.$$

**Lemma 2.2.** *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n \in Z_+$, $\alpha > 0$, $\lambda > 0$, and setting weight function $\omega_{\alpha,\lambda}(x, p, q)$ as:*

$$\omega_{\alpha,\lambda}(x,p,q) = \int_{R_+^n} \frac{1}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} \left( \frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}} \right)^{(n-\lambda)p} \left( \frac{\|x\|_\alpha}{\|y\|_\alpha} \right)^{\frac{\lambda}{q}} dy,$$

*then*

$$\omega_{\alpha,\lambda}(x,p,q) = \|x\|_\alpha^{(n-\lambda)(p-1)} \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right), \tag{18}$$

*where the $B(\cdot,\cdot)$ is $\beta$-function.*

*Proof.* By (17), we have

$$\omega_{\alpha,\lambda}(x,p,q) = \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}} \int_{R_+^n} \frac{1}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} \|y\|_\alpha^{-(n-\lambda+\frac{\lambda}{q})} dy$$

$$= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}} \lim_{r\to+\infty} \int \cdots \int_{y_1,\cdots,y_n>0; y_1^\alpha+\cdots+y_n^\alpha<r^\alpha}$$

$$\times \frac{\left[r((\frac{y_1}{r})^\alpha + \cdots + (\frac{y_n}{r})^\alpha)^{\frac{1}{\alpha}}\right]^{-(n-\lambda+\frac{\lambda}{q})}}{\left[\|x\|_\alpha + r((\frac{y_1}{r})^\alpha + \cdots + (\frac{y_n}{r})^\alpha)^{\frac{1}{\alpha}}\right]^\lambda} y_1^{1-1} \cdots y_n^{1-1} dy_1 \cdots dy_n$$

$$= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}} \lim_{r\to+\infty} \frac{r^n \Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \int_0^1 \frac{(ru^{\frac{1}{\alpha}})^{-(n-\lambda+\frac{\lambda}{q})}}{\left(\|x\|_\alpha + ru^{\frac{1}{\alpha}}\right)^\lambda} u^{\frac{n}{\alpha}-1} du$$

$$= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}} \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \lim_{r\to+\infty} \int_0^r \frac{1}{(\|x\|_\alpha + u)^\lambda} u^{\lambda-\frac{\lambda}{q}-1} du$$

$$= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}} \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{\lambda}{p}-1} du$$

$$= \|x\|_\alpha^{(n-\lambda)(p-1)} \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\frac{\lambda}{p}, \lambda - \frac{\lambda}{p}\right) = \|x\|_\alpha^{(n-\lambda)(p-1)} \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right),$$

hence (18) is valid.

**Lemma 2.3.** *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n \in Z_+$, $\alpha > 0$, $\lambda > 0$, $0 < \varepsilon < \lambda(q-1)$ and setting $\tilde{\omega}_{\alpha,\lambda}(x, q, \varepsilon)$ as:*

$$\tilde{\omega}_{\alpha,\lambda}(x,q,\varepsilon) = \int_{R_+^n} \frac{1}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} \|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}} dy,$$

*then we have*

$$\tilde{\omega}_{\alpha,\lambda}(x,q,\varepsilon)=\|x\|_{\alpha}^{-\frac{\lambda}{q}-\frac{\varepsilon}{q}}\frac{\Gamma^{n}(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B\left(\frac{\lambda}{p}-\frac{\varepsilon}{q},\frac{\lambda}{q}+\frac{\varepsilon}{q}\right). \tag{19}$$

*Proof.* By the method to similar proof of Lemma 2.2, Lemma 2.3 can be proven.

## 3. The Proof of Theorem

By Hölder's inequality, we have

$$\begin{aligned}
A:&=\int_{R_{+}^{n}}\int_{R_{+}^{n}}\frac{f(x)g(y)}{(\|x\|_{\alpha}+\|y\|_{\alpha})^{\lambda}}dxdy\\
&=\int_{R_{+}^{n}}\int_{R_{+}^{n}}\frac{f(x)}{(\|x\|_{\alpha}+\|y\|_{\alpha})^{\frac{\lambda}{p}}}\left(\frac{\|x\|_{\alpha}^{\frac{1}{q}}}{\|y\|_{\alpha}^{\frac{1}{p}}}\right)^{n-\lambda}\left(\frac{\|x\|_{\alpha}}{\|y\|_{\alpha}}\right)^{\frac{\lambda}{pq}}\\
&\quad\times\frac{g(y)}{(\|x\|_{\alpha}+\|y\|_{\alpha})^{\frac{\lambda}{q}}}\left(\frac{\|y\|_{\alpha}^{\frac{1}{p}}}{\|x\|_{\alpha}^{\frac{1}{q}}}\right)^{n-\lambda}\left(\frac{\|y\|_{\alpha}}{\|x\|_{\alpha}}\right)^{\frac{\lambda}{pq}}dxdy\\
&\le\left[\int_{R_{+}^{n}}\int_{R_{+}^{n}}\frac{f^{p}(x)}{(\|x\|_{\alpha}+\|y\|_{\alpha})^{\lambda}}\left(\frac{\|x\|_{\alpha}^{\frac{1}{q}}}{\|y\|_{\alpha}^{\frac{1}{p}}}\right)^{(n-\lambda)p}\left(\frac{\|x\|_{\alpha}}{\|y\|_{\alpha}}\right)^{\frac{\lambda}{q}}dxdy\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_{R_{+}^{n}}\int_{R_{+}^{n}}\frac{g^{q}(x)}{(\|x\|_{\alpha}+\|y\|_{\alpha})^{\lambda}}\left(\frac{\|y\|_{\alpha}^{\frac{1}{p}}}{\|x\|_{\alpha}^{\frac{1}{q}}}\right)^{(n-\lambda)q}\left(\frac{\|y\|_{\alpha}}{\|x\|_{\alpha}}\right)^{\frac{\lambda}{p}}dxdy\right]^{\frac{1}{q}}\\
&=\left[\int_{R_{+}^{n}}f^{p}(x)\left(\int_{R_{+}^{n}}\frac{1}{(\|x\|_{\alpha}+\|y\|_{\alpha})^{\lambda}}\left(\frac{\|x\|_{\alpha}^{\frac{1}{q}}}{\|y\|_{\alpha}^{\frac{1}{p}}}\right)^{(n-\lambda)p}\left(\frac{\|x\|_{\alpha}}{\|y\|_{\alpha}}\right)^{\frac{\lambda}{q}}dy\right)dx\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_{R_{+}^{n}}g^{q}(y)\left(\int_{R_{+}^{n}}\frac{1}{(\|x\|_{\alpha}+\|y\|_{\alpha})^{\lambda}}\left(\frac{\|y\|_{\alpha}^{\frac{1}{p}}}{\|x\|_{\alpha}^{\frac{1}{q}}}\right)^{(n-\lambda)q}\left(\frac{\|y\|_{\alpha}}{\|x\|_{\alpha}}\right)^{\frac{\lambda}{p}}dx\right)dy\right]^{\frac{1}{q}}\\
&=\left(\int_{R_{+}^{n}}f^{p}(x)\omega_{\alpha,\lambda}(x,p,q)dx\right)^{\frac{1}{p}}\left(\int_{R_{+}^{n}}g^{q}(y)\omega_{\alpha,\lambda}(y,q,p)dy\right)^{\frac{1}{q}},
\end{aligned}$$

according to the condition of taking equality in Hölder's inequality, if this inequality takes the form of an equality, then there exist constants $C_1$ and $C_2$, such that they are not all zero, and

$$\begin{aligned} &\frac{C_1 f^p(x)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} \left( \frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}} \right)^{(n-\lambda)p} \left( \frac{\|x\|_\alpha}{\|y\|_\alpha} \right)^{\frac{\lambda}{q}} \\ = &\frac{C_2 g^q(y)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} \left( \frac{\|y\|_\alpha^{\frac{1}{p}}}{\|x\|_\alpha^{\frac{1}{q}}} \right)^{(n-\lambda)q} \left( \frac{\|y\|_\alpha}{\|x\|_\alpha} \right)^{\frac{\lambda}{p}}, \quad a.e. \ in \ R_+^n \times R_+^n, \end{aligned}$$

it follows that

$$\begin{aligned} &C_1 \|x\|_\alpha^{(n-\lambda)(p-1)+n} f^p(x) = C_2 \|y\|_\alpha^{(n-\lambda)(q-1)+n} g^q(y) = C(constant), \\ &a.e. \, in \, R_+^n \times R_+^n, \end{aligned}$$

which contradicts (6), hence we have

$$A < \left( \int_{R_+^n} f^p(x) \omega_{\alpha,\lambda}(x,p,q) dx \right)^{\frac{1}{p}} \left( \int_{R_+^n} g^q(y) \omega_{\alpha,\lambda}(y,q,p) dy \right)^{\frac{1}{q}}.$$

Hence by (18), we obtain

$$\begin{aligned} A &< \left[ \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \right]^{\frac{1}{p}} \left( \int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x) dx \right)^{\frac{1}{p}} \\ &\quad \times \left[ \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \right]^{\frac{1}{q}} \left( \int_{R_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} g^q(y) dy \right)^{\frac{1}{q}} \\ &= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left( \int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x) dx \right)^{\frac{1}{p}} \\ &\quad \times \left( \int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(q-1)} g^q(x) dx \right)^{\frac{1}{q}}. \end{aligned}$$

Hence (7) is valid.

For $0 < a < b < \infty$, setting

$$g_{a,b}(y) = \begin{cases} \|y\|_\alpha^{\lambda-n} \left( \displaystyle\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} dx \right)^{p-1}, & a < \|y\|_\alpha < b \\ 0, & 0 < \|y\|_\alpha \leq a \, \text{or} \, \|y\|_\alpha \geq b \end{cases}$$

$$\tilde{g}(y) = \|y\|_\alpha^{\lambda-n}\left(\int_{R_+^n}\frac{f(x)}{(\|x\|_\alpha+\|y\|_\alpha)^\lambda}dx\right)^{p-1}, \ \ y\in R_+^n,$$

by (6), for sufficiently small $a>0$ and sufficiently large $b>0$, we have

$$0<\int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{(n-\lambda)(q-1)}g_{a,b}^q(y)dy<\infty.$$

Hence by (7), we have

$$\begin{aligned}
&\int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{(n-\lambda)(q-1)}\tilde{g}^q(y)dy\\
&=\int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{\lambda-n}\left(\int_{R_+^n}\frac{f(x)}{(\|x\|_\alpha+\|y\|_\alpha)^\lambda}dx\right)^p dy\\
&=\int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{\lambda-n}\left(\int_{R_+^n}\frac{f(x)}{(\|x\|_\alpha+\|y\|_\alpha)^\lambda}dx\right)^{p-1}\\
&\quad\times\left(\int_{R_+^n}\frac{f(x)}{(\|x\|_\alpha+\|y\|_\alpha)^\lambda}dx\right)dy\\
&=\int_{R_+^n}\int_{R_+^n}\frac{f(x)g_{a,b}(y)}{(\|x\|_\alpha+\|y\|_\alpha)^\lambda}dxdy\\
&<\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left(\int_{R_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{R_+^n}\|y\|_\alpha^{(n-\lambda)(q-1)}g_{a,b}^q(y)dy\right)^{\frac{1}{q}}\\
&=\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left(\int_{R_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{(n-\lambda)(q-1)}\tilde{g}^q(y)dy\right)^{\frac{1}{q}},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{(n-\lambda)(q-1)}\tilde{g}^q(y)dy\\
&\quad<\left[\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\right]^p\int_{R_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx.
\end{aligned}$$

For $a \to 0^+$, $b \to +\infty$, we obtain

$$\int_{R_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} \tilde{g}^q(y)dy \le \left[\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\right]^p \int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx,$$

hence by (6), we obtain

$$0 < \int_{R_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} \tilde{g}^q(y)dy < \infty,$$

hence $\tilde{g}(y)$ satisfies (6). By (7), we have

$$\begin{aligned}
&\int_{R_+^n} \|y\|_\alpha^{\lambda-n} \left(\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} dx\right)^p dy \\
&= \int_{R_+^n}\int_{R_+^n} \frac{f(x)\tilde{g}(y)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} dxdy < \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \\
&\times \left(\int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{R_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} \tilde{g}^q(y)dy\right)^{\frac{1}{q}} \\
&= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left(\int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx\right)^{\frac{1}{p}} \\
&\times \left[\int_{R_+^n} \|y\|_\alpha^{\lambda-n} \left(\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} dx\right)^p dy\right]^{\frac{1}{q}},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\int_{R_+^n} \|y\|_\alpha^{\lambda-n} \left(\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} dx\right)^p dy \\
&< \left[\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\right]^p \int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx.
\end{aligned}$$

Hence (8) is valid.

**Remark.** By (8), we can also obtain (7), hence (8) and (7) are equivalent.

If the constant factor $K_1 := \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\frac{\lambda}{p}, \frac{\lambda}{q})$ in (7) is not the best possible, then there exists a positive constant $K < K_1$, such that

$$\begin{aligned}
&\int_{R_+^n}\int_{R_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} dxdy \\
&< K \left(\int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{R_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} g^q(y)dy\right)^{\frac{1}{q}}.
\end{aligned} \tag{20}$$

In particular, for $0 < \varepsilon < \lambda(q-1)$, setting

$$f(x) = \|x\|_\alpha^{-\frac{(n-\lambda)(p-1)+n+\varepsilon}{p}}, \quad g(y) = \|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}},$$

(20) is still true. By the properties of limit, there exists a sufficiently small $a > 0$, such that

$$\begin{aligned}
&\int_{\|x\|_\alpha > a}\int_{R_+^n} \frac{1}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda}\|x\|_\alpha^{-\frac{(n-\lambda)(p-1)+n+\varepsilon}{p}}\|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}}dxdy \\
&< K\left(\int_{\|x\|_\alpha > a}\|x\|_\alpha^{(n-\lambda)(p-1)}\|x\|_\alpha^{-(n-\lambda)(p-1)-n-\varepsilon}dx\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_{\|y\|_\alpha > a}\|y\|_\alpha^{(n-\lambda)(q-1)}\|y\|_\alpha^{-(n-\lambda)(q-1)-n-\varepsilon}dy\right)^{\frac{1}{q}} \\
&= K\int_{\|x\|_\alpha > a}\|x\|_\alpha^{-n-\varepsilon}dx.
\end{aligned}$$

On the other hand, by (19), we have

$$\begin{aligned}
&\int_{\|x\|_\alpha > a}\int_{R_+^n} \frac{1}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda}\|x\|_\alpha^{-\frac{(n-\lambda)(p-1)+n+\varepsilon}{p}}\|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}}dxdy \\
&= \int_{\|x\|_\alpha > a}\|x\|_\alpha^{-n+\frac{\lambda}{q}-\frac{\varepsilon}{p}}\int_{R_+^n}\frac{1}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda}\|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}}dydx \\
&= \int_{\|x\|_\alpha > a}\|x\|_\alpha^{-n+\frac{\lambda}{q}-\frac{\varepsilon}{p}}\tilde{\omega}_{\alpha,\lambda}(x,q,\varepsilon)dx \\
&= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B\left(\frac{\lambda}{p}-\frac{\varepsilon}{q},\frac{\lambda}{q}+\frac{\varepsilon}{q}\right)\int_{\|x\|_\alpha > a}\|x\|_\alpha^{-n-\varepsilon}dx,
\end{aligned}$$

hence we obtain

$$\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B\left(\frac{\lambda}{p}-\frac{\varepsilon}{q},\frac{\lambda}{q}+\frac{\varepsilon}{q}\right) < K,$$

for $\varepsilon \to 0^+$, we have

$$K_1 = \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right) \le K,$$

this contradicts the fact that $K < K_1$.Hence the constant factor $\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\frac{\lambda}{p},\frac{\lambda}{q})$ in (7) is the best possible.

Since (8) and (7) are equivalent, the constant factor $[\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\frac{\lambda}{p},\frac{\lambda}{q})]^p$ in (8) is also the best possible.

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Hong Yong
Department of Mathematics,
Guangdong University of Business Study,
Guangzhou 510320,
P. R. China