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MULTIPLIERS AND TENSOR PRODUCTS OF VECTOR VALUED $L^p(G, A)$ SPACES

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Abstract. In this paper we define a normed space $A_p^q(G, A)$ and prove some properties of this space. In particular, we show that the space $\mathcal{L}\checkmark(\mathcal{G},\mathcal{A})$ $\mathcal{L}^{\infty}(\mathcal{G},\mathcal{A})$ $\mathcal{L}^{\text{II}}(\mathcal{G},\mathcal{A})$ is isometrically isomorphic to the space $A_q^p(G, A)$ and the space of multipliers from $L^p(G, A)$ to $L^{q'}(G, A^*)$ is isometrically isomorphic to the dual of the space $A_p^q(G, A)$ if G satisfies a property P_p^q.

1. Introduction and Preliminaries

Throughout this paper, we let G be a locally compact Abelian group with Haar measure dt, A be a commutative Banach algebra with identity of norm 1 and X be a Banach space. $C_0(G, X)$ denotes the Banach space of X-valued continuous functions on G vanish at infinity, under the supremum norm

$$\|f\|_{\infty X} = \sup_{t\in G} \|f(t)\|_X \quad \text{for} \quad f \in \mathrm{C_0(G, X)}$$

Let $C_C(G, X)$ be the space of all continuous and X-valued functions on G with compact support. and

$$L^P(G, X) = \{f : G \to X; f \text{ is measurable and } \|f(.)\|_X \in L^P(G)\}$$

with the norm given by

$$\|f\|_{pX} = \left(\int_G \|f(t)\|_X^p \, dt\right)^{1/p}, 1 \le p < \infty.$$

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It follows that $L^p(G,X)$ is a Banach space for $1 \leq p < \infty$. If $X = \mathbb{C}$, the set of complex numbers, then we write that $L^p(G,X) = L^P(G)$. If A is a commutative Banach algebra with identity of norm 1, then the space $L^1(G,A)$ is a commutative Banach algebra under convolution

$$f * g(t) = \int_G f(t-s)\, g(s)\, ds = \int_G f(s)\, g(t-s)\, ds$$

and the norm

$$\|f\|_{1A} = \int_G \|f(t)\|_A\, dt$$

for $f, g \in L^1(G.A)$,([4], [6]).

For $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$, the dual space $L^P(G,X)^*$ is isometrically isomorphic to $L^q(G, X^*)$ if and only if X^* has the Radon-Nikodym property (RNP for brevity) in the wide sense (Lai [9]). (See Theorem 1.2 in [8]). Thus for $f \in L^p(G,X), g \in L^q(G,X^*)$, the dual pair $< f(.), g(.) > \in L^1(G)$ and Hölder inequality implies

$$\int_G |< f(.), g(.) > |dt \leq \|f\|_{pX} \cdot \|g\|_{qX^*} \cdot$$

In [8], Lai proved the following theorems.

Theorem A ([8;Theorem2.2]). *Let* $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1, f \in L^p(G,A)$ *and* $g \in L^q(G,A)$. *Then* $f * g \in C_0(G,A)$ *and*

$$\|f_* g\|_{\infty A} \leq \|f\|_{pA} \cdot \|g\|_{qA}$$

Analogy to the scalar function case ([4] 20.18. Theorem) we can easily obtain the following;

Theorem B *Let* $\frac{1}{p} + \frac{1}{q} > 1$ *and* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. *If* $f \in L^p(G,A)$, $g \in L^q(G,A)$ *then* $f * g \in L^r(G,A)$ *and*

$$\|f_* g\|_{rA} \leq \|f\|_{pA} \cdot \|g\|_{qA}$$

A Banach space V is called a left (right) Banach A-module over a Banach algebra A if V is a left (right) module over A in the algebraic sense for some multiplication, $(a,v) \to a.v$, and satisfies $\|av\| \leq \|a\| . \|v\|$ for all $a \in A, v \in V$. Again, we can assume that V is a Banach A-module. Then the closed linear subspace of V spanned by

$$AV = \{av \,|a \in A, v \in V\}$$

is called the essential part of V and is denoted by V_e. If $V = V_e$, then V is said to be an essential Banach A-module. In [8], Lai proved the following theorem.

Proposition C ([8; Proposition 2.3]). Let X^* have the wide RNP. Then the Banach space $L^p(G, X)$ is an essential $L^1(G, A)$ - module under convolution such that for $f \in L^1(G, A)$ and $g \in L^p(G, X)$ we have

$$\|f_* g\|_{pX} \le \|f\|_{1A} \cdot \|g\|_{pX} \cdot$$

Let V and W be a left and right Banach A-module respectively. Let $V \ {}_{\gamma} W$ denote the projective tensor product (Bonsall-Duncan,[1]) of V and W, and let K be the closed linear subspace of $V \ {}_{\gamma} W$ which is spanned by all the elements of the form

$$av \quad \omega - \nu \quad a\omega, \text{for every } a \in A, v \in V, \omega \in W.$$

Then the A-module tensor product $V \ {}_{A} W$ is defined to be the quotient Banach space $(V \ {}_{\gamma} W)/K$. It is known that every element t of $V \ {}_{A} W$ is defined by

(1.1) $$t = \sum_{i=1}^{\infty} \nu_i \quad w_i, \ \nu_i \in V, W_i \in W, \sum_{i=1}^{\infty} \|\nu_i\| \, \|\mathrm{w_i}\| < \infty.$$

It is a normed space under the norm

$$\|t\| = \inf \left\{ \sum_{i=1}^{\infty} \|v_i\| \, \|\mathrm{w_i}\| < \infty \right\}$$

where the infimum is taken over all possible representations for t.

If V and W are left (right) Banach A-modules, then a multiplier (or module homomorphism) from V to W is a bounded linear operator T from V to W, which commutes with module multiplication i.e. $T(a\nu) = aT(\nu)$ for $a \in A$ and $\nu \in V$. We denote $Hom_A(V, W)$ or $M(V, W)$ as the space of multipliers from V to W.

Now let V and W be a left and right Banach A-module respectively. It is known that W^*, the dual of W, is a left Banach A-module. Rieffel in [11] proved that there is a natural isometric isomorphism

(1.2) $$Hom_A(V, W^*) \cong (V \ {}_{A} W)^*$$

under which the linear functional t on $V \ {}_{A} W$, which corresponds to an operator $T \in Hom_A(V, W^*)$ has the value $< \omega, T(v) >= t(v \quad \omega)$ for $v \quad \omega \in V \ {}_{A} W$ and the ultra weak*-topology on $Hom_A(V, W^*)$ corresponds to the weak*- topology on $(V \ {}_{A} W)^*$ (see Rieffel [11] and also Lai [6], [7]).

Rieffel [11,5.5.Theorem] proved that if G satisfies property P_p^q (see Def. 2, §3) then $L^p(G) \quad {}_G L^q(G) \cong A_p^q$, for $1 \leq p, q < \infty$. By using the before-mentioned Rieffel's technique, in this study we will show that $L^p(G,A) \quad {}_{L^1(G,A))} L^q(G,A^*) \cong A_p^q(G,A)$.

2. The Space $A_p^q(G,A)$

Throughout this section we let G be a locally compact Abelian group, A be a commutative Banach algebra with identity of norm 1.

In view of Theorem B, we can define a bilinear map b from $L^p(G,A)\,\mathrm{x}L^q(G,A)$ into $L^r(G,A)$by

$$b(f,g) = \tilde{f} * g, f \in L^p(G,A), g \in L^q(G,A)$$

where $\tilde{f}(x) = f(-x)$. It is easy to see that $\|b\| \leq 1$. Then, b lifts to a linear map B from $L^p(G,A) \quad {}_\gamma L^q(G,A)$ into $L^r(G,A), B(f \quad g) = \tilde{f} * g$, where $f \in L^p(G,A), g \in L^q(G,A)$ and $\|B\| \leq 1$ by the Theorem 6 in (Bonsall-Duncan, [1]).

Definition 1. *The range of B with the quotient norm will be denoted by* $A_p^q(G,A)$.

Thus $A_p^q(G,A)$ is a Banach space of functions defined on G, which can be viewed as a linear submanifold in $L^r(G,A)$. In view of the fact that every element of $L^p(G,A) \quad {}_\gamma L^q(G,A)$ has an expansion of the form (1.1), we can see that (see [8]) $A_p^q(G,A)$ consists of exactly those functions h on G, which has at least one expansion of the form

$$h = \sum_{i=1}^{\infty} f_i * g_i, f_i \in L^p(G,A), g_i \in L^q(G,A)$$

such that $\sum_{i=1}^{\infty} \|f_i\|_{pA} \cdot \|g_i\|_{qA} < \infty$. $A_p^q(G,A)$ is equipped with the norm

$$|||h||| = \inf\left\{\sum_{i=1}^{\infty} \|f_i\|_{pA} \|g_i\|_{qA} : f_i \in L^p(G,A), g_i \in L^q(G,A)\right\}$$

Applying relation (1.2) to the spaces $V = L^p(G,A)$ and $W = L^q(G,A)$ yields

$$Hom_{L^1(G,A)}\left(L^p(G,A), L^{q'}(G,A^*)\right) \cong \left(L^p(G,A) \quad {}_{L^1(G,A)} L^q(G,A)\right)^*$$

for $1 \le p < \infty$ and $1 \le q < \infty$. We now turn to the problem of representing the dual space of $L^p(G,A) \quad {}_{L^1(G,A)} L^q(G,A)$ as a function space. Lai [8] has shown that

$$Hom_{L^1(G,A)}\left(L^1(G,A), L^p(G,X^*)\right) \cong \left(L^1(G,A) \quad {}_{L^1(G,A)} L^q(G,X)\right)^* \\ \cong L^p(G,X^*)$$

where $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$. Thus we can assume that $p > 1$ and $q > 1$.

3. Multipliers from $L^p(G,A)$ To $L^{q'}(G,A^*)$

In this section, we assume that G is not a compact Abelian group. Let K be the closed linear subspace of $L^p(G,A) \quad {}_\gamma L^q(G,A)$, which is spanned by all the elements of the form

$$(\varphi * f) \quad g - g \quad (\tilde{\varphi} * f)$$

where $f \in L^p(G,A), g \in L^q(G,A)$ and $\varphi \in L^1(G,A)$. Then, the $L^1(G,A)$-module tensor product $L^p(G,A) \quad {}_{L^1(G,A)} L^q(G,A)$ is defined to be the quotient Banach space

$$L^p(G,A) \quad {}_\gamma L^q(G,A)/K.$$

We denote by $L_s f$ the s-translation of f on G, that is,

$$L_s f(t) = f(t-s) \text{ for } t, s \in G.$$

A bounded linear operator T from $L^p(G,A)$ to $L^{q'}(G,A^*)$ is invariant if T commutes with the translation operators $L_s\,(s \in G)$. That is, $L_s T = T L_s, s \in G$.

Lemma 1. ([12; lemma 1]. *Let G be a non-compact locally compact Abelian group. Then*

$$\lim_{s \to +\infty} \|f + L_s f\|_{pX} = 2^{1/p}. \|f\|_{pX}$$

for $f \in L^p(G,X), 1 \le p < \infty$.

Theorem 1. *If T is a bounded invariant operator from $L^p(G,A)$ to $L^{q'}(G,A^*)$ and $p > q'$ then $T \equiv 0$.*

Proof. Assume that $T \not\equiv 0$. Since T is a linear bounded operator, there exists $c > 0$, such that

(3.1) $$\|Tf\|_{q'A^*} \le c. \|f\|_{pA}, \mathrm{f} \in \mathrm{L^p(G,A)}$$

Since T is an invariant linear operator, we write

$$\|Tf + L_s Tf\|_{q'A^*} \leq c. \|f + L_s f\|_{pA}.$$

Also by Lemma 1 we have

$$2^{1/q'} \|Tf\|_{q'A^*} \leq c.2^{1/p} \|f\|_{pA}.$$

Hence, we find

$$\|Tf\|_{q'A^*} \leq c.2^{\frac{1}{p}-\frac{1}{q'}} \|f\|_{pA}$$

and $c.2^{1/p-1/q'} < c$ if $p < q'$. But, this is a contradiction because c is the smallest constant satisfying the inequality (3.1). Therefore, $T \equiv 0$.

Theorem 2. *If* $\frac{1}{p} + \frac{1}{q} < 1$ *and* $1 \leq p, q < \infty$ *then*

$$L^p(G, A) \quad {}_{L^1(G,A)} L^q(G, A) = \{0\}$$

Proof. If $\frac{1}{p} + \frac{1}{q} < 1$ then, $p > q'$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Moreover, we have

$$\left(L^p(G, A) \quad {}_{L^1(G,A)} L^q(G, A)\right)* \equiv Hom_{L^1(G,A)}\left(L^p(G, A), L^{q'}(G, A*)\right) \tag{3.2}$$

(see Theorem 1.4 in Rieffel, [11]). From (3.2), Theorem 1 and the Hahn-Banach theorem, we obtain

$$L^p(G, A) \quad {}_{L^1(G,A)} L^q(G, A) = \{0\}.$$

This completes the proof.

It is known that the X-valued space $C_c(G, X)$ of continuous functions with compact support in G is dense in $L^p(G, X)$, $1 \leq p < \infty$. The following lemma is easily proved.

Lemma 2. *Let* $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} \geq 1$ *and* $q' = 2p$. *Given any* $\varphi \in C_c(G, A)$, *define* T_φ *by* $T_\varphi(f) = f_* \varphi, f \in L^p(G, A)$. *Then,* $T_\varphi \in Hom_{L^1(G,A)} \left(L^p(G, A), L^{q'}(G, A^*)\right)$ *and the inequality*

$$|||T_\varphi|||_{q',A} \; |||\varphi|||_{q,A}$$

holds.

Proof. For all $f \in L^p(G, A)$ and $g \in L^1(G, A)$ we have

$$T_\varphi(g_* f) = (g_* f)_* \varphi = g_*(f_* \varphi) = g_* T_\varphi(f).$$

That means T_φ is a $L^1(G,A)$ -homomorphism. Since $q' = 2p$, we have $\frac{1}{q'} = \frac{1}{p} + \frac{1}{q} - 1$ and $\frac{1}{p} + \frac{1}{q} \geq 1$. Hence, if we apply Theorem B, we have

$$f * \varphi \in L^{q'}(G,A) \text{ and } \|T_\varphi(f)\|_{q',A} \leq \|f\|_{p,A} \, |||\varphi|||_{q,A},$$

for $f \in L^p(G,A)$ and $\varphi \in C_c(G,A) \subset L^q(G,A)$. Therefore, we find

$$|||T_\varphi|||_{q',A} \cdot \quad |||\varphi|||_{q,A},$$

where $|||.|||_{q',A}$ and $|||.|||_{q,A}$ are operator norms on $L^{q'}(G,A)$ and $L^q(G,A)$, respectively. Thus T_φ is continuous. Consequently,

$$T_\varphi \in Hom_{L^1(G,A)}\left(L^p(G,A), L^{q'}(G,A^*)\right).$$

Definition 2. *A locally compact Abelian group G is said to satisfy property P_p^q if every element of $Hom_{L^1(G,A)}\left(L^p(G,A), L^{q'}(G,A^*)\right)$ can be approximated in the ultraweak operator topology by operators of the form $T_\varphi, \varphi \in C_c(G,A)$.*

Theorem 3. *Let G be a locally compact Abelian group. If $q' = 2p$, $\frac{1}{p}+\frac{1}{q}-1 = \frac{1}{r}$ and $\frac{1}{p} + \frac{1}{q} \geq 1$ then the following statements are equivalent:*
(a) *G satisfies property P_p^q.*
(b) *The kernel of B is K and*

$$L^p(G,A) \quad {}_{L^1(G,A)} L^q(G,A) \equiv A_p^q(G,A).$$

Proof. Assume that G satisfies property P_p^q. It is obvious that $K \subset KerB$. To show $KerB \subset K$, it suffices to show $K^\perp \subset (KerB)^\perp$. Let $F \in K^\perp$. From the isometric isomorphism

$$K^\perp \cong \left(L^p(G,A) \quad {}_{L^1(G,A)} L^q(G,A)\right)^* \cong Hom_{L^1(G,A)}\left(L^p(G,A), L^{q'}(G,A^*)\right),$$

there is a multiplier $T \in Hom_{L^1(G,A)}\left(L^p(G,A), L^{q'}(G,A^*)\right)$ corresponding F such that

$$< t, F >= \sum_{i=1}^{\infty} < g_i, Tf_i > \tag{3.3}$$

where $t \in KerB, t = \sum_{i=1}^{\infty} f_i \quad g_i$ and $\sum_{i=1}^{\infty} \|f_i\|_{pA} \cdot \|g_i\|_{qA} < \infty$. Furthermore, since G satisfies property P_p^q, there exist operators net $\left(T_{\varphi_j}\right), \varphi_j \in C_c(G,A)$ such that

$$\lim_j \sum_{i=1}^{\infty} < g_i, T_{\varphi_j} f_i > = \lim_j \sum_{i=1}^{\infty} < g_i, f_i * \varphi_j > = \sum_{i=1}^{\infty} < g_i, T f_i >. \tag{3.4}$$

We also note that

$$\sum_{i=1}^{\infty} < g_i, f_{i*} \varphi_j >= \sum_{i=1}^{\infty} < \tilde{f}_i * g_i, \varphi_j >. \tag{3.5}$$

Again, if we use the Hölder inequality and the equality (3.5) then we obtain

$$\left| \sum_{i=1}^{\infty} < g_i, f_{i*} \varphi_j > \right| = \left| \sum_{i=1}^{\infty} < f_{i*} g_i, \varphi_j > \right| \leq \left\| \sum_{i=1}^{\infty} f_{i*} g_i \right\|_{rA} . \left\| \varphi_j \right\|_{r' A*} = 0. \tag{3.6}$$

Hence, if we combine (3.4) and (3.6) then

$$< t, F >= \sum_{i=1}^{\infty} < g_i, T_{f_i} >= 0.$$

Therefore, $< t, F >= 0$ for all $t \in KerB$. That means $F \in (KerB)^{\perp}$. Hence $K = KerB$. This proves that

$$L^p (G, A) \quad {}_{L^1(G,A)} \, L^q (G, A) \cong A_p^q (G, A) .$$

Suppose conversely that $KerB = K$. We will illustrate that the set $N = \{T_{\varphi} \left| \varphi \in C_c (G, A) \right. \}$ is everywhere dense in $Hom_{L^1(G,A)} \left(L^p (G, A) , L^{q'} (G, A^*) \right)$ in the ultraweak* operator topology. Let M be the set of all linear functionals which corresponds to the operators T_{φ}. If we prove that M is everywhere dense in $\left(L^p (G, A) \quad {}_{L^1(G,A)} \, L^q (G, A) \right)^*$ in the weak*-topology then we complete the proof. Since

$$\left(L^p (G, A) \quad {}_{L^1(G,A)} \, L^q (G, A) \right)^* \cong (KerB)^{\perp} ,$$

$< t, F >= 0$ for all $t \in KerB$ and $F \in M$ i.e., $t \in M^{\perp}$. That means $KerB \subset M^{\perp}$. Conversely, if we use (3.5) to obtain that

$$< \sum_{i=1}^{\infty} \tilde{f}_i * g_i, \varphi > =< t, F >= 0. \tag{3.7}$$

Also, by using the equality (3.7) and the Hahn-Banach theorem, we find that $\sum_{i=1}^{\infty} \tilde{f}_i * g_i = 0$. Therefore, $M^{\perp} \subset KerB$. Consequently, $M^{\perp} = KerB$. This proves the assertion.

Corollary 1. *Let G be a locally compact Abelian group and* $q' = 2p$, $\frac{1}{p}+\frac{1}{q} \geq 1$, $\frac{1}{p}+\frac{1}{q}-1 = \frac{1}{r}, \frac{1}{q}+\frac{1}{q'} = 1$. *If G satisfies the property* P_p^q, *then we have the identification*

$$Hom_{L^1(G,A)}\left(L^p(G,A), L^{q'}(G,A^*)\right) \cong \left(A_p^q(G,A)\right)^*$$

Proof. By Theorem 3 and the isometric isomorphism (1.2), one can obtain that

$$Hom_{L^1(G,A)}\left(L^p(G,A), L^{q'}(G,A^*)\right) \cong \left(L^p(G,A) \quad {}_{L^1(G,A)} L^q(G,A)\right)^*$$

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