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UNIFORM CONVERGENCE THEOREM FOR THE H_1-INTEGRAL REVISITED

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Abstract. In this note we show that the uniform convergence theorem for the H_1-integral is false.

1. Introduction

Most notions and notations we use is taken from [1]. Moreover for partial divisions D_1 and D_2 of a compact interval $[a,b]$ we write $D_2 \sqsupseteq D_1$, if for every $([s,t],\eta) \in D_2$ there is $([u,v],\xi) \in D_1$ such that $[s,t] \subset [u,v]$. Recall from [2] that a function $f\colon [a,b] \to \mathbb{R}$ is *H_1-integrable on* $[a,b]$ to a number $A \in \mathbb{R}$, if there exists a positive function δ on $[a,b]$ such that for every $\varepsilon > 0$ there exists a division D_0 of $[a,b]$ such that $\left|(D)\sum f(\xi)(v-u)-A\right| < \varepsilon$ for every δ-fine division $D \sqsupseteq D_0$. (This definition is equivalent to the one from [1], see [3, Theorem 2.4]. The difference is in the definition of the relation $\sqsupseteq$.) In this case we will say that A is the *H_1-integral of f on* $[a,b]$ and write $A = \int_a^b f$.

I. J. L. Garces and P. Y. Lee claimed to prove the uniform convergence theorem for the H_1-integral [1, Theorem 4]. Alas, we have the following theorem.

Theorem 1. *The uniform convergence theorem does not hold for the H_1-integral.*

Proof. Let (G_n) be a sequence of open dense subsets of $[0,1]$, whose intersection E is a null set. For each $n \in \mathbb{N}$ denote by h_n the characteristic function of $[0,1] \setminus G_n$. By [2, Lemma 4], each function h_n is H_1-integrable.

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The series $f = \sum_{n\in\mathbb{N}} h_n/2^n$ is uniformly convergent on $[0,1]$. Suppose that f is H_1-integrable using a positive function δ on $[0,1]$. For each $n \in \mathbb{N}$ let

$$E_n = \{x \in E : \delta(x) > n^{-1}\}.$$

The set E is a dense $\mathcal{G}_\delta$ set, so it is residual. Thus there is an $n \in \mathbb{N}$ and an open interval I such that E_n is dense in I. Since E is a null set and $f > 0$ outside of E, there is an $m \geq n$ such that the measure of the closure of the set

$$F_m = \{x \in I \setminus E : f(x) > m^{-1} \text{ and } \delta(x) > m^{-1}\}$$

is positive, say M.

Define $\varepsilon = M/(2m)$. By assumption, there is a division D_0 of $[0,1]$ such that

$$\Big|(D)\sum f(\xi)(v-u) - \int_a^b f\Big| < \varepsilon \tag{1}$$

for every δ-fine division $D \sqsupseteq D_0$. Without loss of generality we may assume that a subset of D_0, say D_I, is a division of I. Every interval from D_I can be written as the union of a finite family of nonoverlapping intervals of length less than m^{-1}. Denote by $\mathcal{A}$ the family of all these intervals. Clearly if

$$\mathcal{B} = \{J \in \mathcal{A} : J \cap F_m \neq \emptyset\},$$

then $\sum_{J\in\mathcal{B}} |J| \geq M$.

For each $J \in \mathcal{B}$ we can pick an $x_J \in J \cap F_m$ and, as E_n is dense in I, a $y_J \in J \cap E_n$. Let $D_1 \sqsupseteq D_0$ be a δ-fine division of $[0,1] \setminus \bigcup_{J\in\mathcal{B}} J$. Both $D_2 = \{(J, x_J) : J \in \mathcal{B}\}$ and $D_3 = \{(J, y_J) : J \in \mathcal{B}\}$ are δ-fine partial divisions of $[0,1]$ and $D_2, D_3 \sqsupseteq D_I$. So, $D_4 = D_1 \cup D_2$ and $D_5 = D_1 \cup D_3$ are δ-fine divisions of $[0,1]$ such that $D_4, D_5 \sqsupseteq D_0$.

For all $J \in \mathcal{B}$ we have $f(x_J) > m^{-1}$ and $f(y_J) = 0$. Thus

$$(D_4)\sum f(\xi)(v-u) - (D_5)\sum f(\xi)(v-u) > m^{-1}\sum_{J\in\mathcal{B}} |J| \geq M/m = 2\varepsilon,$$

contrary to (1). Consequently, f is not H_1-integrable. ■

I. J. L. Garces and P. Y. Lee proved the controlled convergence theorem for the H_1-integral [1, Theorem 6]: if an equi-H_1-integrable sequence of functions (f_n) is pointwise convergent to some function f, then f is H_1-integrable and $\int f = \lim \int f_n$. In view of Theorem 1, we obtain the following corollary.

Corollary 2. *There is a uniformly convergent sequence of H_1-integrable functions which is not equi-H_1-integrable.*

So, [1, Theorem 4] (the one which turned out to be false) does not follow from [1, Theorem 6], contrary to the remark at the bottom of page 444 of [1].

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