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SINGULAR LIMIT OF A CLASS OF NON-COOPERATIVE REACTION-DIFFUSION SYSTEMS

D. Hilhorst, M. Mimura and R. Weidenfeld

Abstract. We consider a two component reaction-diffusion system with a small parameter ϵ

$$\begin{cases} u_t = d_u \Delta u + \dfrac{1}{\epsilon}(u^m v - a u^n), \\ v_t = d_v \Delta v - \dfrac{1}{\epsilon} u^m v, \end{cases}$$

where m and n are positive integers, together with zero-flux boundary conditions. It is known that any nonnegative solution becomes spatially homogeneous for large time. In particular, when $n > m \geq 1$, $(u^\epsilon, v^\epsilon)(t) \to (0,0)$ as $t \to \infty$, while when $m \geq n \geq 1$, there exists some positive constant v_∞^ϵ such that $(u^\epsilon, v^\epsilon)(t) \to (0, v_\infty^\epsilon)$ as $t \to \infty$. In order to find the value of v_∞^ϵ, we derive a limiting problem when $\epsilon \to 0$ under some conditions on the values of m, n and on the initial functions (u_0, v_0), by which an approximate value of v_∞^ϵ can be obtained.

1. Introduction

Among many classes of reaction-diffusion (RD) systems, we restrict ourselves to the following rather specific two component RD system :

$$\begin{cases} u_t = d_u \Delta u + k u^m v - a u^n, \\ v_t = d_v \Delta v - k u^m v, \end{cases} \tag{1.1}$$

where u, v are the concentrations of U, V, respectively, which are governed by the following cubic autocatalytic chemical reaction processes :

$$\begin{cases} mU + V \longrightarrow (m+1)U \\ nU \longrightarrow P. \end{cases}$$

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For the system (1.1), the positive constants d_u and d_v are the diffusion rates for u and v respectively, k and a are the reaction rates which are positive constants and m, n are some positive integers. In the specific case where $m = n = 1$, (1.1) is a diffusive epidemic model where u and v are respectively the population densities of infective and susceptable species [1]. When $m = 2$, $n = 1$, it is called the Gray-Scott model and describes an autocatalytic chemical process [2]. Fundamental problems for (1.1) involve the global existence, uniqueness and asymptotic behavior of nonnegative solutions in a smooth bounded domain (in $\mathbb{R}^N$) together with the boundary and initial conditions

$$\text{(1.2)} \qquad \frac{\partial u}{\partial \nu}(x,t) = \frac{\partial v}{\partial \nu}(x,t) = 0, \quad \text{for all } (x,t) \in \partial \quad \times \mathbb{R}^+,$$

$$\text{(1.3)} \qquad u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 \quad x \in \quad ,$$

where ν stands for the outward normal unit vector to ∂ . If $a = 0$, (1.1) reduces to

$$\text{(1.4)} \qquad \left\{ \begin{array}{l} u_t = d_u \Delta u + k u^m v, \\ v_t = d_v \Delta v - k u^m v, \end{array} \right.$$

which is called a consumer and resource system with balance law. There are many papers devoted to the system (1.4) with (1.2), (1.3) (e.g. [3, 4, 5, 6, 7, 8, 9, 10]). Indeed, we know that as $t \to \infty$, $(u,v)(t)$ converges to $(u_\infty, 0)$ uniformly in $\bar{\ }$ where u_∞ is explicitly given by $u_\infty = < u_0 + v_0 >$. Here $< w >$ is the spatial average of w over . Furthermore, it is proved by [10] that for $m > 1$ there exists some constant $K > 0$ such that

$$\|(u(t) - u_\infty, v(t))\|_{L^\infty(\ )} \leq K t^{-\frac{1}{m-1}} \quad \text{as } t \to \infty.$$

On the other hand, if $a > 0$, the asymptotic state depends on the values of m and n. If $n > m \geq 1$, $(u,v)(t)$ converges to $(0,0)$ uniformly in $\bar{\ }$ as $t \to \infty$. On the contrary, if $m \geq n \geq 1$, there exists a positive constant v_∞ such that $(u,v)(t)$ converges to $(0, v_\infty)$ uniformly in $\bar{\ }$ as $t \to \infty$ [11]. That is, every solution of (1.1)-(1.2) becomes spatially homogeneous for large time. We therefore conclude that the fundamental problems stated above have been already solved. However, from qualitative points of view, we still have the following question on (1.1)-(1.3):

Question 1: when $m \geq n \geq 1$, how does the asymptotic state v_∞ depend on the initial functions u_0, v_0, on k, a and on the domain ?

This question has not yet been solved, except in some special cases. Consider first a limiting situation where the reaction rates k and a are both sufficiently small (or, in other words, the diffusion rates are very large), so that (1.1) can be rewritten as

$$\text{(1.5)} \qquad \begin{cases} u_t = \dfrac{1}{\epsilon} d_u \Delta u + u^m v - a u^n, \\ v_t = \dfrac{1}{\epsilon} d_v \Delta v - u^m v. \end{cases}$$

Here we may set $k = 1$. For sufficiently small $\epsilon > 0$, the two-timing method reveals that the solution (u, v) becomes immediately spatially homogeneous and then its time evolution is described by the solution of the initial value problem for the following system of ordinary differential equations :

$$\text{(1.6)} \qquad \begin{cases} U_t = U^m V - a U^n, \\ V_t = -\, U^m V, \end{cases}$$

together with the initial conditions

$$\text{(1.7)} \qquad (U, V)(0) = (< u_0 >, < v_0 >).$$

We will show in Section 4 that there exists some positive constant V_∞ such that as $t \to \infty$, the solution $(U, V)(t)$ of (1.6), (1.7) converges to $(0, V_\infty)$, where V^∞ approximately gives the value v_∞ for the original problem (1.1)-(1.3). For a more precise discussion, we refer to the papers by[12, 13].

The aim of this paper is to answer Question 1, assuming another limiting situation which is opposite to (1.5). Let us rewrite (1.1) as

$$\text{(1.8)} \qquad \begin{cases} u_t = d_u \Delta u + \dfrac{1}{\epsilon}(u^m v - a u^n), \\ v_t = d_v \Delta v - \dfrac{1}{\epsilon} u^m v. \end{cases}$$

We study the limiting behavior as $\epsilon \to 0$ of solutions (u^ϵ, v^ϵ) of System (1.8) together with the boundary and initial conditions (1.2) and (1.3). We assume that the initial functions u_0 and v_0 satisfy the hypothesis $\|u_0\|^{m-n}_{L^\infty(\ )} \|v_0\|_{L^\infty(\ )} < a$ and derive the limiting system corresponding to (1.8) as $\epsilon \to 0$, which in turn yields the asymptotic limit of the constant v^ϵ_∞ as $\epsilon \to 0$, where v^ϵ_∞ is the asymptotic limit of $v^\epsilon(t)$ as $t \to \infty$.

More precisely, we prove a compactness property for the sequence $\{(u^\epsilon, v^\epsilon)\}$ and a strong decay property for the function $u^\epsilon(t)$. This leads us to prove the convergence of a subsequence of $\{v^\epsilon\}$ to a function v solution of a Neumann Problem for the heat equation. In order to characterize the initial condition of the limiting problem, we prove that until a time of order $\epsilon \ln 1/\epsilon$ the difference in the $L^2(\ )$-norm between the pairs $(u^\epsilon, v^\epsilon)(t)$ and $(U, V)(t/\epsilon)$ where (U, V) is the solution of (1.6) is of order of ϵ^β where β is a positive constant. Thus, we identify the initial function of the limiting problem with the asymptotic limit as $t \to \infty$ of $V(t)$, which in turn proves the convergence of the whole sequence $\{v^\epsilon\}$. Then the limit as $\epsilon \to 0$ of the constant v^ϵ_∞ is obtained by using the decay property of u^ϵ together with the fact that the average of the limiting function v does not depend on time.

The contents of this paper is as follows: In Section 2, we state the main results and in Sections 3-7, we prove some lemmas as well as the main results. Finally, in Section 8, we present concluding remarks about the system (1.8) together with (1.2) and (1.3).

2. Results

We may use a spatial rescaling which amounts to setting $d_u = 1$ and $d_v = d$ and consider the following ϵ-family of problems :

$$(P^\epsilon)\begin{cases} u_t = \Delta u + \dfrac{1}{\epsilon}(u^m v - a u^n) & \text{in } Q := \quad \times (0,\infty) \\ v_t = d\Delta v - \dfrac{1}{\epsilon} u^m v & \text{in } Q, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial \quad \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{for all } x \in \quad , \end{cases}$$

where is a smooth bounded domain of $\mathbb{R}^N$, $m \geq n \geq 1$, d and a are positive constants and $u_0, v_0 \in C^1(\bar{\ })$ are both nonnegative functions. In the sequel we use the notation $Q_T := \quad \times (0,T)$.

It is well known (see [6], [11]) that there exists a unique global bounded non-negative smooth solution pair (u^ϵ, v^ϵ) of Problem (P^ϵ). We make the hypothesis

$$H_a \quad : \quad M_1^{m-n} M_2 < a,$$

where

$$M_1 := \|u_0\|_{L^\infty(\ )} \quad \text{and} \quad M_2 := \|v_0\|_{L^\infty(\ )}.$$

The main result of this paper is the following :

Theorem 2.1 *Let $T > 0$ be fixed arbitrarily. As $\epsilon \to 0$*

$$u^\epsilon \to 0 \quad \text{in } C(\bar{\ } \times [\mu,\infty)) \cap L^2(Q_T) \tag{2.1}$$

for all $\mu > 0$ and there exists a function $v \in L^2(Q_T)$ such as

$$v^\epsilon \to v \quad \text{in } L^2(Q_T), \tag{2.2}$$

where the function v is the unique classical solution of the problem

$$(P^0)\begin{cases} v_t = d\Delta v & \text{in } Q,\\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial \quad \times (0,\infty),\\ v(x,0) = \bar{V}(x) & \text{for all } x \in \quad , \end{cases}$$

and

$$\bar{V}(x) = \lim_{t\to\infty} V(x,t),$$

where (U,V) is the unique solution of the initial value problem (Q^0)

$$(Q^0)\begin{cases} U_t = U^m V - aU^n & \text{in } Q,\\ V_t = -\,U^m V & \text{in } Q,\\ U(x,0) = u_0(x) \quad V(x,0) = v_0(x) & \text{for all } x \in \quad . \end{cases}$$

The one-dimensional case of this result is also numerically confirmed by (Fig. 1-1). In order to prove this result, we introduce a new time variable $\tau = \dfrac{t}{\epsilon}$ and set

$$U^\epsilon(x,\tau) := u^\epsilon(x,t) \quad V^\epsilon(x,\tau) := v^\epsilon(x,t).$$

Then U^ϵ and V^ϵ satisfy the problem

$$(Q^\epsilon)\quad\begin{cases} U_t = \epsilon\Delta U + U^m V - aU^n & \text{in } Q,\\ V_t = \epsilon d\Delta V - U^m V & \text{in } Q,\\ \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0 & \text{on } \partial \quad \times (0,\infty),\\ U(x,0) = u_0(x) \quad V(x,0) = v_0(x) & \text{for all } x \in \quad . \end{cases}$$

We recall [11] that

$$(u^\epsilon, v^\epsilon)(t) \to (0, v^\epsilon_\infty) \quad \text{in } C(\bar{\ }) \text{ as } t \to \infty, \tag{2.3}$$

for some positive constant v^ϵ_∞.

The second result which we prove is the following :

Theorem 2.2. *Let $(0, v^\epsilon_\infty)$ be the equilibrium solution of (P^ϵ). Then*

$$v^\epsilon_\infty \to \frac{1}{|\ \ |}\int \bar{V}(x)dx \quad \text{as} \quad \epsilon \to 0. \tag{2.4}$$

This theorem tells us that for small $\epsilon > 0$, the value v_∞^ϵ is approximately given by the spatial average of $\bar{V}(x)$ which is the asymptotically stable critical point of (Q^0).

Remark. In the case that $m = n$, the condition H_a becomes $\|v_0\|_{L^\infty(\ )} < a$. Suppose that it is not satisfied ; then Theorem 2.2 does not hold. As a counter example, we consider the one-dimensional problem in the interval $\ = (0,1)$ and choose u_0 with support in $[0, \frac{1}{2}]$ and $v_0 = 3a$ on . Then, the study of the ODE system (Q^ϵ) shows that $V(x,t) = v_0 = 3a$ for $x \in (\frac{1}{2}, 1]$ and all $t > 0$ so that

$$\int_0^1 \bar{V}(x)dx \geq \frac{3a}{2},$$

whereas if $m = n$,

$$v_\infty^\epsilon < a.$$

In the appendix, we study two special cases *without assuming Hypothesis* H_a. As the first case, we take $a = 0$. Then the $L^1(\ )$ norm of $(u^\epsilon + v^\epsilon)(t)$ is preserved in time and equal to the average over of $(u_0 + v_0)$. Thus the asymptotic behavior of $(u^\epsilon, v^\epsilon)(t)$ as $t \to \infty$ is well known. More precisely, we prove the following result:

Theorem 2.3. *Let* (u^ϵ, v^ϵ) *be the solution of* (P^ϵ) *with* $a = 0$*. Then*

$$v^\epsilon \to 0 \quad \text{in } L^2(Q_T) \text{ as } \epsilon \to 0, \tag{2.5}$$

and

$$u^\epsilon \to u \quad \text{in } L^2(Q_T) \text{ as } \epsilon \to 0,$$

where u *is the unique solution of the problem*

$$\begin{cases} u_t = \Delta u & \text{in } \ \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \ \times (0,T), \\ u(x,0) = u_0(x) + v_0(x) & \text{for all } x \in \ . \end{cases}$$

The second case which we consider is the case that $n > m \geq 1$. Then we have that (see [11])

$$(u^\epsilon, v^\epsilon)(t) \to (0,0) \quad \text{as } t \to \infty.$$

We prove the following result :

Theorem 2.4. *Fix $T > 0$ arbitrarily and suppose that $n > m \geq 1$ and that $u_0(x) > 0$ for all $x \in$. Then*

$$u^\epsilon(t),\ v^\epsilon(t) \to 0 \quad \text{in } L^2(Q_T) \text{ as } \epsilon \to 0. \tag{2.6.}$$

The proof of these two theorems are shown in the Appendix.

3. Decay of u^ϵ and Precompactness of $\{v^\epsilon\}$

We start with the following lemma :

Lemma 3.1. *Let (u^ϵ, v^ϵ) be the solution of Problem (P^ϵ). Then*

$$0 \leq \frac{1}{\epsilon}\int_0^\infty \int (u^\epsilon)^m v^\epsilon \leq \int v_0, \tag{3.1}$$

and

$$\int u_0 \leq \frac{a}{\epsilon}\int_0^\infty \int (u^\epsilon)^n \leq \int (u_0 + v_0). \tag{3.2}$$

Proof. Integrating the second equation in (P^ϵ) over $\times (0,t)$ gives

$$\int v^\epsilon(t) - \int v_0 = -\frac{1}{\epsilon}\int_0^t \int (u^\epsilon)^m v^\epsilon,$$

in which we let $t \to \infty$ to deduce (3.1). Furthermore, adding up the two parabolic equations in (P^ϵ) and integrating over $\times (0,t)$ gives

$$\int (u^\epsilon + v^\epsilon)(t) - \int (u_0 + v_0) = -\frac{a}{\epsilon}\int_0^t \int (u^\epsilon)^n,$$

and letting $t \to \infty$ we deduce, also using that $u^\epsilon(t) \to 0$ as $t \to \infty$, that

$$\frac{a}{\epsilon}\int_0^\infty \int (u^\epsilon)^n = \int (u_0 + v_0) - \lim_{t\to\infty}\int v^\epsilon(t).$$

Also since

$$\frac{d}{dt}\int v^\epsilon(t) = -\frac{1}{\epsilon}\int (u^\epsilon)^m v^\epsilon \leq 0,$$

we obtain

$$\int v^\epsilon(t) \le \int v_0.$$

Finally (3.2) follows from the inequality

$$0 \le \lim_{t\to\infty} \int v^\epsilon(t) \le \int v_0. \qquad \blacksquare$$

Next we show the following result :

Lemma 3.2. *Let* (u^ϵ, v^ϵ) *be a solution of* (P^ϵ)*. Then*

$$0 \le v^\epsilon(x,t) \le M_2, \tag{3.3}$$

and

$$0 \le u^\epsilon(x,t) \le \begin{cases} M_1 e^{-\dfrac{\delta t}{\epsilon}} & \text{if } n=1, \\ \dfrac{M_1}{\left(1+(n-1)\delta M_1^{n-1}\dfrac{t}{\epsilon}\right)^{\frac{1}{n-1}}} & \text{if } n>1, \end{cases} \tag{3.4}$$

for all $(x,t) \in Q$*, where* $\delta := a - \|u_0\|_{L^\infty(\ )}^{m-n} \|v_0\|_{L^\infty(\ )} = a - M_1^{m-n} M_2 > 0$.

Proof. The second inequality in (3.3) follows from the maximum principle. Next we prove the second inequality in (3.4). Define $\mathcal{L}^\epsilon$ by

$$\mathcal{L}^\epsilon(w) := w_t - \Delta w - \frac{1}{\epsilon}(w^m v^\epsilon - a w^n)$$

for a smooth function w and solve the following initial value problem :

$$\begin{cases} \bar{u}_t^\epsilon = -\dfrac{\delta}{\epsilon}(\bar{u}^\epsilon)^n, \\ \bar{u}^\epsilon(0) = M_1. \end{cases}$$

We find that

$$\bar{u}^\epsilon(t) = \begin{cases} M_1 e^{-\frac{\delta t}{\epsilon}} & \text{if } n=1, \\ \dfrac{M_1}{\left(1+(n-1)\delta M_1^{n-1}\dfrac{t}{\epsilon}\right)^{\frac{1}{n-1}}} & \text{if } n>1. \end{cases}$$

Indeed, if $n = 1$, we have that

$$\bar{u}^\epsilon_t = (M_1 e^{-\frac{\delta t}{\epsilon}})_t = -M_1 \frac{\delta}{\epsilon} e^{-\frac{\delta t}{\epsilon}} = -\frac{\delta}{\epsilon}\bar{u}^\epsilon,$$

whereas if $n > 1$, we have that

$$\begin{aligned}
\bar{u}^\epsilon_t &= \left(\frac{M_1}{\left(1+(n-1)\delta M_1^{n-1}\frac{t}{\epsilon}\right)^{\frac{1}{n-1}}}\right)_t \\
&= \left(-\frac{1}{n-1}(n-1)\delta M_1^{n-1}\frac{1}{\epsilon}\right)\frac{M_1}{\left(1+(n-1)\delta M_1^{n-1}\frac{t}{\epsilon}\right)^{\frac{1}{n-1}+1}} \\
&= -\frac{\delta}{\epsilon}\frac{M_1^n}{\left(1+(n-1)\delta M_1^{n-1}\frac{t}{\epsilon}\right)^{\frac{n}{n-1}}} \\
&= -\frac{\delta}{\epsilon}(\bar{u}^\epsilon)^n.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{L}^\epsilon(\bar{u}^\epsilon) &= (\bar{u}^\epsilon)_t - \Delta\bar{u}^\epsilon + \frac{1}{\epsilon}(\bar{u}^\epsilon)^n(a-(\bar{u}^\epsilon)^{m-n}v^\epsilon) \\
&= -\frac{\delta}{\epsilon}(\bar{u}^\epsilon)^n + \frac{1}{\epsilon}(\bar{u}^\epsilon)^n(a-(\bar{u}^\epsilon)^{m-n}v^\epsilon) \\
&= \frac{1}{\epsilon}(\bar{u}^\epsilon)^n\Big(a-\delta-(\bar{u}^\epsilon)^{m-n}v^\epsilon\Big).
\end{aligned}$$

Since $\bar{u}^\epsilon \le M_1$ and by the definition of δ, we find that

$$a-\delta-(\bar{u}^\epsilon)^{m-n}v^\epsilon \ge a-\delta-M_1^{m-n}M_2 = 0,$$

so that

$$\mathcal{L}^\epsilon(\bar{u}^\epsilon) \ge 0.$$

Since

$$\frac{\partial\bar{u}^\epsilon}{\partial\nu} = 0 \quad \text{on } \partial \quad \times \mathbb{R}^+,$$

and

$$\bar{u}^\epsilon(0) = M_1 \ge u_0(x),$$

the comparison principle (see for instance [14]) insures that

$$u^\epsilon(x,t) \le \bar{u}^\epsilon(t) \quad \text{for all } (x,t) \in Q.$$

■

Corollary 3.3. *We have that*

$$u^\epsilon \to 0 \quad \text{as } \epsilon \to 0 \text{ in } C(^{-} \times [\mu, \infty))$$

for all $\mu > 0$ *and*

$$u^\epsilon \to 0 \quad \text{as } \epsilon \to 0 \text{ in } L^2(Q_T),$$

for all $T > 0$.

Proof. Let μ positive be arbitrary. We deduce from Lemma 3.2 that

$$\sup_{^{-} \times [\mu,\infty)} |u^\epsilon(x,t)| \leq \begin{cases} M_1 e^{-\frac{\delta\mu}{\epsilon}} & \text{if } n = 1, \\ \dfrac{M_1}{\left(1 + (n-1)\delta M_1^{n-1} \dfrac{\mu}{\epsilon}\right)^{\frac{1}{n-1}}} & \text{if } n > 1, \end{cases}$$

which converges to zero as $\epsilon \to 0$. Moreover we have that for all $T \geq \mu > 0$

$$\begin{aligned} \int_0^T \int (u^\epsilon)^2 \ &= \int_0^\mu \int (u^\epsilon)^2 + \int_\mu^T \int (u^\epsilon)^2 \\ &\leq \mu| \ \ |M_1^2 + (T-\mu)| \ \ |M_1^2 \begin{cases} e^{-\frac{2\delta\mu}{\epsilon}} & \text{if } n = 1, \\ \dfrac{1}{\left(1 + (n-1)\delta M_1^{n-1} \dfrac{\mu}{\epsilon}\right)^{\frac{2}{n-1}}} & \text{if } n > 1, \end{cases} \end{aligned}$$

in which we let ϵ tend to 0 to deduce

$$\limsup_{\epsilon \to 0} \int_0^T \int (u^\epsilon)^2 \leq \mu| \ \ |M_1^2$$

for all $\mu > 0$ so that

$$\limsup_{\epsilon \to 0} \int_0^T \int (u^\epsilon)^2 = 0,$$

which completes the proof. ■

Next we will show a compactness property for the sequence $\{v^\epsilon\}$.

Lemma 3.4. *There exists a positive constant* C_1 *such that*

$$\int_0^\infty \int |\nabla v^\epsilon|^2 \leq C_1.$$

Proof. We multiply the second equation of (P^ϵ) by v^ϵ and integrate by parts to obtain

$$\frac{1}{2}\int (v^\epsilon(t))^2 + d\int_0^t \int |\nabla v^\epsilon|^2 \le \frac{1}{2}\int v_0,$$

where we then let $t \to \infty$. ■

Lemma 3.5. *The sequence* $\{v^\epsilon\}$ *is relatively compact in* $L^2(0,T;L^2(\ ))$*. In particular, there exists a subsequence which we denote again by* $\{v^\epsilon\}$ *and a function* v *such that*

$$v^\epsilon \to v \quad \text{strongly in } L^2(Q_T),$$

as $\epsilon \to 0$.

Proof. By Lemma 3.4, we find that $\{v^\epsilon\}$ is bounded in $L^2(0,T;H^1(\ ))$. This implies that

$$\{\Delta v^\epsilon\} \quad \text{is bounded in } L^2(0,T;(H^1(\ ))').$$

Furthermore, Lemma 3.1 gives

$$\{\frac{1}{\epsilon}(u^\epsilon)^m v^\epsilon\} \quad \text{is bounded in } L^1(0,T;L^1(\ )).$$

In particular, since $H^s(\ ) \subset L^\infty(\ )$ for s large enough

$$L^1(\ ) \subset (L^\infty(\ ))' \subset (H^s(\ ))',$$

holds. It follows that

$$\{\frac{1}{\epsilon}(u^\epsilon)^m v^\epsilon\} \quad \text{is bounded in } L^1(0,T;(H^s(\ ))')$$

which together with the fact that $(H^1(\ ))' \subset (H^s(\ ))'$ for $s \ge 1$, implies that

$$\{v_t^\epsilon\} \quad \text{is bounded in } L^1(0,T;(H^s(\ ))').$$

Since also

$$\{v^\epsilon\} \quad \text{is bounded in } L^2(0,T;H^1(\ ))$$

and by the embeddings $H^1(\ ) \subset L^2(\ ) \subset (H^s(\ ))')$ where the first embedding is compact, it follows from [15, Corollary 4]sim that

$$\{v^\epsilon\} \quad \text{is precompact in } L^2(0,T;L^2(\ )),$$

which completes the proof. ∎

The above results are sufficient to prove that v satisfies the parabolic equation and the homogeneous Neumann boundary condition in Problem (P^0). However we cannot prove yet that v satisfies the initial condition in Problem (P^0) and therefore we cannot prove either at this point that the function v is uniquely defined.

4. The System of Ordinary Differential Equations

In this section we study the system :

$$(IVP) \quad \begin{cases} U_t = U^m V - aU^n & \text{for } t > 0, \\ V_t = -\, U^m V & \text{for } t > 0, \\ U(0) = u_0 \quad V(0) = v_0, \end{cases}$$

where $0 \leq u_0 \leq M_1$ and $0 \leq v_0 \leq M_2$ are fixed constants. We only suppose that the constants m and n are such that $m, n \geq 1$.

Lemma 4.1. *Problem (IVP) has a unique solution (U, V) such that for any $t \geq 0$*

$$0 \leq U(t) \leq u_0 + v_0 \leq M_1 + M_2$$

and

$$0 \leq V(t) \leq v_0 \leq M_2$$

hold. Moreover

$$(U, V) \to (0, \bar{V}) \quad \text{as } t \to \infty.$$

Proof. Since $u_0, v_0 \geq 0$, we have that $U, V \geq 0$. Also since $V_t \leq 0$ and

$$(U + V)_t = -aU^n \leq 0,$$

it follows that

$$V(t) \leq v_0 \;,\; U(t) \leq u_0 + v_0.$$

Since $V(t)$ is nonincreasing and bounded from below there exists a constant $\bar{V} \in [0, v_0]$ such that

$$V(t) \to \bar{V} \text{ as } t \to \infty.$$

Similarly there exists a constant $\bar{U}$ such that

$$U(t) + V(t) \to \bar{U} + \bar{V} \text{ as } t \to \infty.$$

Therefore

$$U(t) \to \bar{U} \text{ as } t \to \infty,$$

and $\bar{U} \in [0, u_0 + v_0]$. Setting $U^t(s) = U(t+s)$ and $V^t(s) = V(t+s)$ for $s \in [0,1]$, we deduce that

$$U^t \to \bar{U} \ , \ V^t \to \bar{V} \text{ in } C([0,1]),$$

as $t \to \infty$. Integrating the differential equations for U and V gives

$$\begin{cases} U(t+1) - U(t) = \displaystyle\int_t^{t+1} \Big(U^m V - a U^n \Big), \\ V(t+1) - V(t) = - \displaystyle\int_t^{t+1} U^m V, \end{cases}$$

which we rewrite as

$$\begin{cases} U^t(1) - U^t(0) = \displaystyle\int_0^1 \Big((U^t)^m V^t - a (U^t)^n \Big), \\ V^t(1) - V^t(0) = - \displaystyle\int_0^1 (U^t)^m V^t. \end{cases}$$

Letting $t \to \infty$, we deduce that

$$\begin{cases} 0 = (\bar{U})^m \bar{V} - a (\bar{U})^n, \\ 0 = -(\bar{U})^m \bar{V}, \end{cases}$$

so that $(\bar{U})^n = 0$ and thus $\bar{U} = 0$. ∎

Remark. In the special case that $m = n$, we have that

$$\begin{cases} U(t) + V(t) = u_0 + v_0 - a \displaystyle\int_0^t U^m(s) ds, \\ V(t) = v_0 e^{- \int_0^t U^m(s) ds}, \end{cases}$$

which implies, letting $t \to \infty$, the following equalities involving $\bar{V}$:

$$\begin{cases} \bar{V} = u_0 + v_0 - a \displaystyle\int_0^\infty U^m(s) ds, \\ \bar{V} = v_0 e^{- \int_0^\infty U^m(s) ds}. \end{cases}$$

Setting $I = \int_0^\infty U^m(s)ds$, one can compute I from the identity

$$u_0 + v_0 - aI = v_o e^{-I},$$

and then deduce that

$$\bar{V} = u_0 + v_0 - aI = v_o e^{-I}.$$

5. Gradient Estimates for the System of Ordinary Differential Equations

Again we only suppose that $m, n \geq 1$ and we consider the system

$$(Q^0) \quad \begin{cases} U_t = U^m V - aU^n & \text{in } Q, \\ V_t = -\, U^m V & \text{in } Q, \\ U(x,0) = u_0(x), \quad V(x,0) = v_0(x) & \text{for all } x \in \quad . \end{cases}$$

By Lemma 4.1, we have that for each $x \in$

$$V(x,t) \to \bar{V}(x) \quad \text{as } t \to \infty.$$

Since $V(x,t) \leq M_2$, the Lebesgue monotone convergence theorem implies the following result :

Lemma 5.1. *For all $p \in [1, \infty)$ we have that*

$$V(x,t) \to \bar{V}(x) \quad \text{in } L^p(\quad) \text{ as } t \to \infty.$$

Next we prove the following lemma :

Lemma 5.2. *There exist two positive constants C_2 and C_3 such that*

$$\|\nabla U(t)\|_{L^\infty(\quad)}, \|\nabla V(t)\|_{L^\infty(\quad)} \leq C_2 e^{C_3 t} \quad \text{for all } t > 0 \tag{5.1}$$

Proof. We set

$$W = U + V,$$

so that

$$U = W - V.$$

We have that

$$\begin{cases} V_t = -U^m V & \text{in } Q, \\ W_t = -aU^n & \text{in } Q, \end{cases}$$

and therefore

$$V(t) = v_0 e^{-\displaystyle\int_0^t U^m(s)ds},$$

$$W(t) = (u_0 + v_0) - a\int_0^t U^n(s)ds.$$

Thus

$$\nabla V(t) = \nabla v_0 e^{-\displaystyle\int_0^t U^m(s)ds} - v_0 m\Big(\int_0^t \Big(U^{m-1}\nabla U\Big)ds\Big)e^{-\displaystyle\int_0^t U^m(s)ds},$$

$$\nabla W(t) = \nabla(u_0 + v_0) - an\int_0^t U^{n-1}(s)\nabla U(s)ds,$$

which imply that

$$|\nabla V(t)| \leq |\nabla v_0| + C\int_0^t |\nabla U(s)|ds,$$

$$|\nabla W(t)| \leq |\nabla(u_0 + v_0)| + C\int_0^t |\nabla U(s)|ds,$$

and then

$$|\nabla V(t)| \leq |\nabla v_0| + C\int_0^t (|\nabla V(s)| + |\nabla W(s)|)ds,$$

$$|\nabla W(t)| \leq |\nabla(u_0 + v_0)| + C\int_0^t (|\nabla V(s)| + |\nabla W(s)|)ds.$$

Next we add up those two inequalities to deduce

$$|\nabla V(t)| + |\nabla W(t)| \leq |\nabla v_0| + |\nabla(u_0 + v_0)| + C\int_0^t (|\nabla V(s)| + |\nabla W(s)|)ds,$$

and finally we obtain

$$|\nabla V(t)| + |\nabla W(t)| \leq \tilde{C}e^{Ct},$$

which completes the proof. ■

6. Asymptotic Limit of Problem (Q^ϵ)

In this section we study the limiting behavior as $\epsilon \to 0$ of the solution (U^ϵ, V^ϵ) of Problem (Q^ϵ) :

$$(Q^\epsilon)\quad\begin{cases} U_t = \epsilon\Delta U + U^m V - aU^n & \text{in } Q, \\ V_t = \epsilon d\Delta V - U^m V & \text{in } Q, \\ \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0 & \text{on } \partial\quad \times (0,\infty), \\ U(x,0) = u_0(x) \quad V(x,0) = v_0(x) & \text{for all } x \in \quad . \end{cases}$$

Here we suppose that one of the two following hypotheses is satisfied :
(i) $m \geq n \geq 1$ and $\|u_0\|^{m-n}_{L^\infty(\ )}\|v_0\|_{L^\infty(\ )} < a$,
or
(ii) $n > m \geq 1$.

We first prove the following estimates :

Lemma 6.1. *There exists a positive constant* $\tilde{M}_1$ *such that*

$$\begin{cases} 0 \leq U^\epsilon(x,t) \leq \tilde{M}_1, \\ 0 \leq V^\epsilon(x,t) \leq M_2, \end{cases} \tag{6.1}$$

for all $(x,t) \in \quad \times [0,\infty)$. *Moreover there exists a positive constant* C_4 *such that for all* $t \geq 0$

$$\int_0^t \int |\nabla U^\epsilon|^2 \leq \frac{C_4}{\epsilon}, \tag{6.2}$$

$$\int_0^t \int |\nabla V^\epsilon|^2 \leq \frac{C_4}{\epsilon}. \tag{6.3}$$

Proof. The comparison principle insures that

$$0 \leq V^\epsilon(x,t) \leq M_2.$$

Suppose that $n > m \geq 1$. We remark that there exists M_3 such that

$$r^m M_2 - ar^n \leq 0 \quad \text{for all } r \geq M_3.$$

Then applying a comparison principle, one has that

$$U^\epsilon(x,t) \leq \tilde{M}_1 \quad \text{for all } (x,t) \in Q, \tag{6.4}$$

where $\tilde{M}_1 = \max(M_1, M_3)$.

In the case that $m \geq n \geq 1$, we recall that

$$U^\epsilon(x,t) = u^\epsilon(x,\epsilon t) \quad V^\epsilon(x,t) = v^\epsilon(x,\epsilon t),$$

for all $(x,t) \in \quad \times [0,\infty)$. Then Lemma 3.2 gives

$$0 \leq u^\epsilon(x,\epsilon t) \leq M_1 \leq \tilde{M}_1,$$

which proves the inequalities (6.1).

Then by multiplying the second equation for in (Q^ϵ) by V^ϵ and integrating by part, we obtain

$$\frac{1}{2}\int (V^\epsilon)^2(t) + \epsilon d \int_0^t \int |\nabla V^\epsilon|^2 + \int_0^t \int (U^\epsilon)^n (V^\epsilon)^2 = \frac{1}{2}\int v_0^2,$$

so that

$$\epsilon d \int_0^t \int |\nabla V^\epsilon|^2 \leq \frac{1}{2}\int v_0^2.$$

Next adding the equations for U^ϵ and V^ϵ, we have

$$\begin{aligned} U_t^\epsilon + V_t^\epsilon &= \epsilon \Delta U^\epsilon + \epsilon d \Delta V^\epsilon - a(U^\epsilon)^n \\ &= \epsilon \Delta (U^\epsilon + V^\epsilon) + \epsilon(d-1)\Delta V^\epsilon - a(U^\epsilon)^n, \end{aligned}$$

so that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \Big(U^\epsilon \quad + V^\epsilon\Big)^2 + \epsilon \int \Big|\nabla(U^\epsilon + V^\epsilon)\Big|^2 + a\int (U^\epsilon)^n \Big(U^\epsilon + V^\epsilon\Big) \\ &\qquad = \epsilon(d-1)\int \nabla V^\epsilon . \nabla(U^\epsilon + V^\epsilon), \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}\int \Big(U^\epsilon + V^\epsilon\Big)^2(t) + \frac{\epsilon}{2}\int_0^t \int \Big| \quad \nabla(U^\epsilon + V^\epsilon)\Big|^2 \\ &\qquad\qquad \leq \frac{1}{2}\int (u_0 + v_0)^2 + \epsilon C(d) \int_0^t \int \Big|\nabla V^\epsilon\Big|^2 \leq C. \end{aligned}$$

Thus we conclude that

$$\epsilon \int_0^t \int \Big|\nabla U^\epsilon\Big|^2 \leq 2\epsilon \int_0^t \int \Big|\nabla(U^\epsilon + V^\epsilon)\Big|^2 + 2\epsilon \int_0^t \int \Big|\nabla V^\epsilon\Big|^2 \leq C,$$

which completes the proof. ■

Next we prove the following result :

Theorem 6.2. *There exists positive constants α, β and C_5 such that for all* $t \in [0, \ln \frac{1}{\epsilon^\alpha}]$

$$\int_0^t \int \Big[\Big(U^\epsilon - U\Big)^2 + \Big(V^\epsilon - V\Big)^2\Big](x,s)dxds \le C_5 \epsilon^\beta. \tag{6.5}$$

Proof. We set $\tilde{U}^\epsilon = U^\epsilon - U$ and $\tilde{V}^\epsilon = V^\epsilon - V$ and we define

$$f_m(r,s) = \begin{cases} \dfrac{r^m - s^m}{r - s} & \text{if } r \neq s, \\ mr^{m-1} & \text{if } r = s, \end{cases}$$

and

$$f_n(r,s) = \begin{cases} \dfrac{r^n - s^n}{r - s} & \text{if } r \neq]texts, \\ nr^{n-1} & \text{if } r = s. \end{cases}$$

Then we have that

$$\begin{aligned} \tilde{U}^\epsilon_t &= (U^\epsilon - U)_t \\ &= \epsilon \Delta U^\epsilon + (U^\epsilon)^m V^\epsilon - a(U^\epsilon)^n - U^m V + aU^n \\ &= \epsilon \Delta U^\epsilon + V^\epsilon((U^\epsilon)^m - U^m) + U^m(V^\epsilon - V) - a((U^\epsilon)^n - U^n) \\ &= \epsilon \Delta U^\epsilon + \Big(V^\epsilon f_m(U^\epsilon, U) - a f_n(U^\epsilon, U)\Big)\tilde{U}^\epsilon + U^m \tilde{V}^\epsilon, \end{aligned}$$

and

$$\begin{aligned} \tilde{V}^\epsilon_t &= (V^\epsilon - V)_t \\ &= \epsilon d \Delta V^\epsilon - (U^\epsilon)^m V^\epsilon + U^m V \\ &= \epsilon d \Delta V^\epsilon - (U^\epsilon)^m \tilde{V}^\epsilon - V f_m(U^\epsilon, U)\tilde{U}^\epsilon. \end{aligned}$$

Thus $\tilde{U}^\epsilon$ and $\tilde{V}^\epsilon$ satisfy the problem :

$$\begin{cases} \tilde{U}^\epsilon_t = \epsilon \Delta U^\epsilon + \Big(V^\epsilon f_m(U^\epsilon, U) - a f_n(U^\epsilon, U)\Big)\tilde{U}^\epsilon + U^m \tilde{V}^\epsilon & \text{in } Q, \\ \tilde{V}^\epsilon_t = \epsilon d \Delta V^\epsilon - (U^\epsilon)^m \tilde{V}^\epsilon - V f_m(U^\epsilon, U)\tilde{U}^\epsilon & \text{in } Q, \\ \dfrac{\partial U^\epsilon}{\partial \nu} = \dfrac{\partial V^\epsilon}{\partial \nu} = 0 & \text{on } \partial \quad \times (0, \infty), \\ \tilde{U}^\epsilon(x,0) = 0 \quad \tilde{V}^\epsilon(x,0) = 0 & \text{for all } x \in \quad . \end{cases}$$

Multiplying the equations respectively by $\tilde{U}^\epsilon$ and $\tilde{V}^\epsilon$ and integrating over gives

$$\frac{1}{2}\frac{d}{dt}\int (\tilde{U}^\epsilon)^2 \;= \epsilon\int \Delta U^\epsilon \tilde{U}^\epsilon + \int \Big(V^\epsilon f_m(U^\epsilon,U) - a f_n(U^\epsilon,U)\Big)(\tilde{U}^\epsilon)^2 + \int U^m \tilde{V}^\epsilon \tilde{U}^\epsilon,$$
$$\frac{1}{2}\frac{d}{dt}\int (\tilde{V}^\epsilon)^2 = \epsilon d\int \Delta V^\epsilon \tilde{V}^\epsilon - \int (U^\epsilon)^m (\tilde{V}^\epsilon)^2 - \int V f_m(U^\epsilon,U)\tilde{U}^\epsilon \tilde{V}^\epsilon,$$

and then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int (\tilde{U}^\epsilon)^2 = &-\epsilon\int \nabla U^\epsilon . \nabla\tilde{U}^\epsilon \\ &+\int \Big(V^\epsilon f_m(U^\epsilon,U) - a f_n(U^\epsilon,U)\Big)(\tilde{U}^\epsilon)^2 + \int U^m \tilde{V}^\epsilon \tilde{U}^\epsilon,\end{aligned} \tag{6.6}$$

$$\frac{1}{2}\frac{d}{dt}\int (\tilde{V}^\epsilon)^2 = -\epsilon d\int \nabla V^\epsilon . \nabla\tilde{V}^\epsilon - \int (U^\epsilon)^m (\tilde{V}^\epsilon)^2 - \int V f_m(U^\epsilon,U)\tilde{U}^\epsilon \tilde{V}^\epsilon. \tag{6.7}$$

By the lemmas 4.1 and 6.1, one finds that there exists a positive constant C such that

$$0 \le U^\epsilon, V^\epsilon, U, V \le C \quad \text{in} \quad \times [0,\infty).$$

Therefore summing (6.6) and (6.7) gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \Big((\tilde{U}^\epsilon)^2 + (\tilde{V}^\epsilon)^2\Big) \le &-\epsilon \int \nabla U^\epsilon . \nabla\tilde{U}^\epsilon - \epsilon d\int \nabla V^\epsilon . \nabla\tilde{V}^\epsilon \\ &+C\int (\tilde{U}^\epsilon)^2 + C\int (\tilde{V}^\epsilon)^2 + C\int |\tilde{U}^\epsilon \tilde{V}^\epsilon|,\end{aligned}$$

so that, by Young's inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \Big((\tilde{U}^\epsilon)^2 + (\tilde{V}^\epsilon)^2\Big) \le &-\epsilon\int \nabla U^\epsilon . \nabla(U^\epsilon - U) \\ &-\epsilon d\int \nabla V^\epsilon . \nabla(V^\epsilon - V) + 2C\int \Big((\tilde{U}^\epsilon)^2 + (\tilde{V}^\epsilon)^2\Big).\end{aligned} \tag{6.8}$$

By the Lemmas 5.2 and 6.1, we have that

$$\begin{aligned}\epsilon\Big|\int_0^t \int \nabla U^\epsilon . \nabla U\Big| &\le \epsilon\Big(\int_0^t \int |\nabla U^\epsilon|^2\Big)^{\frac{1}{2}} \Big(\int_0^t \int |\nabla U|^2\Big)^{\frac{1}{2}} \\ &\le \epsilon\Big(\frac{C_4}{\epsilon}\Big)^{\frac{1}{2}} |\quad|^{\frac{1}{2}} \Big(\int_0^t C_2^2 e^{2C_3 t}\Big)^{\frac{1}{2}} \\ &\le \tilde{C}\sqrt{\epsilon} e^{C_3 t},\end{aligned}$$

and similarly

$$\epsilon\Big|\int_0^t\int \nabla V^\epsilon.\nabla V\Big| \le \epsilon\Big(\int_0^t\int |\nabla V^\epsilon|^2\Big)^{\frac{1}{2}}\Big(\int_0^t\int |\nabla V|^2\Big)^{\frac{1}{2}}$$
$$\le \tilde{C}\sqrt{\epsilon}e^{C_3 t}.$$

Then, integrating (6.8) over $(0,t)$ yields

$$\int \Big((\tilde{U}^\epsilon(t))^2 + (\tilde{V}^\epsilon(t))^2\Big) \le \tilde{C}\sqrt{\epsilon}e^{C_3 t} + C\int_0^t\int \Big((\tilde{U}^\epsilon)^2 + (\tilde{V}^\epsilon)^2\Big). \tag{6.9}$$

Setting $Y(t) = \displaystyle\int_0^t\int \Big((\tilde{U}^\epsilon)^2 + (\tilde{V}^\epsilon)^2\Big)$ and $h(t) = \tilde{C}\sqrt{\epsilon}e^{C_3 t}$, we have proved that

$$Y'(t) \le CY(t) + h(t).$$

Applying Gronwall's inequality, we deduce

$$\begin{aligned} Y(t) &\le \textstyle\int_0^t h(\tau)e^{C(t-\tau)}d\tau \\ &\le \tilde{C}'\sqrt{\epsilon}e^{C't}. \end{aligned}$$

Let $\alpha \in (0, \min(\dfrac{1}{2C'}, \dfrac{1}{2C_3}))$ be arbitrary. We have shown that for all $t \in [0, \ln\dfrac{1}{\epsilon^\alpha}]$

$$\int_0^t\int \Big((U^\epsilon - U)^2 + (V^\epsilon - V)^2\Big) \le \tilde{C}'\epsilon^{\frac{1}{2}-C'\alpha}. \tag{6.10}$$

This completes the proof of Theorem 6.2. ∎

Substituting the inequality (6.10) into (6.9), we deduce that

$$\int \Big(V^\epsilon(x, \ln\frac{1}{\epsilon^\alpha}) - V(x, \ln\frac{1}{\epsilon^\alpha})\Big)^2 \to 0 \quad \text{as } \epsilon \to 0.$$

Since, by Lemma 5.1, $V(x,t) \to \bar{V}(x)$ in $L^2(\ \ )$ as $t \to \infty$, we have proved the following result :

Corollary 6.3. *Let* $\tau(\epsilon) = \ln\dfrac{1}{\epsilon^\alpha}$. *Then*

$$\int \Big(V^\epsilon(x, \tau(\epsilon)) - \bar{V}(x)\Big)^2 \to 0 \quad \text{as } \epsilon \to 0.$$

7. Proofs of Theorem 2.1 and Theorem 2.2

We recall that by Corollary 3.3 and Lemma 3.5, there exists a subsequence of $\{(u^\epsilon, v^\epsilon)\}$ which we denote again by $\{(u^\epsilon, v^\epsilon)\}$ such that

$$(u^\epsilon, v^\epsilon) \to (0, v) \quad \mathrm{in} L^2(Q_T) \mathrm{as} \epsilon \to 0.$$

We first prove the following result :

Lemma 7.1. Set $\tau(\epsilon) = \ln \dfrac{1}{\epsilon^\alpha}$. We have that

$$\int_{\epsilon\tau(\epsilon)}^T \int \frac{(u^\epsilon)^m}{\epsilon} \to 0 \quad \mathrm{as} \epsilon \to 0.$$

Proof. By Lemma 3.2, we have that
(i) if $n = 1$,

$$\begin{aligned} \int_{\epsilon\tau(\epsilon)}^T \int \frac{(u^\epsilon)^m}{\epsilon} &\le \int_{\epsilon\tau(\epsilon)}^T \frac{M_1^m| \ |}{\epsilon} e^{-\frac{\delta m t}{\epsilon}} \\ &\le \frac{M_1^m| \ |}{m\delta} e^{-\delta m \tau(\epsilon)} \to 0 \quad \text{as } \epsilon \to 0. \end{aligned}$$

(ii) if $n > 1$,

$$\begin{aligned} \int_{\epsilon\tau(\epsilon)}^T \int \frac{(u^\epsilon)^m}{\epsilon} &\le \int_{\epsilon\tau(\epsilon)}^T \frac{M_1^m| \ |}{\epsilon} \frac{1}{\Big(1 + (n-1)\delta M_1^{n-1} \dfrac{t}{\epsilon}\Big)^{\frac{m}{n-1}}} \\ &= \frac{M_1^m| \ |}{\epsilon} \Big(\frac{\epsilon}{(m-n+1)\delta M_1^{n-1}} \Big) \\ &\qquad \int_{\epsilon\tau(\epsilon)}^T \frac{d}{dt} \Big[\frac{1}{\Big(1 + (n-1)\delta M_1^{n-1} \dfrac{t}{\epsilon}\Big)^{\frac{m}{n-1} - 1}} \Big] \\ &\le \frac{M_1^{m-n+1}| \ |}{\delta(m-n+1)} \frac{1}{\Big(1 + (n-1)\delta M_1^{n-1} \tau(\epsilon)\Big)^{\frac{m-n+1}{n-1}}} \\ &\to 0 \quad \text{as } \epsilon \to 0. \end{aligned}$$

■

Proof. of Theorem 2.1 Let $T > 0$ be arbitrary and φ be an arbitrary smooth function such that $\dfrac{\partial \varphi}{\partial \nu} = 0$ on $\partial \ \times (0, T)$ and $\varphi(T) = 0$. Then

$$\int_{\epsilon\tau(\epsilon)}^T \int (v_t^\epsilon - d\Delta v^\epsilon)\varphi = -\frac{1}{\epsilon} \int_{\epsilon\tau(\epsilon)}^T \int (u^\epsilon)^m v^\epsilon \varphi,$$

which implies that

$$\int_{\epsilon\tau(\epsilon)}^{T}\int v^{\epsilon}\Big(\varphi_t + d\Delta\varphi\Big) = \int_{\epsilon\tau(\epsilon)}^{T}\int \frac{1}{\epsilon}(u^{\epsilon})^m v^{\epsilon}\varphi - \int v^{\epsilon}(x,\epsilon\tau(\epsilon))\varphi(x,\epsilon\tau(\epsilon)).$$

By Lemma 7.1, we have that

$$\lim_{\epsilon\to 0}\int_{\epsilon\tau(\epsilon)}^{T}\int \frac{1}{\epsilon}(u^{\epsilon})^m v^{\epsilon}\varphi \to 0,$$

whereas by Corollary 6.3

$$\int v^{\epsilon}(x,\epsilon\tau(\epsilon))\varphi(x,\epsilon\tau(\epsilon)) \to \int \bar{V}(x)\varphi(x,0)$$

as $\epsilon \to 0$. Furthermore, since

$$v^{\epsilon}\chi_{(\epsilon\tau(\epsilon),T)} \le M_2,$$

and since there exists a subsequence which we denote again by $\{v^{\epsilon}\}$ such that

$$v^{\epsilon} \to v \quad \text{a.e. in } Q_T,$$

we deduce from Lebesgue dominated convergence theorem that

$$v^{\epsilon}\chi_{(\epsilon\tau(\epsilon),T)} \to v \quad \text{in } L^1(Q_T).$$

Therefore v satisfies

$$\int_0^T\int v\Big(\varphi_t + d\Delta\varphi\Big) = -\int \bar{V}(x)\varphi(x,0)$$

for all $\varphi \in C^{2,1}(\bar{Q_T})$ such that $\dfrac{\partial\varphi}{\partial\nu} = 0$ on $\partial \quad \times [0,T]$ and $\varphi(T) = 0$. One can then easily deduce that v is the unique classical solution of Problem (P^0). ∎

Next we turn to the proof of Theorem 2.2. We first prove a key lemma which shows that the whole process occurs in a very short time.

Lemma 7.2. *Fix $\mu > 0$ be arbitrary. There holds*

$$\Big\|\int \Big(v^{\epsilon} - v^{\epsilon}_{\infty}\Big)\Big\|_{L^{\infty}(\mu,\infty)} \to 0 \quad \text{as } \epsilon \to 0. \tag{7.1}$$

Proof. We have that for $T \ge t \ge \mu$

$$\begin{aligned}\int \Big(v^{\epsilon}(T) - v^{\epsilon}(t)\Big) &= \int_t^T \frac{d}{dt}\int v^{\epsilon}(s)ds\\ &= -\int_t^T\int \frac{1}{\epsilon}(u^{\epsilon})^m(s)v^{\epsilon}(s)ds,\end{aligned}$$

which implies, letting $T \to \infty$, that

$$\Big|\int (v^\epsilon(t) - v^\epsilon_\infty)\Big| = \int_t^\infty \int \frac{1}{\epsilon}(u^\epsilon)^m(s)v^\epsilon(s)ds.$$

Furthermore we have that
(i) if $n = 1$ and $t \geq \mu$

$$\begin{aligned}\frac{1}{\epsilon}\int_t^\infty \int (u^\epsilon)^m v^\epsilon &\leq \frac{M_1^m M_2|\ \ |}{\epsilon}\int_t^\infty e^{-\frac{\delta m s}{\epsilon}}ds \\ &\leq \frac{M_1^m M_2|\ \ |}{\delta m}e^{-\frac{\delta m t}{\epsilon}} \\ &\leq \frac{M_1^m M_2|\ \ |}{\delta m}e^{-\frac{\delta m \mu}{\epsilon}},\end{aligned}$$

(ii) if $n > 1$ and $t \geq \mu$

$$\begin{aligned}\frac{1}{\epsilon}\int_t^\infty \int (u^\epsilon)^m v^\epsilon &\leq \frac{M_1^m M_2|\ \ |}{\epsilon}\int_t^\infty \Big(\frac{1}{\Big(1+(n-1)M_1^{n-1}\delta\frac{s}{\epsilon}\Big)^{\frac{m}{n-1}}}\Big)ds \\ &= -\frac{M_1^{m-n+1}M_2|\ \ |}{\delta(m-n+1)}\int_t^\infty \frac{d}{ds}\Big[\Big(1+(n-1)M_1^{n-1}\delta\frac{s}{\epsilon}\Big)^{-\frac{m-n+1}{n-1}}\Big]ds \\ &= \frac{M_1^{m-n+1}M_2|\ \ |}{\delta(m-n+1)}\frac{1}{\Big(1+(n-1)M_1^{n-1}\delta\frac{t}{\epsilon}\Big)^{\frac{m-n+1}{n-1}}} \\ &\leq \frac{M_1^{m-n+1}M_2|\ \ |}{\delta(m-n+1)}\frac{1}{\Big(1+(n-1)M_1^{n-1}\delta\frac{\mu}{\epsilon}\Big)^{\frac{m-n+1}{n-1}}}.\end{aligned}$$

Therefore

$$\begin{aligned}&\Big\|\int \Big(v^\epsilon - v^\epsilon_\infty\Big)\Big\|_{L^\infty(\mu,\infty)} \\ &\leq \begin{cases}\dfrac{M_1^m M_2|\ \ |}{\delta m}e^{-\frac{\delta m\mu}{\epsilon}} & \text{if } n = 1, \\ \dfrac{M_1^{m-n+1}M_2|\ \ |}{\delta(m-n+1)}\dfrac{1}{\Big(1+(n-1)M_1^{n-1}\delta\frac{\mu}{\epsilon}\Big)^{\frac{m-n+1}{n-1}}} & \text{if } n > 1,\end{cases} \\ &\to 0 \quad \text{as } \epsilon \to 0,\end{aligned}$$

which completes the proof. ∎

Proof of Theorem 2.2. Let $T > 0$ be arbitrary. Since

$$\int v(t) = \int \bar{V} \quad \text{for all } t \geq 0,$$

we have that, by Lemma 7.2,

$$
\begin{aligned}
\Big(\int \bar{V} - | \quad |v^\epsilon_\infty\Big)^2 &= \int_T^{T+1} \Big(\int \bar{V} - | \quad |v^\epsilon_\infty\Big)^2 dt \\
&= \int_T^{T+1} \Big(\int v(t) - | \quad |v^\epsilon_\infty\Big)^2 dt \\
&\leq 2\int_T^{T+1} \Big(\int v(t) - \int v^\epsilon(t)\Big)^2 dt \\
&\qquad +2\int_T^{T+1} \Big(\int v^\epsilon(t) - | \quad |v^\epsilon_\infty\Big)^2 dt \\
&\leq 2| \quad \int_T^{T+1} \int \Big(v(t) - v^\epsilon(t)\Big)^2 dt \\
&\qquad +2\int_T^{T+1} \Big(\int v^\epsilon(t) - | \quad |v^\epsilon_\infty\Big)^2 dt
\end{aligned}
$$

which tends to zero as $\epsilon \to 0$. ■

8. Concluding Remarks

In this paper, we have discussed the singular limit analysis of a two-component reaction-diffusion system with very fast reaction terms, namely Problem (P^ϵ). For this problem, it was already known that when $m \geq n \geq 1$, there exists some constant v^ϵ_∞ such that the nonnegative solution $(u^\epsilon, v^\epsilon)(t)$ tends to $(0, v^\epsilon_\infty)$ as $t \to \infty$. Assuming that $\|u_0\|^{m-n}_{L^\infty(\;)}\|v_0\|_{L^\infty(\;)} < a$, we have shown that as $\epsilon \to 0$, u^ϵ tends to zero and v^ϵ tends to v, where v is the solution of the heat equation

$$v_t = d\Delta v,$$

together with the homogeneous Neumann boundary condition and the initial condition $v(x,0) = \bar{V}(x)$ where $\bar{V}(x)$ is given by the asymptotic state

$$\lim_{t\to\infty}(U,V)(x,t) = (0, \bar{V}(x)),$$

and where (U,V) is the solution of the system of ordinary differential equations

$$
\left\{
\begin{array}{lll}
U_t = U^m V - aU^n & \text{in} & \times \mathbb{R}^+, \\
V_t = -U^m V & \text{in} & \times \mathbb{R}^+,
\end{array}
\right. \tag{8.1}
$$

with

$$(U,V)(x,0) = (u_0(x), v_0(x)).$$

Our second result shows that the asymptotic state v^ϵ_∞ of the reaction-diffusion Problem (P^ϵ) is approximately given by the spatial average of $\bar{V}(x)$ over .

For the case where $\|u_0\|^{m-n}_{L^\infty(\ )}\|v_0\|_{L^\infty(\ )} > a$, we have not yet discussed this singular limit problem as $\epsilon \to 0$. The difficulty is, as it was shown in the introduction, that the transient behavior of solutions is totally different from the previous case. In fact, it exhibits spatio-temporal patterns such as expending rings or splitting spots. Moreover, from pattern formation viewpoints, the analysis of the transient behavior of u is important but we will also leave this case as a future work for us.

Appendix A: The Case That $a = 0$

In this section we consider the problem

$$(R^\epsilon) \quad \begin{cases} u_t^\epsilon = \Delta u^\epsilon + \dfrac{1}{\epsilon}(u^\epsilon)^m v^\epsilon & \text{in } Q, \\ v_t^\epsilon = d\Delta v^\epsilon - \dfrac{1}{\epsilon}(u^\epsilon)^m v^\epsilon & \text{in } Q, \\ \dfrac{\partial}{\partial \nu}u^\epsilon = \dfrac{\partial}{\partial \nu}v^\epsilon = 0 & \text{on } \partial\ \ \times (0,\infty), \\ u^\epsilon(x,0) = u_0(x) \quad v^\epsilon(x,0) = v_0(x) & \text{for all } x \in\ \ , \end{cases}$$

with the same hypothesis on m, d, u_0, v_0 and as before (but of course without the assumption H_a). First we present some preliminary results.

Lemma A.1. *For all $(x,t) \in Q$, we have that*

$$0 \leq u^\epsilon(x,t), \quad 0 \leq v^\epsilon(x,t) \leq M_2. \tag{A.1}$$

Proof. (A.1) immediately follows from the comparison principle. ■

Lamma A.2. *The following inequality holds*

$$\frac{1}{\epsilon}\int_0^\infty \int (u^\epsilon)^m v^\epsilon \leq \int v_0. \tag{A.2}$$

Proof. Integrate the partial differential equation for v^ϵ in (R^ϵ) on $\ \times (0,t)$ and let $t \to \infty$. The result of Lemma A.2 immediately follows. ■

Lemma A.3. *We have that for all $t > 0$*

$$d\int_0^T \int |\nabla v^\epsilon|^2 \leq \int (v_0)^2 \tag{A.3}$$

and there exist a constant C_6 such that

$$\int_0^T \int |\nabla u^\epsilon|^2 \leq C_6, \quad \int_0^T \int (u^\epsilon)^2 \leq C_6. \tag{A.4.}$$

Proof. We deduce (A.3) from the inequality

$$\frac{1}{2}\frac{d}{dt}\int (v^\epsilon)^2 + d\int |\nabla v^\epsilon|^2 \leq 0.$$

Then adding up the partial differential equations for u^ϵ and for v^ϵ in (R^ϵ) gives

$$(u^\epsilon + v^\epsilon)_t = \Delta(u^\epsilon + v^\epsilon) + (d-1)\Delta v^\epsilon,$$

which permits to prove the estimatesv (A.4) similarly as in the proof of Lemma 6.1. ∎

Lemma A.4. *The sequences* $\{u^\epsilon\}$ *and* $\{v^\epsilon\}$ *are precompact in* $L^2(Q_T)$.

Proof. By the lemmas A.1 and A.3, the sequences

$$\{u^\epsilon\} \quad \text{and} \quad \{v^\epsilon\} \quad \text{are bounded in } L^2(0,T;H^1(\quad)),$$

and by Lemma A.2 we have that

$$\{\frac{1}{\epsilon}(u^\epsilon)^m v^\epsilon\} \quad \text{is bounded in } L^1(Q_T).$$

Then we follow the argument of Lemma 3.5 to conclude that the sequences $\{u^\epsilon\}$ and $\{v^\epsilon\}$ are precompact in $L^2(Q_T)$. ∎

Lemma A.5. *Let* $\mu \in (0,T)$ *be arbitrary. There exists* $\eta = \eta(\mu) > 0$ *such that*

(A.5) $$u^\epsilon(x,t) \geq \eta \quad \text{for all } (x,t) \in \quad \times [\mu, T];$$

in particular we have that

(A.6) $$u^\epsilon > 0 \quad \text{in} \quad \times (\mu, T).$$

Proof. Let $\tilde{u}$ be the solution of the following problem :

$$\begin{cases} u_t = \Delta u & \text{in } \quad \times (0,T), \\ \dfrac{\partial}{\partial \nu} u = 0 & \text{in } \partial \quad \times (0,T), \\ u(x,0) = u_0(x) & \text{for all } x \in \quad . \end{cases}$$

The comparison principle shows that

$$u^\epsilon(x,t) \geq \tilde{u}(x,t) \quad \text{for all } (x,t) \in Q.$$

Since for all $\mu \in (0,T)$ there exists $\eta = \eta(\mu) > 0$ such that

$$\tilde{u}(x,t) \geq \eta \quad \text{for all } (x,t) \in \quad \times (\mu, T),$$

we deduce the result of Lemma A.5. ■

Proof of Theorem 2.3. Let $\mu \in (0,T)$ and $\eta = \eta(\mu)$ be as in Lemma A.5. For all $t \geq \mu$ we have that

$$v_t^\epsilon = d\Delta v^\epsilon - \frac{1}{\epsilon}(u^\epsilon)^m v^\epsilon \leq d\Delta v^\epsilon - \frac{1}{\epsilon}\eta^m v^\epsilon,$$

so that

$$\begin{aligned} \tfrac{1}{2}\textstyle\int \ (v^\epsilon)^2(T) + d\int_\mu^T \int \ |\nabla v^\epsilon|^2 + \frac{1}{\epsilon}\displaystyle\int_\mu^T \int \eta^m (v^\epsilon)^2 &\leq \tfrac{1}{2}\textstyle\int \ (v^\epsilon)^2(\mu) \\ &\leq \frac{| \quad | M_2^2}{2}; \end{aligned}$$

therefore

$$\int_\mu^T \int (v^\epsilon)^2 \leq \frac{| \quad | M_2^2 \epsilon}{2\eta^m},$$

so that $v^\epsilon \to 0$ in $L^2(\quad \times (\mu, T))$ as $\epsilon \to 0$. Furthermore we have that

$$\begin{aligned} \int_0^T \int (v^\epsilon)^2 &= \int_0^\mu \int (v^\epsilon)^2 + \int_\mu^T \int (v^\epsilon)^2 \\ &\leq \mu | \quad | M_2^2 + \frac{| \quad | M_2^2 \epsilon}{2\eta^m}. \end{aligned}$$

Letting $\epsilon \to 0$ yields

$$\limsup_{\epsilon \to 0} \int_0^T \int (v^\epsilon)^2 \leq \mu | \quad | M_2^2,$$

for all $\mu \in (0,T)$ so that

$$\limsup_{\epsilon \to 0} \int_0^T \int (v^\epsilon)^2 \leq 0,$$

which implies the convergence of v^ϵ to 0 in $L^2(Q_T)$ as $\epsilon \to 0$.

Next we prove the convergence property for u^ϵ. Since

$$\begin{cases} (u^\epsilon + v^\epsilon)_t = \Delta u^\epsilon + d\Delta v^\epsilon & \text{in} \quad \times (0,T), \\ \dfrac{\partial}{\partial \nu}(u^\epsilon + v^\epsilon) = 0 & \text{on } \partial \quad \times (0,T), \\ (u^\epsilon + v^\epsilon)(x,0) = u_0(x) + v_0(x) & \text{for all } x \in \quad , \end{cases}$$

we have that for all $\varphi \in C^{2,1}(\;)$ such that $\dfrac{\partial\varphi}{\partial\nu} = 0$ on $\partial\;\;\times(0,T)$ and $\varphi(T) = 0$,

$$\int_0^T \int \Big[(u^\epsilon + v^\epsilon)\varphi_t + (u^\epsilon + dv^\epsilon)\Delta\varphi\Big] + \int (u_0 + v_0)\varphi(0) = 0. \tag{A.7}$$

Now suppose that a subsequence of $\{u^\epsilon\}$ converges to a function u strongly in $L^2(Q_T)$ as $\epsilon \to 0$. Then u satisfies the integral identity

$$\int_0^T \int \Big[u\varphi_t + u\Delta\varphi\Big] + \int (u_0 + v_0)\varphi(0) = 0, \tag{A.8}$$

which implies the result of Theorem 2.3. ■

Appendix B: The Case Where $n > m \geq 1$ and $u_0 > 0$

Proof of Theorem 2.4. By Lemma 6.1, we have that

$$0 \leq U^\epsilon \leq \tilde{M}_1 \quad \text{and} \quad 0 \leq V^\epsilon \leq M_2,$$

and since

$$U^\epsilon(x,t) = u^\epsilon(x,\epsilon t) \quad \text{and} \quad V^\epsilon(x,t) = v^\epsilon(x,\epsilon t) \quad \text{for all } (x,t) \in Q,$$

it follows that

$$0 \leq u^\epsilon \leq \tilde{M}_1 \quad \text{and} \quad 0 \leq v^\epsilon(x,t) \leq M_2.$$

As in Lemma 3.1 one can prove the inequality

$$\frac{a}{\epsilon}\int_0^T \int (u^\epsilon)^n \leq \int (u_0 + v_0), \tag{B.1}$$

so that

$$\int_0^T \int (u^\epsilon)^n \to 0 \quad \text{as } \epsilon \to 0, \tag{B.2}$$

which since $L^n(Q_T) \subset L^2(Q_T)$ implies that

$$\int_0^T \int (u^\epsilon)^2 \to 0 \quad \text{as } \epsilon \to 0. \tag{B.3}$$

Moreover we have that by Corollary 6.3

$$\int \Big(V^\epsilon(x,\tau(\epsilon)) - \bar{V}(x)\Big)^2 \to 0 \quad \text{as } \epsilon \to 0, \tag{B.4}$$

where $\tau(\epsilon) = \ln \dfrac{1}{\epsilon^\alpha}$. Next we prove that if $u_0(x) > 0$ then $\bar{V}(x) = 0$. Suppose on the contrary that $\bar{V}(x) > 0$; since

$$U(x,t) \to 0 \quad \text{as } t \to \infty, \tag{B.5}$$

we have that for t large enough

$$\begin{aligned} U_t(x,t) &= U^m(x,t)V(x,t) - aU^n(x,t), \\ &= U^m(x,t)\Big(V(x,t) - aU^{n-m}(x,t)\Big), \\ &\geq U^m(x,t)\frac{\bar{V}(x)}{2} > 0, \end{aligned}$$

which contradicts (B.5) (note that this result was proved by Hoshino [11] for the solution of Problem (Q^ϵ)). Thus we have that

$$\bar{V}(x) = 0 \quad \text{for all } x \in \quad ,$$

so that

$$\int V^\epsilon(x,\tau(\epsilon)) \to 0 \quad \text{as } \epsilon \to 0, \tag{B.6}$$

and since $V^\epsilon(x,\tau(\epsilon)) = v^\epsilon(x,\epsilon\tau(\epsilon))$, (B.6) yields

$$\int v^\epsilon(x,\epsilon\tau(\epsilon)) \to 0 \quad \text{as } \epsilon \to 0. \tag{B.7}$$

Moreover fix $\mu > 0$; the inequality $\dfrac{d}{dt}\displaystyle\int v^\epsilon \leq 0$ implies that

$$\int v^\epsilon(x,\mu) \to 0 \quad \text{as } \epsilon \to 0. \tag{B.8}$$

Furthermore we have that

$$\begin{aligned} \int_0^T \int (v^\epsilon)^2 &= \int_0^\mu \int (v^\epsilon)^2 + \int_\mu^T \int (v^\epsilon)^2 \\ &\leq \mu| \quad |M_2^2 + (T-\mu)M_2 \int v^\epsilon(\mu), \end{aligned}$$

which by the equation B.8 implies that

$$\limsup_{\epsilon\to 0}\int_0^T\int (v^\epsilon)^2 \;\; \le \mu| \;\; |M_2^2 + (T-\mu)M_2 \limsup_{\epsilon\to 0}\int v^\epsilon(\mu)$$
$$\le \mu| \;\; |M_2^2,$$

for all $\mu > 0$; consequently

$$\limsup_{\epsilon\to 0}\int_0^T\int (v^\epsilon)^2 = 0, \tag{B.9}$$

which completes the proof. ■

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D. Hilhorst, and R. Weidenfeld
Laboratoire de Mathématiques (UMR 8628),
Université de Paris-Sud, 91405
Orsay Cedex, FRANCE

M. Mimura
Department of Mathematical and Life Sciences,
Graduate School of Science,
Hiroshima University, 1-3-1,
Kagamiyama, Higashi-Hiroshima 739-8526,
JAPAN