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SELF-SIMILAR SOLUTIONS OF A SEMILINEAR HEAT EQUATION

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Abstract. In this note we classify positive solutions of an equation

$$\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u - |u|^{p-1}u = 0 \quad \text{in} \quad \mathbf{R}^N,$$

where $1 < p < (N+2)/N$.

Under the assumption that $|x|^{2/(p-1)}u(x)$ is uniformly bounded in $\mathbf{R}^N$, we show that as $r = |x|$ tends to ∞, $r^{2/(p-1)}u(r\sigma)$ converges uniformly to a continuous function $A(\sigma)$ on S^{N-1}. Conversely we also show that given any nonnegative continuous function $A(\sigma)$ on S^{N-1}, there exists a unique positive solution with that property.

1. Introduction

In 1999, when H. Brezis was visiting Korea, he asked whether we can classify self-similar solutions of a semilinear heat equation. This problem may be related to the work by Chen, Matano and Veron [3], where they classified singular solutions of the equation

$$\Delta u = |u|^{p-1}u \tag{1.1}$$

and investigated an asymptotic behavior of solutions as both $r \to 0$ and $r \to \infty$.

In this note, we investigate some properties of positive solutions of the equation

$$\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u - |u|^{p-1}u = 0 \quad \text{in} \quad \mathbf{R}^N, \tag{1.2}$$

where $1 < p < (N+2)/N$. We notice that if $p \geq (N+2)/N$, (1.2) does not have any positive solution decaying to zero. Although, the equation (1.2) looks

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close to (1.1) but it is technically more difficult and possesses quite a different asymptotic profile. In fact, this equation lacks a priori estimates and we need to put an additional assumption (see Theorem 2). Even radial solutions are not easy to classify if they change sign. Thus we here consider only positive solutions of (1.2).

The equation (1.2) arises naturally in the study of the asymptotic behavior of solutions of a semilinear parabolic equation

$$v_t = \Delta v - |v|^{p-1}v \quad \text{in} \quad \mathbf{R}^N \times (0,\infty). \tag{1.3}$$

We observe that if $v(x,t)$ solves (1.3), then the rescaled functions

$$v_\lambda(x,t) = \lambda^{2/(p-1)}v(\lambda x, \lambda^2 t), \quad \lambda > 0 \tag{1.4}$$

define a one parameter family of solutions to (1.3). A solution v is said to be self-similar when $v_\lambda(x,t) = v(x,t)$ for every $\lambda > 0$. It can be easily verified that v is a self-similar solution to (1.3) if and only if v has the form

$$v(x,t) = t^{-1/(p-1)}u(x/\sqrt{t}), \tag{1.5}$$

where u satisfies (1.2).

The asymptotic behavior of solutions of (1.3) is usually determined by the limiting profile of (1.4) as $\lambda \to \infty$, which becomes a self-similar solution (see [4]). Henceforth the classification of solutions of (1.2) is also valuable in this respect.

The positive radial solutions are fairly well-understood even though the property of more general (sign-changing) solutions is not revealed yet. For more details, we refer to [2], [4] and [8].

Here, we claim that every positive solution of (1.2) such that

$$\limsup_{r\to\infty} r^{2/(p-1)}u(r\sigma)$$

is uniformly bounded on the unit sphere S^{N-1} must satisfy

$$\lim_{r\to\infty} r^{2/(p-1)}u(r\sigma) = A(\sigma) \tag{1.6}$$

for some continuous function $A(\sigma)$ on S^{N-1} and conversely for every nonnegative continuous function $A(\sigma)$ on S^{N-1}, there exists a unique positive solution of (1.2) satisfying (1.6).

Similar results hold for self-similar solutions of a semilinear heat equation with source and we put a remark in section 5.

2. Existence

In treating self-similar solutions, we shall use the standard method introduced in [4] (see also [5] and [6]).

Given a nonnegative (nontrivial) continuous function $A(\sigma)$ on S^{N-1}, we consider a positive solution of

$$v_t = \Delta v - |v|^{p-1}v \tag{2.1}$$

with an initial unction

$$v(x,0) = A(\frac{x}{|x|})|x|^{-\alpha} \quad \text{for} \quad x \neq 0. \tag{2.2}$$

Here

$$\alpha = \frac{2}{p-1}.$$

First, we remark that $v(x,0)$ is not locally integrable and the standard existence theory can not be applied. Secondly, we note that the equation (2.1) has a singular stationary solution given by

$$W(x) = c(p,N)|x|^{-\alpha}$$

for some positive constant $c(p,N)$. Nevertheless the existence of such a solution is guaranteed by taking a monotone limit of a family of positive solutions $v_k(x,t)$ of (2.1) with the truncated initial function

$$v_k(x,0) = \min\{k, A(\frac{x}{|x|})|x|^{-\alpha}\}, \tag{2.3}$$

which is integrable in $\mathbf{R}^N$ and its existence and uniqueness is given in [1] for example. These truncated solutions are bounded by an a priori estimate

$$C(|x|^2 + t)^{-\alpha/2}$$

for some positive constant C larger than the maximum of $A(\sigma)$ on S^{N-1}; and thus the monotone limit exists and becomes smooth.

Moreover, the above solution is minimal among all the positive solutions satisfying (2.2) (see [4] for details) and thus becomes self-similar. In fact, let $v(x,t)$ be the minimal solution. Then given $\lambda > 0$, obviously

$$\lambda^{2/(p-1)}v(\lambda x, \lambda^2 t) \geq v(x,t).$$

One also has for every $\mu > 0$

$$\mu^{2/(p-1)}v(\mu\lambda x, \mu^2\lambda^2 t) \geq v(\lambda x, \lambda^2 t).$$

Simply taking $\mu = 1/\lambda$, we obtain

$$v(x,t) \geq \lambda^{2/(p-1)}v(\lambda x, \lambda^2 t).$$

As remarked earlier since $v(x,t)$ is self-similar, v has the form

$$v(x,t) = t^{-1/(p-1)} w(x/\sqrt{t}), \tag{2.4}$$

where w satisfies (1.2). Moreover, we also have

$$\begin{aligned} \lim_{r\to\infty} r^\alpha w(r\sigma) &= \lim_{t\to 0} t^{-1/(p-1)} w(\frac{\sigma}{\sqrt{t}}) \\ &= \lim_{t\to 0} v(\sigma,t) \\ &= A(\sigma). \end{aligned}$$

Thus summarizing the above arguments, we may conclude that

Theorem 1. *Given any nonnegative continuous function $A(\sigma)$ on S^{N-1}, there exists a unique positive solution of (1.2) satisfying (1.6).*

Remark. When $A(\sigma)$ is identically zero, the solution corresponds to a very singular solution and its existence and uniqueness is known (see [2]). The uniqueness for the general case will be proved in the next section.

3. Uniqueness

When $A(\sigma)$ is strictly positive for all $\sigma \in S^{N-1}$, the uniqueness can be proved by applying the maximum principle as we will see below. In fact, let U and u be any two solutions. We may assume that $U \geq u$.

We define

$$l = \min\{k \geq 1 | ku(x) \geq U(x), \quad x \in \mathbf{R}^N\}. \tag{3.1}$$

The set on the right hand side of (3.1) is not empty, because we may take k sufficiently large, in order that the inequality holds due to the boundary assumption (1.6). The proof of uniqueness is reduced to showing that l is not greater than 1.

Suppose $l > 1$ to the contrary. From the boundary behavior at infinity, $lu(x)$ must touch $U(x)$ in a compact subset of $\mathbf{R}^N$. But $lu(x)$ is a super-solution and can not touch $U(x)$ from above. Hence one can slightly reduce the factor l and still has the same inequality in (3.1), which leads to a contradiction.

The above method does not work when $A(\sigma)$ vanishes somewhere on S^{N-1}. Hence, we provide another argument, which holds in general.

We may assume that $A(\sigma)$ is not identically zero. Let U and u be two solution with $U \geq u$ and $\lim_{|x|\to\infty} |x|^\alpha (U-u) = 0$. We recall that the problem

$$\begin{gathered} \Delta w + \frac{1}{2} x \cdot \nabla w + \frac{1}{p-1} w - |w|^{p-1} w = 0 \\ \lim_{|x|\to\infty} |x|^\alpha w = 0 \end{gathered} \tag{3.2}$$

has a unique positive (radial) fast orbit

$$u_0(x) = f(r) = Ae^{-r^2/4} r^{\alpha - N}\{1 + O(1/r^2)\}$$

and that $u \geq u_0$.

Let $v = U - u$, then v solves

$$\begin{aligned}
&\Delta v + \tfrac{1}{2}x \cdot \nabla v + \tfrac{1}{p-1}v - v^p = U^p - u^p - (U-u)^p \geq 0, \\
&\lim_{|x|\to\infty} |x|^\alpha v = 0,
\end{aligned}$$

and becomes a subsolution. Now, given any $\epsilon > 0$, we compare v with a positive solution u_ϵ of (3.2) with

$$\lim_{|x|\to\infty} |x|^\alpha u = \epsilon$$

(Such a solution exists uniquely and is radial and called a slow orbit). We easily see that

$$0 \cdot \ v \cdot \ u_\epsilon.$$

Taking $\epsilon \to 0$, we find that u_ϵ decays to u_0 and $0 \cdot \ v \cdot \ u_0$, which implies that v decays exponentially.

Now, we work in the weighted space $L^2(K)$ with $K(x) = e^{-|x|^2/4}$. Define

$$Lu \equiv -\frac{1}{K}\nabla \cdot (K\nabla u),$$

then L is a self-adjoint operator on $L^2(K)$. Since

$$Lu_0 = -\frac{1}{p-1}u_0 + u_0^p,$$

$$Lv = -\frac{1}{p-1}v + U^p - u^p,$$

and $< Lu_0, v >=< Lv, u_0 >$, we get

$$\int Ku_0^p v dx = \int K(U^p - u^p)u_0 dx.$$

Since

$$\int Ku^{p-1}u_0 v dx \geq \int Ku_0^p v dx$$

and

$$\int K(U^p - u^p)u_0 dx \geq \int Ku^{p-1}(U-u)u_0 dx = \int Ku^{p-1}u_0 v dx,$$

we must have $U \equiv u$, which proves the uniqueness.

4. BEHAVIOR NEAR INFINITY

We consider positive solutions of the equation

$$\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u - |u|^{p-1}u = 0 \quad \text{in} \quad \mathbf{R}^N, \tag{4.1}$$

where $1 < p < (N+2)/N$.

Under the assumption that

$$\limsup_{r\to\infty} r^{\alpha}u(r\sigma) < \infty \tag{4.2}$$

uniformly on $\sigma \in S^{N-1}$ we show that $r^{\alpha}u(r\sigma)$ converges uniformly to a continuous function $A(\sigma)$ on the unit sphere S^{N-1} as $r \to \infty$. For $N = 1$, the equation (4.1) becomes an ordinary differential equation and it is easy to analyze asymptotic profiles and thus it is enough to assume that $\lim_{|x|\to\infty} u(x) = 0$ in this case (see [2]). But we do not know whether the same results hold for higher dimensions. In any case we can construct a solution such that $\lim_{r\to\infty} r^{\alpha}u(r\sigma_1) = 0$ and $\lim_{r\to\infty} r^{\alpha}u(r\sigma_2) = \infty$ for some $\sigma_1, \sigma_2 \in S^{N-1}$ simply by taking a monotone limit of solutions (1.2) and (1.6) with continuous $A(\sigma)$ vanishing on some part of S^{N-1} and glowing to infinity on some other part. We need to rule out these solutions.

The goal of this section is to show that a set of continuous functions $f_r(\sigma) = r^{\alpha}u(r\sigma)$ is equicontinuous on the unit sphere S^{N-1} as r varies over $[1, \infty)$ and then the Arzela-Ascoli theorem implies that the uniform limit exists as $r \to \infty$ (the uniqueness of such a limit is already guaranteed in section 3). We notice that the assumption (4.2) is imposed for the equiboundedness of the set $\{f_r\}$.

The standard regularity theory for an elliptic equation is not applied in this case because of the term "$x \cdot \nabla u$". The key ingredient of proof is transforming (4.1) into a parabolic equation and then applying a regularity theory for a parabolic equation instead.

Theorem 2. *Every positive solution of* (4.1) *with the assumption* (4.2) *satisfies*

$$\lim_{r\to\infty} r^{2/(p-1)}u(r\sigma) = A(\sigma) \tag{4.3}$$

for some continuous function $A(\sigma)$ *on the unit sphere* S^{N-1}.

Proof. Let $v(x,t) = |y|^{\alpha}u(y)$, $y = x/\sqrt{t}$, $x \in \mathbf{R}^N$, $t > 0$. Then the assumption (4.2) implies that $v(x,t)$ is uniformly bounded. An elementary calculation gives

$$v_t = -\frac{\alpha}{2}|y|^{\alpha}u(y)\frac{1}{t} - \frac{1}{2}|y|^{\alpha}y \cdot \nabla_y u(y)\frac{1}{t}, \tag{4.4}$$

$$v_{x_i} = \alpha|y|^{\alpha-2}y_i u(y)\frac{1}{\sqrt{t}} + |y|^{\alpha}u_{y_i}(y)\frac{1}{\sqrt{t}}, \tag{4.5}$$

$$\begin{aligned}\Delta v \quad &= \alpha(\alpha-2)|y|^{\alpha-2}u(y)\frac{1}{t} + N\alpha|y|^{\alpha-2}u(y)\frac{1}{t} \\ &\quad +2\alpha|y|^{\alpha-2}y\cdot\nabla_y u(y)\frac{1}{t} + |y|^{\alpha}\Delta_y u(y)\frac{1}{t} \\ &= \alpha(\alpha-2)|x|^{-2}v + N\alpha|x|^{-2}v + 2\alpha|y|^{\alpha}y\cdot\nabla_y u(y)|x|^{-2} \\ &\quad +|y|^{\alpha}(-\frac{1}{2}y\cdot\nabla_y u - \frac{1}{p-1}u + u^p)\frac{1}{t}.\end{aligned} \tag{4.6}$$

From the equation (4.4) we have

$$|y|^{\alpha}y\cdot\nabla_y u(y) = -2tv_t - \alpha v$$

and using the above equation we may write the equation (4.6) as

$$\Delta v = k|x|^{-2}v + \phi(x,t)v_t + |y|^{\alpha}u^p(y)\frac{1}{t},$$

with $k = N\alpha + \alpha(\alpha-2) - 2\alpha^2$ and $\phi(x,t) = 1 - 4\alpha t/|x|^2$.

Rewrite the above equation in the form of

$$v_t = a(x,t)\Delta v + b(x,t)v + c(x,t),$$

where

$$\begin{aligned} a(x,t) \quad &= \frac{1}{\phi(x,t)}, \\ b(x,t) \quad &= -\frac{k}{\phi(x,t)|x|^2}, \\ c(x,t) \quad &= -\frac{(\frac{|x|}{\sqrt{t}})^{\alpha+2}u^p(\frac{x}{\sqrt{t}})}{\phi(x,t)|x|^2}. \end{aligned}$$

Let

$$D = \{(x,t)|\frac{1}{2} \cdot \ |x| \cdot \ 2, \quad 0 < t \cdot \ \frac{1}{32\alpha}\},$$

then $1 < a(x,t) \cdot \ 2$ and $a(x,t), b(x,t), c(x,t)$ are continuous and uniformly bounded in D. Notice that

$$|y|^{\alpha+2}u^p(y) = (|y|^{\alpha}u(y))^p.$$

The standard regularity theory implies that $v(x,t)$ is uniformly Holder continuous in D, see [7]. In particular $v(x,t)$ is uniformly Holder continuous for $|x| = 1$ as t varies over $(0, \frac{1}{32\alpha}]$. Thus for $\sigma_1, \sigma_2 \in S^{N-1}$,

$$|v(\sigma_1,t) - v(\sigma_2,t)| \cdot \;\; M|\sigma_1 - \sigma_2|^\delta$$

for some positive constants M and δ, which implies

$$(1/\sqrt{t})^\alpha |u(\sigma_1/\sqrt{t}) - u(\sigma_2/\sqrt{t})| \cdot \;\; M|\sigma_1 - \sigma_2|^\delta.$$

Hence $\{f_r(\sigma) = r^\alpha u(r\sigma)\}$ is equicontinuous on S^{N-1} as r varies over $[1,\infty)$. Therefore $\lim_{r\to\infty} r^\alpha u(r\frac{x}{|x|})$ exists and becomes a continuous function on S^{N-1}. The uniqueness of this limit is shown in section 3.

5. Remark on Bad Sign Case

We may consider self-similar solutions of

$$v_t = \Delta v + |v|^{p-1}v \tag{5.1}$$

and classify positive solutions of

$$\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + |u|^{p-1}u = 0 \quad \text{in} \quad \mathbf{R}^N. \tag{5.2}$$

In this case if $p \cdot \;\; (N+2)/N$, (5.2) does not have any positive solution and we need to impose the condition $p > (N+2)/N$.

The result in section 4 still holds without any changes but the existence part is a little different. When we consider the Cauchy problem, the initial data (2.2) is locally integrable and a global solution exists for small initial data. But solutions may blow up for large initial data as shown in [9]. Moreover the uniqueness result also breaks down in this case, see [9] for details.

References

1. H. Brezis and A. Friedman, Nonlinear parabolic equations involving measures as initial conditions *J. Math. pures et appl.* **62** (1983), 73-97.
2. H. Brezis, L. A. Peletier, and D. Terman, A very singular solution of the heat equation with absorption, *Arch. Rat. Mech. Anal.* **95** (1986), 185-209.
3. X-Y. Chen, H. Matano, and L. Veron, Anisotropic singularities of solutions of non-linear elliptic equations in $\mathbf{R}^2$, *J. Funct. Anal.* **83(1)** (1989), 50-97.
4. M. Kwak, A semilinear heat equation with a singular initial data *Proc. Roy. Soc. Edinburgh Sect. A*, **128** (1998), 745-758.

5. M. Kwak, A porous media equation with absorption. I Long time behaviour, *J. Math. Anal. Appl.* **223** (1998), 96-110.

6. M. Kwak, A porous media equation with absorption. II Uniqueness of the very singular solution, *J. Math. Anal. Appl.* **223** (1998), 111-125.

7. G. M. Lieberman, *Second order parabolic differential equations*, World Scientific, 1996.

8. Y. Naito and T. Suzuki, Radial symmetry of self-similar solutions for semilinear heat equations, *J. Differential Equations*, **163(2)** (2000), 407-428.

9. Y. Naito, *Non-uniqueness of solutions to the Cauchy problem for semilinear heat equations with singular initial data*, Preprint, 2002.

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