TAIWANESE JOURNAL OF MATHEMATICS
Vol. 8, No. 1, pp. 135-154, March 2004
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ON SOME NONLINEAR DISSIPATIVE EQUATIONS WITH SUB-CRITICAL NONLINEARITIES

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Abstract. We study the Cauchy problem for the nonlinear dissipative equations

$$(1)\qquad \begin{cases} \partial_t u + \alpha\,(-\Delta)^{\frac{\rho}{2}} u + \beta\,|u|^{\sigma} u + \gamma\,|u|^{\varkappa} u = 0, \ x \in \mathbf{R}^n, t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases}$$

where $\alpha, \beta, \gamma \in \mathbf{C}$, Re $\alpha > 0$, $\rho > 0$, $\varkappa > \sigma > 0$. We are interested in the critical case, $\sigma = \frac{\rho}{n}$ and sub critical cases $0 < \sigma < \frac{\rho}{n}$. We assume that the initial data u_0 are sufficiently small in a suiatble norm, $\left|\int u_0\,(x)\,dx\right| > 0$ and Re$\beta\delta(\alpha,\rho,\sigma) > 0$, where

$$\delta(\alpha,\rho,\sigma) = \int |G\,(x)|^{\sigma}\, G\,(x)\, dx$$

and $G\,(x) = \mathcal{F}^{-1} e^{-\alpha|\xi|^{\rho}}$. In the sub critical case we assume that σ is close to $\frac{\rho}{n}$. Then we prove global existence in time of solutions to the Cauchy problem (1) satisfying the time decay estimate

$$\|u\,(t)\|_{\mathbf{L}^\infty} \cdot \begin{cases} C\,(1+t)^{-\frac{1}{\sigma}} (\log\,(2+t))^{-\frac{1}{\sigma}} \ \text{ if } \sigma = \frac{\rho}{n}, \\ C\,(1+t)^{-\frac{1}{\sigma}} \ \text{ if } \sigma \in \left(0, \frac{\rho}{n}\right). \end{cases}$$

1. Introduction

We study the Cauchy problem for the nonlinear dissipative equations with fractional power of the negative Laplacian and complex coefficients

$$(2)\qquad \begin{cases} \partial_t u + \alpha\,(-\Delta)^{\frac{\rho}{2}} u + \beta\,|u|^{\sigma} u + \gamma\,|u|^{\varkappa} u = 0, \ x \in \mathbf{R}^n, t > 0, \\ u\,(0,x) = u_0\,(x)\,, \ x \in \mathbf{R}^n, \end{cases}$$

Received April 15, 2003.
Communicated by S. B. Hsu.
2000 *Mathematics Subject Classification*: 35Q35.
Key words and phrases: Nonlinear dissipative equations, Sub-critical nonlinearities.

where $\alpha, \beta, \gamma \in \mathbf{C}$, Re $\alpha > 0$, $\rho > 0$, $\varkappa > \sigma > 0$. Furthermore we assume that Re $\beta\delta(\alpha, \rho, \sigma) > 0$ and initial data are sufficiently small in $\mathbf{L}^{\infty} \cap \mathbf{L}^{1,a}$ where $\mathbf{L}^{p}$, $1 \cdot p \cdot \infty$, is the usual Lebesgue space, the weighted Lebesgue space $\mathbf{L}^{1,b}$ is defined by

$$\mathbf{L}^{1,b} = \left\{ \phi \in \mathbf{L}^{1}; \|\phi\|_{\mathbf{L}^{1,b}} = \left\| \langle x \rangle^{b} \phi \right\|_{\mathbf{L}^{1}} < \infty \right\}, \ b \geq 0$$

and

$$\delta(\alpha, \rho, \sigma) = \int |G(x)|^{\sigma} G(x)\, dx, G(x) = \mathcal{F}^{-1}\left(e^{-\alpha|\xi|^{\rho}}\right).$$

We restrict our attention to the critical and subcritical cases $0 < \sigma \cdot \frac{\rho}{n}$ and prove global existence in time for small solutions to the Cauchy problem (2). In the super critical case $\sigma > \frac{\rho}{n}$ the problem is easier, and it was studied in [15] under the restrictions $\sigma > \rho > \frac{1}{4}$, $n = 1$. Equation (2) with $\rho = 2$, $\sigma = \frac{2}{n}$ is known as the complex Landau-Ginzburg equation. Local existence in time for the solutions to the Cauchy problem (2) with $\rho = 2$ was studied by many authors (see, e.g. [7],[8]) and references cited therein). In the case $\rho \neq 2$, local existence in time can be easily shown by the contraction mapping principle in $\mathbf{L}^{2}$ framework. Nonlinear dissipative equations with a fractional power of the negative Laplacian in the principal part were studied extensively (see, e.g., [1],[2],[20],[23] and references cited therein). Blow-up in finite time of positive solutions to the Cauchy problem

$$\partial_t u + (-\Delta)^{\frac{\rho}{2}} u - u^{1+\sigma} = 0, \ u(0, x) = u_0(x) > 0 \tag{3}$$

was proved in [5],[24] for the case $0 < \sigma < \frac{2}{n}$, $\rho = 2$, in [10],[18] for the case $\sigma = \frac{2}{n}$, $\rho = 2$, and in paper [22] for the case $0 < \rho \cdot 2$, $0 < \sigma \cdot \frac{\rho}{n}$. Their proofs of blow-up results are based on the positivity of linear evolution operator $\mathcal{F}^{-1} e^{-|\xi|^{\rho}}$, associated with equation (3), for $0 < \rho \cdot 2$ (see book [25]), and do not work for the case $\rho > 2$, since $\mathcal{F}^{-1} e^{-|\xi|^{\rho}}$ is not necessarily positive. Large time behavior of positive solutions was studied extensively for a particular case of (2) written as

$$\partial_t u - \Delta u + u^{1+\sigma} = 0,$$

with any $\sigma > 0$ (see [16]) for the super critical case $\sigma > \frac{2}{n}$, [6] for the critical case $\sigma = \frac{2}{n}$ and [3],[4],[9],[17] for the sub critical case $\sigma \in \left(0, \frac{2}{n}\right)$. Global existence in time for small solutions to (3) in the super critical case $\sigma > \frac{2}{n}$, $\rho = 2$ was shown in [5]. Large time asymptotic behavior of small solutions to (2) with $\rho = 2, \gamma = 0$ was investigated in details in [12],[13],[14] under the condition that

$$\begin{aligned} \mathrm{Re}\beta \left((2+\sigma) |\alpha|^2 + \sigma\alpha^2 \right)^{-\frac{n}{2}} &= |\beta| \left| (2+\sigma) |\alpha|^2 + \sigma\alpha^2 \right|^{-\frac{n}{2}} \\ \times \cos &\left(\arg \beta - \frac{n}{2} \arctan \frac{\sin(2 \arg \alpha)}{1 + \frac{2}{\sigma} + \cos(2 \arg \alpha)} \right) > 0. \end{aligned}$$

The critical case $\sigma = \frac{2}{n}, n = 1$ was studied in [12], when the initial data are small in $\mathbf{H}^{1,0} \cap \mathbf{H}^{0,1}$, where $\mathbf{H}^{m,s}$ is the weighted Sobolev space

$$\mathbf{H}^{m,s} = \left\{ \phi \in \mathbf{L}^2 ; \|\phi\|_{\mathbf{H}^{m,s}} = \| \langle i\nabla \rangle^m \langle x \rangle^s \phi \|_{\mathbf{L}^2} < \infty \right\},$$

$m, s \geq 0$, $\langle x \rangle = \sqrt{1 + x^2}$. In [13] the critical case $\sigma = \frac{2}{n}$, was studied for any space dimensions $n \geq 1$, when the initial data are small in $\mathbf{L}^\infty \cap \mathbf{L}^{1,a}$, $a \in (0, 1)$, In [14] it was considered the sub critical case $1 < \sigma < \frac{2}{n}$, when $2 - n\sigma$ is small, $n \geq 1$, the initial data are small in $\mathbf{L}^\infty \cap \mathbf{L}^{1,a}$, $a \in (0, 1)$.

As far as we know the global existence in time for solutions to the Cauchy problem (2) with $\rho \neq 2$ in critical and sub critical cases is not known. Our global result stated below is applicable, in particular, to the problem

$$\partial_t u + (-\Delta)^{\frac{\rho}{2}} u + \lambda u^{1+\sigma} = \mu u^{1+\varkappa}, \; u(0, x) = u_0(x) > 0, \tag{4}$$

with $0 < \sigma < \varkappa \cdot \frac{\rho}{n}, \lambda, \mu > 0$. As we mentioned above when $\lambda = 0$, $\mu > 0$, $0 < \rho \cdot 2$ the solutions of (4) blow up in finite time, and when $\lambda > 0$, $\mu = 0$, $0 < \rho < \infty$ the solution exists globally in time. Our results stated below show that the dissipation term $u^{1+\sigma}$ in equation (4) is stronger than the blow-up term $u^{1+\varkappa}$. Note that the problem of asymptotic behavior of solutions to (4) is still open for the sub critical case $0 < \varkappa < \sigma \cdot \frac{\rho}{n}$ even if $\rho = 2$.

Let $\mathcal{F}\phi$ or $\hat{\phi}$ be the Fourier transform of ϕ defined by $\hat{\phi}(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-ix\xi} \phi(x)\, dx$ and $\mathcal{F}^{-1}\phi(x) = (2\pi)^{-\frac{n}{2}} \int e^{ix\xi} \phi(x)\, dx$ is the inverse Fourier transform of ϕ. By $\mathbf{C}(\mathbf{I}; \mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$.

Now we state the results of this paper. Denote $g(t) = 1 + |\theta|^\sigma \eta \log(1 + t)$, $\eta = \rho \mathrm{Re}\beta\delta(\alpha, \rho, \sigma)$,

$$\chi_\sigma(t) = g(t) \text{ if } \sigma = \tfrac{\rho}{n}, \text{ and}$$
$$\chi_\sigma(t) = 1 + \tfrac{n\sigma|\theta|^\sigma \eta}{\rho - n\sigma} t^{1 - \frac{n\sigma}{\rho}} \text{ if } \sigma \in \left(0, \tfrac{\rho}{n}\right).$$

Theorem 1.1. *Assume that* $\mathrm{Re}\alpha > 0$, $0 < \sigma < \varkappa \cdot \frac{\rho}{n}$ *and*

$$u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^{1,a}, \; a \in (0, \min(1, \rho)).$$

Furthermore we suppose that

$$\mathrm{Re}\beta\delta(\alpha, \rho, \sigma) > 0, \; |\hat{u}_0(0)| = \theta (2\pi)^{-\frac{n}{2}} > 0.$$

Then there exists a positive ε *such that if* $\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^{1,a}} \cdot \varepsilon$, $|\hat{u}_0(0)| \geq C\varepsilon$ *and the value* σ *is close to* $\frac{\rho}{n}$, *so that* $\frac{\rho}{n} - \sigma \cdot C\varepsilon^\sigma$. *then the Cauchy problem (2)* *has a unique global solution* $u(t,x) \in \mathbf{C}\left([0,\infty);\mathbf{L}^{\infty} \cap \mathbf{L}^{1,a}\right)$ *satisfying the time decay estimate*

$$\left\| u(t) - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) \chi_{\sigma}^{-\frac{1}{\sigma}}(t)\, e^{i\psi(t)} \right\|_{\mathbf{L}^{\infty}} \le \begin{cases} C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{\sigma}} g^{-\frac{1}{\sigma}}(t) \text{ if } \sigma = \frac{\rho}{n}, \\ C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{\sigma}} \text{ if } \sigma \in \left(\frac{\rho}{n} - C\varepsilon^{\sigma}, \frac{\rho}{n}\right), \end{cases}$$

and $\psi(t)$ *satisfies*

$$\left|\psi(t) - \arg \widehat{u}_0(0) + |\theta|^{\sigma}\, \widetilde{\eta} \textstyle\int_0^t \chi_{\sigma}^{-\frac{1}{\sigma}}(\tau)(1+\tau)^{-\frac{\sigma}{\rho}n} d\tau\right| \le \begin{cases} C\varepsilon^{1+\sigma} \displaystyle\int_0^t \chi_{\sigma}^{-1-\frac{1}{\sigma}}(\tau) \log \chi_{\sigma}(\tau)(1+\tau)^{-1} d\tau \text{ if } \sigma = \frac{\rho}{n}, \\ C\varepsilon^{1+2\sigma} \displaystyle\int_0^t \chi_{\sigma}^{-\frac{1}{\sigma}}(\tau)(1+\tau)^{-\frac{\sigma}{\rho}n} d\tau \text{ if } \sigma \in \left(\frac{\rho}{n} - C\varepsilon^{\sigma}, \frac{\rho}{n}\right), \end{cases}$$

$\widetilde{\eta} = \mathrm{Im}\beta\delta(\alpha,\rho,\sigma)$.

For the convenience of the reader, we state the global existence result for the super-critical case $\varkappa > \sigma > \frac{\rho}{n}$. In this case we do not need the positivity of $\mathrm{Re}\beta\delta(\alpha,\rho,\sigma)$.

Theorem 1.2. *Assume that* $\mathrm{Re}\alpha > 0$, $\varkappa > \sigma > \frac{\rho}{n}$ *and* $u_0 \in \mathbf{L}^{\infty} \cap \mathbf{L}^{1}$. *Then there exists a positive* ε *such that if* $\|u_0\|_{\mathbf{L}^{\infty}} + \|u_0\|_{\mathbf{L}^{1}} \le \varepsilon$, *then the Cauchy problem* (2) *has a unique global solution* $u(t,x) \in \mathbf{C}\left([0,\infty);\mathbf{L}^{\infty} \cap \mathbf{L}^{1}\right)$ *satisfying the time decay estimate*

$$\|u(t)\|_{\mathbf{L}^{\infty}} \le C\varepsilon(1+t)^{-\frac{n}{\rho}}.$$

Remark 1.1. Our proof of the Theorem 1.1 depends on the positivity of the value $\mathrm{Re}\beta\delta(\alpha,\rho,\sigma)$, which we need to derive better time decay properties of solutions (see Lemma 2.3 below). If $\rho = 2$ then we can calculate explicitly the value of $\delta(\alpha,\rho,\sigma)$ (see also [13,][14])

$$\begin{aligned} \delta(\alpha,2,\sigma) &= \left(\frac{1}{(4\pi t)^{\frac{n}{2}\sigma+\frac{n}{2}}|\alpha|^{\frac{n}{2}\sigma}\alpha^{\frac{n}{2}}} \int e^{-\frac{x^2\sigma}{8}\left(\frac{1}{\alpha}+\frac{1}{\overline{\alpha}}\right)-\frac{x^2}{4\alpha}} dx\right) \\ &= \frac{2^{\frac{n}{2}}}{(4\pi)^{\frac{n}{2}\sigma}} \left(\frac{|\alpha|^{n-\frac{n}{2}\sigma}}{\left((2+\sigma)|\alpha|^2+\sigma\alpha^2\right)^{\frac{n}{2}}}\right). \end{aligned}$$

We do not need the positivity of the kernel $G(x) = \mathcal{F}^{-1}\left(e^{-\alpha|\xi|^{\rho}}\right)$ *of the Green operator which was essentially used previously in proving the blow-up results. The condition that the value* σ *should be close to* $\frac{\rho}{n}$, *so that* $\frac{\rho}{n} - \sigma \le C\varepsilon^{\sigma}$ *is rather technical and is caused by the application of the contraction mapping principle for proving global existence of solutions.*

Remark 1.2. We consider only the case of the linear operator of the form $\alpha(-\Delta)^{\frac{\rho}{2}}$ in equation (2) for simplicity. Our method is also applicable to a more general case of pseudodifferential operators.

We organize the rest of our paper as follows. We prove preliminary lemmas in Section 2. In Lemma 2.1 we obtain estimates of the Green operator in the Lebesgue spaces $\mathbf{L}^p$, $1 \le p \le \infty$ and $\mathbf{L}^{1,a}$. Then in Lemma 2.2 we estimate the Green operator in our basic norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\left((1+t)^{\frac{n}{\rho}}\|\phi(t)\|_{\mathbf{L}^\infty} + \|\phi(t)\|_{\mathbf{L}^1} + (1+t)^{-\frac{a}{\rho}}\|\phi(t)\|_{\mathbf{L}^{1,a}}\right),$$

where $a \in (0,1)$. Large time behavior of the mean value of the nonlinearity $\beta|u|^{\sigma}u$ in equation (2) is evaluated in Lemma 2.3. Section 3 is devoted to the proof of Theorem 1.1. Theorem 1.2 is proved in Section 4.

2. Preliminaries

We write the solution of the linear Cauchy problem

$$\begin{cases} \partial_t u + \alpha(-\Delta)^{\frac{\rho}{2}} u = f(t,x), \ x \in \mathbf{R}^n, t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases}$$

with $\mathrm{Re}\alpha > 0$, $\rho > 0$, by virtue of the Duhamel formula

$$u(t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau,$$

where the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\phi = \mathcal{F}^{-1}e^{-\alpha|\xi|^{\rho}t}\hat{\phi}(\xi) = t^{-\frac{n}{\rho}}\int G\left(t^{-\frac{1}{\rho}}(x-y)\right)\phi(y)dy \tag{5}$$

with a kernel $G(x) = \mathcal{F}^{-1}\left(e^{-\alpha|\xi|^{\rho}}\right)$.

We first collect some preliminary estimates of the Green operator $\mathcal{G}(t)$ in the norms $\|\phi\|_{\mathbf{L}^p}$ and $\|\phi\|_{\mathbf{L}^{1,a}}$, where $\|\phi\|_{\mathbf{L}^{1,a}} = \|\langle\cdot\rangle^{a}\phi\|_{\mathbf{L}^1}$, $a \in (0,1)$, $1 \le p \le \infty$.

Lemma 2.1. *Suppose that the function* $\phi \in \mathbf{L}^{\infty} \cap \mathbf{L}^{1,a}$, *where* $a \in (0,1)$. *Then the estimates*

$$\left\| \mathcal{G}(t)\phi \right\|_{\mathbf{L}^p} \le C \left\| \phi \right\|_{\mathbf{L}^p}$$

and

$$\left\| |\cdot|^{\omega} \left(\mathcal{G}(t)\phi - \vartheta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}} (\cdot) \right) \right) \right\|_{\mathbf{L}^p} \le C t^{\frac{n}{\rho}\left(\frac{1}{p}-1\right)+\frac{\omega - a}{\rho}} \left\| \phi \right\|_{\mathbf{L}^{1,a}}$$

are valid for all $t > 0$, *where*

$$1 \le p \le \infty,\ 0 \le \omega \le a,\ \vartheta = \int \phi(x)\, dx.$$

Proof. Note that the kernel $G(x) = \mathcal{F}^{-1}\left(e^{-\alpha|\xi|^{\rho}} \right)$ in representation (5) is a smooth function $G(x) \in \mathbf{C}^{\infty}(\mathbf{R}^n)$ and decays at infinity so that

$$\sup_{x \in \mathbf{R}^n} \langle x \rangle^{n+\mu+k} \left| \partial_{x_j}^k G(x) \right| \le C, \tag{6}$$

for all $1 \le j \le n$, $k = 0, 1$, where $0 < \mu < \min(1, \rho)$ (since $\rho > 0$ is a fractional number). Indeed we have

$$\left| \partial_{x_j}^k G(x) \right| = \left| \mathcal{F}^{-1}\left(\xi_j^k e^{-\alpha|\xi|^{\rho}} \right) \right| \le C \left\| \xi_j^k e^{-\alpha|\xi|^{\rho}} \right\|_{\mathbf{L}^1} \le C.$$

and

$$\begin{aligned} \left| |x_l|^{\mu} x_l^{n+k} \partial_{x_j}^k G(x) \right| &= \left| \mathcal{F}^{-1}\left(|\partial_{\xi_l}|^{\mu} \partial_{\xi_l}^{n+k} \left(\xi_j^k e^{-\alpha|\xi|^{\rho}} \right) \right) \right| \\ &\le C \left\| |\partial_{\xi_l}|^{\mu} \partial_{\xi_l}^{n+k} \left(\xi_j^k e^{-\alpha|\xi|^{\rho}} \right) \right\|_{\mathbf{L}^1_{\xi}} \end{aligned}$$

for $k = 0, 1$, where the fractional derivative $\left| \partial_{\xi_j} \right|^{\mu}$ for $\mu \in (0,1)$ is defined by

$$|\partial_{\xi_l}|^{\mu} \phi(\xi) = \mathcal{F}_l^{-1}\left(|x_l|^{\mu} \widehat{\phi}(x) \right) = C \int_{\mathbf{R}} \left(\phi\left(\widetilde{\xi}(y) \right) - \phi(\xi) \right) |y|^{-1-\mu} dy,$$

where $\xi = (\xi_1, ..., \xi_n)$, $\widetilde{\xi}(y) = (\xi_1, ..., \xi_l - y, ..., \xi_n)$ (see [21]). We have with $\phi(\xi) = \partial_{\xi_l}^{n+k}\left(\xi_j^k e^{-\alpha|\xi|^{\rho}} \right)$

$$|\partial_{\xi_l}|^{\mu} \phi(\xi) = C \int_{\mathbf{R}} \left(\phi\left(\widetilde{\xi}(y) \right) - \phi(\xi) \right) |y|^{-1-\mu} dy = I_1 + I_2,$$

where

$$I_1 = \int_{\frac{|\xi|}{2} \le |y|} \left(\phi\left(\widetilde{\xi}(y) \right) - \phi(\xi) \right) |y|^{-1-\mu} dy,$$

$$I_2 = \int_{|y| \le \frac{|\xi|}{2}} \left(\phi\left(\widetilde{\xi}(y)\right) - \phi(\xi) \right) |y|^{-1-\mu} dy.$$

Since

$$|\phi(\xi)| = \left| \partial_{\xi_l}^{n+k} \left(\xi_j^k e^{-\alpha|\xi|^\rho} \right) \right| \le C |\xi|^{\rho-n} e^{-C|\xi|^\rho}$$

for all $\xi \in \mathbf{R}^n$, $1 \le j, l \le n$, $k = 0, 1$, we estimate the first summand I_1 by the Young's inequality

$$\begin{aligned}
\|I_1\|_{\mathbf{L}^1_\xi} &\le C \int \tfrac{dy}{|y|^{1+\mu}} \int_{|\xi| \le 2|y|} \left(\left|\phi\left(\widetilde{\xi}(y)\right)\right| + |\phi(\xi)| \right) d\xi \\
&\le C \int \tfrac{dy}{|y|^{1+\mu}} \int_{|\xi| \le 2|y|} \left(\left|\widetilde{\xi}(y)\right|^{\rho-n} e^{-C|\widetilde{\xi}(y)|^\rho} + |\xi|^{\rho-n} e^{-C|\xi|^\rho} \right) d\xi \\
&\le C \int_{|y| \le 1} |y|^{\rho-1-\mu} dy + C \int_{|y|>1} |y|^{-1-\mu} dy \le C
\end{aligned}$$

since $\mu < \rho$. In the case $|y| \le \frac{|\xi|}{2}$ we have with $\xi^* = (\xi_1, ..., \xi_l - \lambda y, ..., \xi_n)$, $\lambda \in (0,1)$

$$\begin{aligned}
\left| \phi\left(\widetilde{\xi}(y)\right) - \phi(\xi) \right| &\le C |y| \left|\partial_{\xi_l} \phi(\xi^*)\right| \\
&\le C |y| |\xi^*|^{\rho-n-1} e^{-C|\xi^*|^\rho} \le C |y| |\xi|^{\rho-n-1} e^{-C|\xi|^\rho}
\end{aligned}$$

for all $\xi \in \mathbf{R}^n$, $|y| \le \frac{|\xi|}{2}$, since $|\xi^*| \le |\xi| + |y| \le \frac{3}{2}|\xi|$ and $|\xi^*| \ge |\xi| - |y| \ge \frac{1}{2}|\xi|$. Whence we get

$$\begin{aligned}
\|I_2\|_{\mathbf{L}^1_\xi} &\le C \left\| \int_{|y| \le \frac{|\xi|}{2}} |\xi|^{\rho-n-1} e^{-C|\xi|^\rho} |y|^{-\mu} dy \right\|_{\mathbf{L}^1_\xi} \\
&\le C \left\| |\xi|^{\rho-n-\mu} e^{-C|\xi|^\rho} \right\|_{\mathbf{L}^1_\xi} \left\| |\xi|^{\mu-1} \int_{|y| \le \frac{|\xi|}{2}} |y|^{-\mu} dy \right\|_{\mathbf{L}^\infty_\xi} \le C.
\end{aligned}$$

Therefore estimate (6) is true. By virtue of (6) we find

$$t^{-\frac{n}{\rho}} \left\| G\left(t^{-\frac{1}{\rho}}(\cdot)\right) \right\|_{\mathbf{L}^1} = \|G(\cdot)\|_{\mathbf{L}^1} \le C \left\| \langle x \rangle^{-n-\mu} \right\|_{\mathbf{L}^1_x} \le C,$$

hence by Young's inequality we obtain the first estimate of the lemma

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^p} \le C \left\| t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) \right\|_{\mathbf{L}^1} \|\phi\|_{\mathbf{L}^p} \le C \|\phi\|_{\mathbf{L}^p}$$

for all $t > 0$, where $1 \le p \le \infty$.

We write

$$\begin{aligned}
&|x|^\omega \left(\mathcal{G}(t)\phi - \vartheta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(x)\right) \right) \\
&= t^{-\frac{n}{\rho}} \int |x|^\omega \left(G\left(t^{-\frac{1}{\rho}}(x-y)\right) - G\left(t^{-\frac{1}{\rho}} x\right) \right) \phi(y) dy
\end{aligned}$$

for any $\omega \in [0,a]$, $a \in (0,1)$ .We have

$$|G(x-y) - G(x)| \cdot \ |G(x-y)| + |G(x)| \cdot \ C\left(\langle x-y\rangle^{-n-\mu} + \langle x\rangle^{-n-\mu}\right)$$

for all $x, y \in \mathbf{R}^n$, and applying the Lagrange finite differences Theorem, in view of (6) we obtain

$$|G(x-y) - G(x)| \cdot \ |y|\,|\nabla_x G(x^*)| \cdot \ C\,|y|\,\langle x^*\rangle^{-n-\mu-1} \cdot \ C\,|y|\,\langle x\rangle^{-n-\mu-1}$$

for all $x, y \in \mathbf{R}^n$, $|y| \cdot \ \frac{|x|}{2}$, therefore

$$|x|^{\omega}\,|G(x-y) - G(x)| \cdot \ C\,|x|^{\omega}\,|y|^{a}\,\langle x\rangle^{-n-\mu-a} \cdot \ C\,|y|^{a}\,\langle x\rangle^{-n-\mu}$$

for all $x, y \in \mathbf{R}^n$, $|y| \cdot \ \frac{|x|}{2}$. And if $|y| \geq \frac{|x|}{2}$ we get

$$\begin{aligned} |x|^{\omega}\,|G(x-y) - G(x)| &\cdot \ C\,|y|^{\omega}\,(|G(x-y)| + |G(x)|) \\ &\cdot \ C\,|y|^{a}\left(\langle x-y\rangle^{-n-\mu} + \langle x\rangle^{-n-\mu}\right). \end{aligned}$$

Thus we obtain

$$\begin{aligned} &|x|^{\omega}\left(G\left(t^{-\frac{1}{\rho}}(x-y)\right) - G\left(t^{-\frac{1}{\rho}}x\right)\right) \\ &\cdot \ C t^{\frac{\omega-a}{\rho}}\,|y|^{a}\left(\left\langle t^{-\frac{1}{\rho}}(x-y)\right\rangle^{-n-\mu} + \left\langle t^{-\frac{1}{\rho}}x\right\rangle^{-n-\mu}\right) \end{aligned}$$

for all $x, y \in \mathbf{R}^n$. Whence

$$\begin{aligned} &\left\|\left|\cdot\right|^{\omega}\left(\mathcal{G}(t)\,\phi - \vartheta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right)\right)\right\|_{\mathbf{L}^p} \\ &\cdot \ C t^{-\frac{n}{\rho}+\frac{\omega-a}{\rho}}\left\|\int\left(\left\langle t^{-\frac{1}{\rho}}(x-y)\right\rangle^{-n-\mu} + \left\langle t^{-\frac{1}{\rho}}x\right\rangle^{-n-\mu}\right)|y|^{a}\,\phi(y)dy\right\|_{\mathbf{L}^p} \\ &\cdot \ C t^{-\frac{n}{\rho}+\frac{\omega-a}{\rho}}\left\|\left\langle t^{-\frac{1}{\rho}}(\cdot)\right\rangle^{-n-\mu}\right\|_{\mathbf{L}^p}\|\langle\cdot\rangle^{a}\phi\|_{\mathbf{L}^1} \cdot \ C t^{\frac{n}{\rho}\left(\frac{1}{p}-1\right)+\frac{\omega-a}{\rho}}\|\langle\cdot\rangle^{a}\phi\|_{\mathbf{L}^1} \end{aligned}$$

for all $t > 0$, where $1 \cdot \ p \cdot \ \infty$, $\omega \in [0,a]$, $a \in (0,1)$. Thus the second estimate of the lemma is true. Lemma 2.1 is proved. ■

Remark 2.1. The first estimate of Lemma 2.1 was shown in [19] or [23]. Here we give a simpler proof.

In the next lemma we estimate the Green operator in our basic norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\left((1+t)^{\frac{n}{\rho}}\|\phi(t)\|_{\mathbf{L}^\infty} + \|\phi(t)\|_{\mathbf{L}^1} + (1+t)^{-\frac{a}{\rho}}\|\phi(t)\|_{\mathbf{L}^{1,a}}\right),$$

where $a \in (0,1)$. Define the function $g(t)$

$$g(t) = 1 + \Box\log(1+t)$$

with some $\square > 0$.

Lemma 2.2. *Let the function* $f(t,x)$ *have a zero mean value* $\int f(t,x)dx = 0$. *Then the following inequality*

$$\left\| g^k(t) \int_0^t g^{-k}(\tau)\, \mathcal{G}\in(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{X}} \cdot \ C \left\| (1+t) f(t,x) \right\|_{\mathbf{X}}$$

is valid for $k = 0, 1$, *provided that the right-hand side is finite.*

Proof. View of the estimate $g^{-1}(\tau) \cdot \ C$ and Lemma 2.1 we get

$$\left\| \int_0^t g^{-k}(\tau)\, \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^\infty} + \left\| \int_0^t g^{-k}(\tau)\, \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^{1,a}}$$
$$\cdot \ C\varepsilon \int_0^4 (1+\tau)^{-1+\frac{a}{\rho}} d\tau \cdot \ C\varepsilon g^{-k}(t)$$

for all $0 \cdot \ t \cdot \ 4$. We now consider $t > 4$. Via the condition of the lemma for the function $g(t)$ we have the estimate $(1+t)^{-\frac{a}{2\rho}} \cdot \ Cg^{-1}(t)$ and

$$\sup_{\tau\in[\sqrt{t},t]} g^{-1}(\tau) \ \cdot \ C\left(1 + \square \log\left(1+\sqrt{t}\right)\right)^{-1}$$
$$\cdot \ C\left(1 + \tfrac{\square}{2} \log(1+t)\right)^{-1} \cdot \ Cg^{-1}(t),$$

hence by virtue of Lemma 2.1 we obtain

$$\left\| \int_0^t g^{-k}(\tau)\, \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^p}$$
$$\cdot \ C \int_0^{\sqrt{t}} (t-\tau)^{\frac{n}{\rho}\left(\frac{1}{p}-1\right)-\frac{a}{\rho}} \tau^{\frac{a}{\rho}-1} d\tau \sup_{\tau>0} (1+\tau)^{-\frac{a}{\rho}} \left\| (1+\tau) f(\tau) \right\|_{\mathbf{L}^{1,a}}$$
$$+ Cg^{-k}(t) \int_{\sqrt{t}}^{\frac{t}{2}} (t-\tau)^{\frac{n}{\rho}\left(\frac{1}{p}-1\right)-\frac{a}{\rho}} \tau^{\frac{a}{\rho}-1} d\tau \sup_{\tau>0} (1+\tau)^{-\frac{a}{\rho}} \left\| (1+\tau) f(\tau) \right\|_{\mathbf{L}^{1,a}}$$
$$+ Cg^{-k}(t) \int_{\frac{t}{2}}^{t} \tau^{\frac{n}{\rho}\left(\frac{1}{p}-1\right)-1} d\tau \sup_{\tau>0} (1+\tau)^{-\frac{n}{\rho}\left(\frac{1}{p}-1\right)} \left\| (1+\tau) f(\tau) \right\|_{\mathbf{L}^p}$$
$$\cdot \ C\left(t^{\frac{n}{\rho}\left(\frac{1}{p}-1\right)-\frac{a}{2\rho}} + g^{-k}(t)\, t^{\frac{n}{\rho}\left(\frac{1}{p}-1\right)} \right) \left\| (1+t) f(t,x) \right\|_{\mathbf{X}}$$
$$\cdot \ Cg^{-k}(t)\, t^{\frac{n}{\rho}\left(\frac{1}{p}-1\right)} \left\| (1+t) f(t,x) \right\|_{\mathbf{X}}$$

for $1 \cdot \ p \cdot \ \infty$, and

$$\left\| \int_0^t g^{-k}(\tau)\, \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^{1,a}}$$
$$\cdot \ C \int_0^{\sqrt{t}} \tau^{\frac{a}{\rho}-1} d\tau \sup_{\tau>0} (1+\tau)^{-\frac{a}{\rho}} \left\| f(\tau) \right\|_{\mathbf{L}^{1,a}}$$
$$+ Cg^{-k}(t) \int_{\sqrt{t}}^{t} \tau^{\frac{a}{\rho}-1} d\tau \sup_{\tau>0} (1+\tau)^{-\frac{a}{\rho}} \left\| f(\tau) \right\|_{\mathbf{L}^{1,a}}$$
$$\cdot \ C\varepsilon \left(t^{\frac{a}{2\rho}} + g^{-k}(t)\, t^{\frac{a}{\rho}} \right) \left\| (1+t) f(t,x) \right\|_{\mathbf{X}}$$
$$\cdot \ Cg^{-k}(t)\, t^{\frac{a}{\rho}} \left\| (1+t) f(t,x) \right\|_{\mathbf{X}}$$

for all $t > 4$. Hence the result of the lemma follows. Lemma 2.2 is proved. ■

Next lemma will be used in the proof of the theorem to evaluate large time behavior of the mean value of the nonlinearity in equation (2). We use the notation $\eta = \rho\ \mathrm{Re}\beta\delta\left(\alpha,\rho,\sigma\right) > 0$,

$$\delta\left(\alpha,\rho,\sigma\right) = \int \left|G\left(x\right)\right|^{\sigma} G\left(x\right) dx$$

and

$$\chi_{\sigma}\left(t\right) = g\left(t\right) \ \text{ if } \sigma = \frac{\rho}{n} \text{ and } \chi_{\sigma}\left(t\right) = 1 + \frac{n\sigma\left|\theta\right|^{\sigma}\eta}{\rho - n\sigma} t^{1-\frac{\sigma}{\rho}n} \text{ if } \sigma \in \left(0, \frac{\rho}{n}\right),$$

where $g\left(t\right) = 1 + \left|\theta\right|^{\sigma}\eta\log\left(1+t\right)$.

Lamma 2.3. *Assume that* $v_0 \in \mathbf{L}^{\infty} \cap \mathbf{L}^{1,a}$*, the norm* $\left\|v_0\right\|_{\mathbf{L}^{\infty}} + \left\|v_0\right\|_{\mathbf{L}^{1,a}} = \varepsilon$ *is sufficiently small and* $\int v_0\left(x\right)dx = \theta = \left|\int u_0\left(x\right)dx\right|$. *Let function* $v\left(t,x\right)$ *satisfy the estimates*

$$\left\|v\right\|_{\mathbf{L}^{\infty}} \le C\varepsilon\left(1+t\right)^{-\frac{1}{\sigma}}, \quad \left\|v\right\|_{\mathbf{L}^{1}} \le C\varepsilon \quad \text{and}$$
$$\left\|v\left(t\right) - \mathcal{G}\left(t\right)v_0\right\|_{\mathbf{L}^{1}} \le C\varepsilon^{1+\sigma} g^{-k}\left(t\right),$$

where $k = 1$ *in the critical case* $\sigma = \frac{\rho}{n}$ *and* $k = 0$ *in the sub critical case* $0 < \sigma < \frac{\rho}{n}$.

Then the inequality

$$\left|1 + \frac{\sigma}{\theta}\int_0^t d\tau \mathrm{Re}\int \beta\left|v\right|^{\sigma} v(\tau,x)dx - \chi_{\sigma}\left(t\right)\right| \le \begin{cases} C\varepsilon^{2\sigma}\log\chi_{\sigma}\left(t\right) \ \text{ if } \sigma = \dfrac{\rho}{n}, \\ C\varepsilon^{2\sigma}\chi_{\sigma}\left(t\right) \ \text{ if } \sigma \in \left(0, \dfrac{\rho}{n}\right) \end{cases} \tag{7}$$

is valid for all $t > 0$.

Proof. In view of the condition $\left\|v\right\|_{\mathbf{L}^{\infty}} + \left\|v\right\|_{\mathbf{L}^{1}} \le C\varepsilon$ we get

$$\left|\frac{\sigma}{\theta}\int_0^t d\tau \mathrm{Re}\int \beta\left|v\right|^{\sigma} v(\tau,x)dx\right| \le C\varepsilon^{\sigma} t, \tag{8}$$

whence estimate (7) follows for all $0 < t < 1$.

We now consider the case $t \ge 1$. By the second estimate of Lemma 2.1 we get

$$\left\|\mathcal{G}\left(t\right)v_0 - \theta t^{-\frac{n}{\rho}} G\left(x t^{-\frac{1}{\rho}}\right)\right\|_{\mathbf{L}^{1}} \le C\varepsilon t^{-\frac{a}{\rho}},$$

where $\theta = \int v_0(x)\,dx$. Hence we find

$$
\begin{aligned}
&\left\| |v|^{\rho} v - |\theta|^{\sigma} \theta t^{-\frac{1+\sigma}{\rho}n} \left| G\left(xt^{-\frac{1}{\rho}}\right)\right|^{\sigma} G\left(xt^{-\frac{1}{\rho}}\right)\right\|_{\mathbf{L}^1} \\
\cdot\quad & C\left(\|v(t) - \mathcal{G}(t) v_0\|_{\mathbf{L}^1} + \left\|\mathcal{G}(t) v_0 - \theta t^{-\frac{n}{\rho}} G\left(xt^{-\frac{1}{\rho}}\right)\right\|_{\mathbf{L}^1}\right) \\
& \times \left(\|v\|^{\sigma}_{\mathbf{L}^\infty} + \|\mathcal{G} v_0\|^{\sigma}_{\mathbf{L}^\infty} + |\theta|^{\sigma} t^{-\frac{n\sigma}{\rho}} \|G\|^{\sigma}_{\mathbf{L}^\infty}\right) \\
\cdot\quad & C\varepsilon^{1+2\sigma} t^{-\frac{n\sigma}{\rho}} g^{-k}(t) + C\varepsilon^{1+\sigma} t^{-\frac{\sigma n + a}{\rho}}
\end{aligned}
$$

for all $t \geq 1$, where $k = 1$ if $\sigma = \frac{\rho}{n}$ and $k = 0$ if $0 < \sigma < \frac{\rho}{n}$. Since

$$
t^{-\frac{n}{\rho}} \int \left| G\left(xt^{-\frac{1}{\rho}}\right)\right|^{\sigma} G\left(xt^{-\frac{1}{\rho}}\right) dx = \int |G(x)|^{\sigma} G(x)\,dx = \delta(\alpha, \rho, \sigma),
$$

and $\mathrm{Re}\beta\delta(\alpha, \rho, \sigma) = \frac{\eta}{\rho} > 0$ we get

$$
\begin{aligned}
&\left| \mathrm{Re} \int \beta |v|^{\sigma} v(t, x)\,dx - |\theta|^{\sigma} \theta t^{-\frac{\sigma}{\rho}n} \frac{\eta}{\rho} \right| \\
\cdot\quad & C \left\| |v|^{\rho} v - |\theta|^{\sigma} \theta t^{-\frac{1+\sigma}{\rho}n} \left| G\left(xt^{-\frac{1}{\rho}}\right)\right|^{\sigma} G\left(xt^{-\frac{1}{\rho}}\right)\right\|_{\mathbf{L}^1} \\
\cdot\quad & C\varepsilon^{1+2\sigma} t^{-\frac{n\sigma}{\rho}} g^{-k}(t) + C\varepsilon^{1+\sigma} t^{-\frac{\sigma n + a}{\rho}}
\end{aligned}
$$

for all $t \geq 1$, where $0 < \sigma \cdot\ \rho$. Therefore

$$
\begin{aligned}
&\left| \frac{\rho}{\theta} \int_1^t d\tau \mathrm{Re} \int \beta |v|^{\sigma} v(\tau, x) dx - |\theta|^{\sigma} \eta \log t \right| \\
\cdot\quad & \int_1^t \frac{C\varepsilon^{2\sigma} d\tau}{\tau\left(1 + |\theta|^{\sigma} \eta \log(1+\tau)\right)} + C\varepsilon^{\sigma} \int_1^t \tau^{-1-\frac{a}{\sigma}} d\tau \\
\cdot\quad & C\varepsilon^{2\sigma} \log\left(1 + |\theta|^{\sigma} \eta \log(1+t)\right) + C\varepsilon^{\sigma}
\end{aligned}
$$

for all $t \geq 1$. Thus in view of (8) we obtain (7) in the case $\sigma = \frac{\rho}{n}$. In the same manner we have the inequality

$$
\begin{aligned}
&\left| \frac{\sigma}{\theta} \int_0^t d\tau \mathrm{Re} \int \beta |v|^{\sigma} v(\tau, x) dx - |\theta|^{\sigma} \frac{n\sigma\eta}{\rho - n\sigma} t^{1-\frac{\sigma}{\rho}n} \right| \\
\cdot\quad & C\varepsilon^{2\sigma} t^{1-\frac{\sigma}{\rho}n} + C\varepsilon^{\sigma} t^{1-\frac{\sigma n + a}{\rho}}
\end{aligned}
$$

for all $t \geq 0$, which in view of (8) implies (7) in the case $0 < \sigma < \frac{\rho}{n}$. Lemma 2.3 is proved. ■

3. Proof of Theorem 1.1

We make a change of the dependent variable $u(t,x) = v(t,x)\, e^{-\varphi(t)+i\psi(t)}$ as in [11]. Then for the new function $v(t,x)$ we get the following equation

$$\partial_t v + \alpha(-\Delta)^{\frac{\rho}{2}} v + \beta e^{-\sigma\varphi} |v|^{\sigma} v + \gamma e^{-\varkappa\varphi} |v|^{\varkappa} v - (\varphi' - i\psi')v = 0.$$

We assume that

$$\int \left(\beta e^{-\sigma\varphi} |v|^{\sigma} v + \gamma e^{-\varkappa\varphi} |v|^{\varkappa} v - (\varphi' - i\psi')v\right) dx = 0$$

then the mean value of new function $v(t,x)$ satisfies a conservation law: $\frac{d}{dt}\int v(t,x)\, dx = 0$, hence $\widehat{v}(t,0) = \widehat{v}_0(0)$ for all $t > 0$. We can choose $\varphi(0) = 0$ and $\psi(0)$ such that $\hat{v}_0(0) = |\hat{u}_0(0)| = \theta(2\pi)^{-\frac{n}{2}} > 0$. Thus we consider the Cauchy problem for the new dependent variables $(v(t,x), \varphi(t))$

$$(9)\qquad \begin{cases} \partial_t v + \alpha(-\Delta)^{\frac{\rho}{2}} v = -\beta e^{-\sigma\varphi}\left(|v|^{\sigma} - \frac{1}{\theta}\int |v|^{\sigma} v dx\right) v \\ \qquad -\gamma e^{-\varkappa\varphi}\left(|v|^{\varkappa} - \frac{1}{\theta}\int |v|^{\varkappa} v dx\right) v, \\ \partial_t \varphi(t) = \frac{1}{\theta} e^{-\sigma\varphi}\left(\mathrm{Re}\beta \int |v|^{\sigma} v dx + e^{(\sigma-\varkappa)\varphi}\mathrm{Re}\gamma \int |v|^{\varkappa} v dx\right), \\ v(0,x) = v_0(x),\ \varphi(0) = 0. \end{cases}$$

We denote $h(t) = e^{\sigma\varphi(t)}$ and write (9) as

$$(10)\qquad \begin{cases} \partial_t v + \alpha(-\Delta)^{\frac{\rho}{2}} v = F(v,h),\ v(0,x) = v_0(x), \\ \partial_t h = \frac{\sigma}{\theta}\left(\mathrm{Re}\beta \int |v|^{\sigma} v dx + h^{1-\frac{\varkappa}{\sigma}}\mathrm{Re}\gamma \int |v|^{\varkappa} v dx\right),\ h(0) = 1, \end{cases}$$

where

$$F(v,h) = -\beta h^{-1}\left(|v|^{\sigma} - \frac{1}{\theta}\int |v|^{\sigma} v dx\right) v - \gamma h^{-\frac{\varkappa}{\sigma}}\left(|v|^{\varkappa} - \frac{1}{\theta}\int |v|^{\varkappa} v dx\right) v.$$

We note that the mean value of the nonlinearity $\widehat{F(v,h)}(t,0) = 0$ for all $t > 0$. It is expected that the second summand $\gamma h^{-\frac{\varkappa}{\sigma}} |v|^{\varkappa} v$ of the nonlinearity $F(v,h)$ decays in time more rapidly than the first one $\beta h^{-1} |v|^{\sigma} v$. We now prove the existence of the solution $(v(t,x), h(t))$ for the Cauchy problem (10) by the successive approximations $(v_m(t,x), h_m(t))$, $m = 1, 2, ...$, defined as follows

$$(11)\qquad \begin{cases} \partial_t v_m + \alpha(-\Delta)^{\frac{\rho}{2}} v_m = F(v_{m-1}, h_{m-1}), \\ \partial_t h_m = \dfrac{\sigma}{\theta}\left(\mathrm{Re}\beta \displaystyle\int |v_{m-1}|^{\sigma} v_{m-1} dx + h_{m-1}^{1-\frac{\varkappa}{\sigma}}\mathrm{Re}\gamma \int |v_{m-1}|^{\varkappa} v_{m-1} dx\right), \\ v_m(0,x) = v_0(x),\ h_m(0) = 1, \end{cases}$$

for all $m \geq 2$, where $v_1 = \mathcal{G}(t)\, v_0,\ h_1 = \chi_\sigma(t)$,

$$\chi_\sigma(t) = g(t) \ \text{ if } \sigma = \frac{\rho}{n} \text{ and } \chi_\sigma(t) = 1 + \frac{n\sigma\,|\theta|^\sigma\,\eta}{\rho - n\sigma} t^{1-\frac{\sigma}{\rho}n} \ \text{if } \sigma \in \left(0, \frac{\rho}{n}\right),$$

where $g(t) = 1 + |\theta|^\sigma\,\eta \log(1+t)$. We now prove by induction the following estimates

$$\begin{aligned} &\|v_m\|_{\mathbf{X}} \le C\varepsilon, \ \ \|v_m(t) - \mathcal{G}(t)\,v_0\|_{\mathbf{L}^1} \le C\varepsilon^{1+\sigma} g^{-k}(t), \\ &|h_m(t) - \chi_\sigma(t)| \le \begin{cases} C\varepsilon^{2\sigma}\log\chi_\sigma(t) \ \text{ if } \sigma = \frac{\rho}{n}, \\ C\varepsilon^{2\sigma}\chi_\sigma(t) \ \text{ if } \sigma \in \left(\frac{\rho}{n} - C\varepsilon^\sigma, \frac{\rho}{n}\right) \end{cases} \end{aligned} \tag{12}$$

for all $m \geq 1$, where $k = 1$ in the critical case $\sigma = \frac{\rho}{n}$ and $k = 0$ in the sub critical case $0 < \sigma < \frac{\rho}{n}$, the norm $\|\cdot\|_{\mathbf{X}}$ is defined as above by

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\left((1+t)^{\frac{n}{\rho}}\|\phi(t)\|_{\mathbf{L}^\infty} + \|\phi(t)\|_{\mathbf{L}^1} + (1+t)^{-\frac{a}{\rho}}\|\phi(t)\|_{\mathbf{L}^{1,a}}\right),$$

where $a \in (0, \min(1,\rho))$. By virtue of Lemma 2.1 we have

$$\begin{aligned} &\|\mathcal{G}(t)\,v_0\|_{\mathbf{L}^\infty} \le C\varepsilon(1+t)^{-\frac{n}{\rho}}, \ \ \|\mathcal{G}(t)\,v_0\|_{\mathbf{L}^1} \le C\varepsilon, \\ &\left\||\cdot|^a\left(\mathcal{G}(t)\,v_0 - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right)\right)\right\|_{\mathbf{L}^1} \le C\varepsilon \end{aligned}$$

and

$$\left\|t^{-\frac{n}{\rho}}\,|\cdot|^a\,G\left(t^{-\frac{1}{\rho}}(\cdot)\right)\right\|_{\mathbf{L}^1} \le Ct^{\frac{a}{\rho}}.$$

Therefore estimates (12) are valid for $m = 1$. We assume that estimates (12) are true with m replaced by $m-1$. The integral equation associated with (11) is written as

$$\begin{cases} v_m(t) = \mathcal{G}(t)\,v_0 + \displaystyle\int_0^t \mathcal{G}(t-\tau) F\left(v_{m-1}(\tau), h_{m-1}(\tau)\right) d\tau, \\ h_m(t) = 1 + \dfrac{\sigma}{\theta}\displaystyle\int_0^t d\tau \left(\mathrm{Re}\beta \int |v_{m-1}|^\sigma\, v_{m-1}dx\right. \\ \qquad \left. + h_{m-1}^{1-\frac{\varkappa}{\sigma}}\mathrm{Re}\gamma \displaystyle\int |v_{m-1}|^\varkappa\, v_{m-1}dx\right). \end{cases}$$

Note that in the critical case $\sigma = \frac{\rho}{n}$ we have

$$
(13)\qquad
\begin{aligned}
&\|F(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^{\infty}}\\
\cdot\quad & Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\mathbf{L}^{\infty}}^{1+\sigma}\left(1+\frac{1}{\theta}\|v_{m-1}(t)\|_{\mathbf{L}^{1}}\right)\\
\cdot\quad & C\varepsilon^{1+\sigma}(1+t)^{-1-\frac{1}{\sigma}}g^{-1}(t),\\
&\|F(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^{1}}\\
\cdot\quad & Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\mathbf{L}^{\infty}}^{\sigma}\left(\|v_{m-1}(t)\|_{\mathbf{L}^{1}}+\frac{1}{\theta}\|v_{m-1}(t)\|_{\mathbf{L}^{1}}^{2}\right)\\
\cdot\quad & C\varepsilon^{1+\sigma}(1+t)^{-1}g^{-1}(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|F(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^{1,a}}\\
\cdot\quad & Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\mathbf{L}^{\infty}}^{\sigma}\|v_{m-1}(t)\|_{\mathbf{L}^{1,a}}\left(1+\frac{1}{\theta}\|v_{m-1}(t)\|_{\mathbf{L}^{1}}\right)\\
\cdot\quad & C\varepsilon^{1+\rho}(1+t)^{-1+\frac{a}{\rho}}g^{-1}(t)
\end{aligned}
$$

for all $t>0$, provided that $(v_{m-1}(t), h_{m-1}(t))$ satisfies (12). Similarly in the sub critical case $\sigma\in\left(0,\frac{\rho}{n}\right)$ we obtain

$$
(14)\qquad
\begin{aligned}
&\|F(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^{\infty}}\\
\cdot\quad & Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\mathbf{L}^{\infty}}^{1+\sigma}\left(1+\tfrac{1}{\theta}\|v_{m-1}(t)\|_{\mathbf{L}^{1}}\right)\\
\cdot\quad & C\varepsilon^{1+\sigma}(1+t)^{-\frac{1+\sigma}{\rho}n}\left(1+\tfrac{n|\theta|^{\sigma}}{\rho-n\sigma}t^{1-\frac{\sigma}{\rho}n}\right)^{-1}\\
\cdot\quad & C\varepsilon\left(\tfrac{\rho}{n}-\sigma\right)(1+t)^{-1-\frac{n}{\rho}},\\
&\|F(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^{1}}\\
\cdot\quad & Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\mathbf{L}^{\infty}}^{\sigma}\left(\|v_{m-1}(t)\|_{\mathbf{L}^{1}}+\tfrac{1}{\theta}\|v_{m-1}(t)\|_{\mathbf{L}^{1}}^{2}\right)\\
\cdot\quad & C\varepsilon\left(\tfrac{\rho}{n}-\sigma\right)(1+t)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|F(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^{1,a}}\\
\cdot\quad & Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\mathbf{L}^{\infty}}^{\sigma}\|v_{m-1}(t)\|_{\mathbf{L}^{1,a}}\left(1+\frac{1}{\theta}\|v_{m-1}(t)\|_{\mathbf{L}^{1}}\right)\\
\cdot\quad & C\varepsilon\left(\frac{\rho}{n}-\sigma\right)(1+t)^{-1+\frac{\alpha}{\rho}}
\end{aligned}
$$

for all $t>0$. This yields the estimate

$$
\left\|(1+t)\,g^{k}(t)\,F(v_{m-1}(t), h_{m-1}(t))\right\|_{\mathbf{X}}\cdot\ C\varepsilon^{1+\sigma}
$$

if we suppose that $\rho - n\sigma \cdot \ C\varepsilon^{\sigma}$. Since $F\left(v_{m-1}\left(\tau\right), h_{m-1}\left(\tau\right)\right)$ have the zero mean value we get via Lemma 2.2

$$\left\| g^{k}\left(t\right)\int_{0}^{t} g^{-k}\left(\tau\right)\mathcal{G}(t-\tau)F\left(v_{m-1}\left(\tau\right), h_{m-1}\left(\tau\right)\right)d\tau\right\|_{\mathbf{X}} \cdot \ C\varepsilon^{1+\sigma}$$

whence it follows that

$$\left\|v_{m}\right\|_{\mathbf{X}} \cdot \ C\varepsilon, \left\|v_{m}\left(t\right) - \mathcal{G}\left(t\right)v_{0}\right\|_{\mathbf{L}^{1}} \cdot \ C\varepsilon^{1+\sigma}g^{-k}\left(t\right).$$

By virtue of Lemma 2.3 we find that

$$\left|h_{m}\left(t\right) - \chi_{\sigma}\left(t\right)\right| \cdot \ \begin{cases} C\varepsilon^{2\sigma}\log\chi_{\sigma}\left(t\right) \text{ if } \sigma = \frac{\rho}{n}, \\ C\varepsilon^{2\sigma}\chi_{\sigma}\left(t\right) \text{ if } \sigma \in \left(\frac{\rho}{n} - C\varepsilon^{\sigma}, \frac{\rho}{n}\right) \end{cases}$$

for all $t > 0$. Thus by induction we see that estimates (12) are valid for all $m \geq 1$. In the same way by induction we can prove that

$$\begin{aligned} \left\|v_{m} - v_{m-1}\right\|_{\mathbf{X}} \ &\cdot \ \frac{1}{4}\left\|v_{m-1} - v_{m-2}\right\|_{\mathbf{X}}, \\ \sup_{t>0}\chi_{\sigma}^{-1}\left(t\right)\left|h_{m}\left(t\right) - h_{m-1}\left(t\right)\right| \ &\cdot \ \frac{1}{4}\left\|v_{m-1} - v_{m-2}\right\|_{\mathbf{X}} \\ &\quad + \frac{1}{4}\sup_{t>0}\chi_{\sigma}^{-1}\left(t\right)\left|h_{m-1}\left(t\right) - h_{m-2}\left(t\right)\right| \end{aligned}$$

for all $m > 2$. Therefore taking limits

$$\lim_{m\to\infty} v_{m}\left(t, x\right) = v\left(t, x\right) \text{ and } \lim_{m\to\infty} h_{m}\left(t\right) = h\left(t\right)$$

we obtain a unique solution $v\left(t, x\right) \in \mathbf{X}$, $h\left(t\right) = e^{\sigma\varphi(t)} \in \mathbf{C}\left(0, \infty\right)$ satisfying equations

$$(15) \qquad \begin{cases} v\left(t\right) = \mathcal{G}\left(t\right)v_{0} + \int_{0}^{t}\mathcal{G}(t-\tau)F\left(v\left(\tau\right), h\left(\tau\right)\right)d\tau, \\ h\left(t\right) = 1 + \frac{\sigma}{\theta}\int_{0}^{t}d\tau\left(\mathrm{Re}\beta\int\left|v\right|^{\sigma}vdx + h^{1-\frac{\varkappa}{\sigma}}\mathrm{Re}\gamma\int\left|v\right|^{\varkappa}vdx\right), \end{cases}$$

and estimates

$$\left\|v\left(t\right) - \mathcal{G}\left(t\right)v_{0}\right\|_{\mathbf{L}^{1}} \cdot \ C\varepsilon^{1+\sigma}g^{-k}\left(t\right)$$

and

$$(16) \qquad \left|h\left(t\right) - \chi_{\sigma}\left(t\right)\right| \cdot \ \begin{cases} C\varepsilon^{2\sigma}\log\chi_{\sigma}\left(t\right) \text{ if } \sigma = \frac{\rho}{n}, \\ C\varepsilon^{2\sigma}\chi_{\sigma}\left(t\right) \text{ if } \sigma \in \left(\frac{\rho}{n} - C\varepsilon^{\sigma}, \frac{\rho}{n}\right). \end{cases}$$

We also have by applying (13) and (14) to (15)

$$\|v(t)-\mathcal{G}(t)v_0\|_{\mathbf{L}^\infty}\leq\begin{cases}C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{\sigma}}g^{-1}(t)\ \text{ if }\sigma=\frac{\rho}{n},\\ C\varepsilon^{1+\sigma}(1+t)^{-\frac{n}{\rho}}\ \text{ if }\sigma\in\left(\frac{\rho}{n}-C\varepsilon^{\sigma},\frac{\rho}{n}\right),\end{cases}\tag{17}$$

and by the definition of ψ we see that

$$\psi(t)=\arg\widehat{u_0(0)}-\frac{1}{\theta}\int_0^t e^{-\sigma\varphi}\left(\mathrm{Im}\beta\int|v|^{\sigma}vdx+e^{(\sigma-\square)\varphi}\mathrm{Im}\gamma\int|v|^{\square}vdx\right).$$

By the time decay property of the solution v we have

$$\left|\psi(t)-\arg\widehat{u_0(0)}+|\theta|^{\sigma}\widetilde{\eta}\int_0^t h^{-\frac{1}{\sigma}}(\tau)(1+\tau)^{-\frac{\sigma}{\rho}n}d\tau\right|\leq C\varepsilon^{\square}t^{-\frac{\square-\sigma}{\rho}n},$$

where $\widetilde{\eta}=\mathrm{Im}\beta\delta(\alpha,\rho,\sigma)$. Hence by (16)

$$\begin{aligned}&\left|\psi(t)-\arg\widehat{u_0(0)}+|\theta|^{\sigma}\widetilde{\eta}\textstyle\int_0^t\chi_\sigma^{-\frac{1}{\sigma}}(\tau)(1+\tau)^{-\frac{\sigma}{\rho}n}d\tau\right|\\ \leq\ & C|\theta|^{\sigma}\widetilde{\eta}\textstyle\int_0^t\chi_\sigma^{-1-\frac{1}{\sigma}}(\tau)|h(\tau)-\chi_\sigma(\tau)|(1+\tau)^{-\frac{\sigma}{\rho}n}d\tau\\ \leq\ &\begin{cases}C\int_0^t\chi_\sigma^{-1-\frac{1}{\sigma}}(\tau)(\log\chi_\sigma(\tau))(1+\tau)^{-1}d\tau\text{ if }\sigma=\frac{\rho}{n},\\ C\int_0^t\chi_\sigma^{-\frac{1}{\sigma}}(\tau)(1+\tau)^{-\frac{\sigma}{\rho}n}d\tau\text{ if }\sigma\in\left(\frac{\rho}{n}-C\varepsilon^{\sigma},\frac{\rho}{n}\right)\end{cases}\end{aligned}\tag{18}$$

for large $t>0$. Then via formulas $u(t,x)=e^{-\varphi(t)+i\psi(t)}v(t,x)=h^{-\frac{1}{\sigma}}(t)e^{i\psi(t)}v(t,x)$ we find the estimates

$$\begin{aligned}&\left\|u(t)-\theta t^{-\frac{n}{\rho}}G\left(t^{-\frac{1}{\rho}}(\cdot)\right)e^{-\varphi(t)+i\psi(t)}\right\|_{\mathbf{L}^\infty}\\ \leq\ &\left\|u(t)-(\mathcal{G}(t)v_0)e^{-\varphi(t)+i\psi(t)}\right\|_{\mathbf{L}^\infty}+\\ &\left\|\left(\mathcal{G}(t)v_0-\theta t^{-\frac{n}{\rho}}G\left(t^{-\frac{1}{\rho}}(\cdot)\right)\right)e^{-\varphi(t)+i\psi(t)}\right\|_{\mathbf{L}^\infty}\\ \leq\ &\begin{cases}C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{\sigma}}g^{-1}(t)\ \text{ if }\sigma=\frac{\rho}{n},\\ C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{\sigma}}\ \text{ if }\sigma\in\left(\frac{\rho}{n}-C\varepsilon^{\sigma},\frac{\rho}{n}\right),\end{cases}\end{aligned}\tag{19}$$

where we have used the estimate

$$\left\|\left(\mathcal{G}(t)v_0-\theta t^{-\frac{n}{\rho}}G\left(t^{-\frac{1}{\rho}}(\cdot)\right)\right)e^{-\varphi(t)+i\psi(t)}\right\|_{\mathbf{L}^\infty}\leq Ct^{-\frac{1}{\sigma}-\frac{a}{\rho}}\|\phi\|_{\mathbf{L}^{1,a}}$$

and (17). We also have by (16)

$$\left\| \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) h^{-\frac{1}{\sigma}}(t) e^{i\psi(t)} - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) \chi_\sigma^{-\frac{1}{\sigma}}(t) e^{i\psi(t)} \right\|_{\mathbf{L}^\infty}$$
$$\cdot \quad C\varepsilon t^{-\frac{n}{\rho}} \chi_\sigma^{-1-\frac{1}{\sigma}}(t) \left| h(t) - \chi_\sigma(t) \right|$$

whence via (19) it follows that

$$\left\| u(t) - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) \chi_\sigma^{-\frac{1}{\sigma}}(t) e^{i\psi(t)} \right\|_{\mathbf{L}^\infty}$$
$$\cdot \begin{cases} C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{\sigma}} g^{-\frac{1}{\sigma}}(t) \ \text{ if } \sigma = \frac{\rho}{n}, \\ C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{\sigma}} \ \text{ if } \sigma \in \left(\frac{\rho}{n} - C\varepsilon^\sigma, \frac{\rho}{n}\right). \end{cases} \tag{20}$$

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

We consider the function space $\widetilde{X}$ with norm

$$\|\phi\|_{\widetilde{\mathbf{X}}} = \sup_{t>0} \left((1+t)^{\frac{n}{\rho}} \|\phi(t)\|_{\mathbf{L}^\infty} + \|\phi(t)\|_{\mathbf{L}^1} \right)$$

and define the mapping $u = \mathcal{M}v$ as follows

$$u(t) = \mathcal{G}(t) u_0 + \int_0^t \mathcal{G}(t-\tau) F(v(\tau))\, d\tau, \tag{21}$$

where $F(v(\tau)) = \beta |v|^\sigma v + \gamma |v|^\varkappa v$, $v \in \widetilde{\mathbf{X}}$ and $\|v\|_{\widetilde{\mathbf{X}}} \cdot \ C\varepsilon$. Using Lemma 2.1 in (21) we obtain

$$\begin{aligned} \|u(t)\|_{\mathbf{L}^\infty} \cdot \ & \|\mathcal{G}(t) v_0\|_{\mathbf{L}^\infty} + \textstyle\int_0^t \|\mathcal{G}(t-\tau) F(v(\tau))\|_{\mathbf{L}^\infty} d\tau \\ \cdot \ & C\left(\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^1}\right)(1+t)^{-\frac{n}{\rho}} \\ & + C \textstyle\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{\rho}} \left(\|v(\tau)\|_{\mathbf{L}^\infty}^\sigma + \|v(\tau)\|_{\mathbf{L}^\infty}^\Box\right) \|v(\tau)\|_{\mathbf{L}^1} d\tau \\ & + C \textstyle\int_{\frac{t}{2}}^t \left(\|v(\tau)\|_{\mathbf{L}^\infty}^\sigma + \|v(\tau)\|_{\mathbf{L}^\infty}^\Box\right) \|v(\tau)\|_{\mathbf{L}^\infty} d\tau \\ \cdot \ & C\left(\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^1}\right)(1+t)^{-\frac{n}{\rho}} + C\varepsilon^{1+\sigma}(1+t)^{-\frac{n}{\rho}} \end{aligned} \tag{22}$$

and

$$\begin{aligned} \|u(t)\|_{\mathbf{L}^1} \cdot \ & \|\mathcal{G}(t) v_0\|_{\mathbf{L}^1} + \textstyle\int_0^t \|\mathcal{G}(t-\tau) F(v(\tau))\|_{\mathbf{L}^1} d\tau \\ \cdot \ & C\|u_0\|_{\mathbf{L}^1} + C \textstyle\int_0^t \left(\|v(\tau)\|_{\mathbf{L}^\infty}^\sigma + \|v(\tau)\|_{\mathbf{L}^\infty}^\Box\right) \|v(\tau)\|_{\mathbf{L}^1} d\tau \\ \cdot \ & C\|u_0\|_{\mathbf{L}^1} + C\varepsilon^{1+\sigma}, \end{aligned} \tag{23}$$

where we have used the fact that $\sigma, \square > \frac{\rho}{n}$. Hence we have by (22) and (23)

$$\|u\|_{\widetilde{\mathbf{X}}} \cdot \; C \left(\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^1} \right) + C\varepsilon^{1+\sigma} \tag{24}$$

which implies that if the initial function is sufficiently small, then the mapping $\mathcal{M}$ defined by $u = \mathcal{M}v$ is a mapping from $\widetilde{\mathbf{X}}$ to itself. In the same way as in the proof of (23) we find that $\mathcal{M}$ is a contraction mapping. Hence we have a unique solution $u \in \widetilde{\mathbf{X}}$ satisfying the integral equation

$$u(t) = \mathcal{G}(t)\, u_0 + \int_0^t \mathcal{G}(t-\tau) \left(\beta |u|^\sigma u(\tau) + \gamma |u|^\varkappa u(\tau) \right) d\tau. \tag{25}$$

Applying the same arguments as in the proof of (22), it follows that

$$\|u(t) - \mathcal{G}(t)\, u_0\| \cdot \; C\varepsilon^{1+\sigma} (1+t)^{-\frac{n}{\rho}}. \tag{26}$$

The result of the theorem follows from (26) and the second estimate of the Lemma 2.1 with $\omega = 0, a > 0$ and $p = \infty$.

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