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ERGODIC PROPERTIES OF CONTINUOUS PARAMETER ADDITIVE PROCESSES

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To my wife Masae Sato

Abstract. Let $\{T_t : t \in \mathbf{R}\}$ be a measure preserving flow in a probability measure space $(\Omega, \mathcal{A}, \mu)$, and $\{F_t : t \in \mathbf{R}\}$ be a family of real-valued measurable functions on $(\Omega, \mathcal{A}, \mu)$ such that $F_{t+s} = F_t + F_s \circ T_t$ (mod μ) for all $t,\ s \in \mathbf{R}$. In this paper we deduce necessary and sufficient conditions for the existence of a real-valued measurable function f on Ω, with $f \in L_p(\Omega, \mu)$ where $0 \le p \le \infty$, such that $F_t = f \circ T_t - f$ (mod μ) for all $t \in \mathbf{R}$. Related results are also obtained. These may be considered to be continuous parameter refinements of the recent discrete parameter results of Alonso, Hong and Obaya concerning additive real coboundary cocycles.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space. We denote by $L_0(\Omega, \mu)$ the space of all real-valued measurable functions on $(\Omega, \mathcal{A}, \mu)$. Most of the time we will not distinguish between the equivalence class of a function f and the function f itself, and hence statements and relations are assumed to hold modulo sets of measure zero, unless the contrary is explicitly explained. We define a metric d_0 in $L_0(\Omega, \mu)$ by

$$(1) \qquad d_0(f, g) = \int \frac{|f(\omega) - g(\omega)|}{1 + |f(\omega) - g(\omega)|}\, d\mu \quad (=: \|f - g\|_0).$$

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Under this metric, $L_0(\ ,\mu)$ becomes an F-space (cf. Chapter 1 of [18] for the definition of an F-space). If $0 < p < \infty$, then we let

$$L_p(\ ,\mu) = \left\{ f \in L_0(\ ,\mu) : \int |f(\omega)|^p \, d\mu < \infty \right\}.$$

When $0 < p < 1$, it becomes a locally bounded F-space under the metric

$$d_p(f,g) = \int |f(\omega) - g(\omega)|^p \, d\mu \quad \big(=: (\|f-g\|_p)^p\big); \tag{2}$$

and, when $1 \cdot\ p < \infty$, it becomes a Banach space under the L_p-norm

$$\|f\|_p = \left(\int |f(\omega)|^p d\mu \right)^{1/p}. \tag{3}$$

Furthermore, if $L_\infty(\ ,\mu)$ denotes the set of all elements in $L_0(\ ,\mu)$ that are essentially bounded on $\ $, then it becomes a Banach space under the L_∞-norm

$$\|f\|_\infty = \inf\{a \in \mathbf{R} : |f(\omega)| \cdot\ a \text{ for } \mu\text{-a.e. } \omega \in \ \}. \tag{4}$$

We consider a *measure preserving flow* $\{T_t : t \in \mathbf{R}\}$ in $(\ ,\mathcal{A},\mu)$. Thus, $\{T_t : t \in \mathbf{R}\}$ satisfies the following hypotheses:

(i) Each T_t is an invertible measure preserving transformation in $(\ ,\mathcal{A},\mu)$.

(ii) $T_t(T_s\omega) = T_{t+s}\omega$ for every $\omega \in\ $, and for every $t, s \in \mathbf{R}$.

(iii) The mapping $(\omega, t) \mapsto T_t\omega$ is a measurable transformation from $(\overline{\ \times \mathbf{R}, \mathcal{A}\ \ \mathcal{B}(\mathbf{R}), \mu\ \ dt})$ to $(\ ,\mathcal{A},\mu)$, where $(\overline{\ \times \mathbf{R}, \mathcal{A}\ \ \mathcal{B}(\mathbf{R}), \mu\ \ dt})$ denotes the completion of the product measure space $(\ \times \mathbf{R}, \mathcal{A}\ \ \mathcal{B}(\mathbf{R}), \mu\ \ dt)$ of $(\ ,\mathcal{A},\mu)$ and $(\mathbf{R}, \mathcal{B}(\mathbf{R}), dt)$, and where $\mathcal{B}(\mathbf{R})$ and dt stand for the σ-field of all Borel subsets of $\mathbf{R}$ and the Lebesgue measure on $\mathbf{R}$, respectively.

Here we note that, if necessary, the above hypothesis (ii) can be replaced with the following weaker hypothesis (ii)$'$, without any change of the results of the paper:

(ii)$'$ $T_t(T_s\omega) = T_{t+s}\omega$ for μ-a.e. $\omega \in\ $, for every $t, s \in \mathbf{R}$.

We denote by $\mathcal{I}$ the σ-field of subsets of $\ $ defined by

$$\mathcal{I} = \{A \in \mathcal{A} : \mu(A \triangle T_t^{-1}A) = 0 \text{ for } t \in \mathbf{R}\}, \tag{5}$$

where $A \triangle T_t^{-1}A$ stands for the symmetric difference of A and $T_t^{-1}A$. A set A in $\mathcal{I}$ is called *invariant* (with respect to $\{T_t\}$). The flow $\{T_t\}$ is called *ergodic* if $A \in \mathcal{I}$ implies either $\mu(A) = 0$ or $\mu(\ \setminus A) = 0$. Incidentally, we recall that a measure preserving transformation T in $(\ ,\mathcal{A},\mu)$ is called *ergodic* if either $\mu(A) = 0$ or

$\mu(\quad \setminus A) = 0$ holds for every $A \in \mathcal{A}$ with $\mu(A \triangle T^{-1}A) = 0$. For other basic notions and definitions in ergodic theory we refer the reader e.g. to Krengel's book [12].

For each $t \in \mathbf{R}$, let $\widehat{T}_t : L_0(\quad, \mu) \to L_0(\quad, \mu)$ denote the operator defined by

$$\widehat{T}_t f(\omega) = f(T_t \omega). \tag{6}$$

Clearly, $\widehat{T}_t$ becomes an invertible linear isometry in $L_0(\quad, \mu)$. Furthermore, we see that

$$\widehat{T}_t \widehat{T}_s = \widehat{T}_{t+s} \quad \text{for } t, s \in \mathbf{R}, \tag{7}$$

and

$$\|\widehat{T}_t f\|_p = \|f\|_p \quad \text{for every } f \in L_p(\quad, \mu), \quad \text{with } 0 \cdot \ p \cdot \ \infty. \tag{8}$$

From the measurability hypothesis (iii) of the flow $\{T_t\}$ it follows (cf. e.g. §1.6 of [12]) that

$$\lim_{t \to 0} \|\widehat{T}_t f - f\|_1 = 0 \quad \text{for every } f \in L_1(\quad, \mu). \tag{9}$$

Thus, an approximation argument implies that

$$\lim_{t \to 0} \|\widehat{T}_t f - f\|_p = 0 \quad \text{for every } f \in L_p(\quad, \mu), \text{ where } 0 \cdot \ p < \infty. \tag{10}$$

By a *process* we mean a family $\{F_t : t \in \mathbf{R}\}$ of real-valued measurable functions on $(\quad, \mathcal{A}, \mu)$. The process $\{F_t\}$ is called *additive* (with respect to the flow $\{T_t\}$) if it satisfies the following conditions:

(i) $F_{t+s}(\omega) = F_t(\omega) + F_s(T_t \omega)$ for μ-a.e. $\omega \in \quad$, for every $t, s \in \mathbf{R}$.

(ii) The mapping $t \mapsto F_t$ from $\mathbf{R}$ to $L_0(\quad, \mu)$ is continuous with respect to the metric d_0.

Examples. (a) Let $1 \cdot \ p < \infty$. For an f in $L_p(\quad, \mu)$, if we define

$$F_t = \int_0^t \widehat{T}_s f \, ds \quad (t \in \mathbf{R}),$$

then $\{F_t : t \in \mathbf{R}\}$ becomes an additive process in $L_p(\quad, \mu)$.

(b) Let $0 \cdot \ p \cdot \ \infty$. For an f in $L_p(\quad, \mu)$, if we define

$$F_t = \widehat{T}_t f - f \quad (t \in \mathbf{R}),$$

then $\{F_t : t \in \mathbf{R}\}$ becomes again an additive process in $L_p(\ \ , \mu)$.

The purpose of this paper is to investigate the ergodic properties of the additive process $\{F_t\}$. In particular, we are interested to obtain necessary and sufficient conditions for the existence of a function f in $L_p(\ \ , \mu)$, where $0 \cdot \ p \cdot \ \infty$, such that

$$F_t = \widehat{T}_t f - f \quad \text{for all } t \in \mathbf{R}. \tag{11}$$

To this end, using $\{T_t\}$ and $\{F_t\}$, we introduce in Section 2 a skew-product flow $\{\vartheta_t : t \in \mathbf{R}\}$ of nonsingular transformations in the probability measure space

$$(K_{\mathbf{R}}, \mathcal{A}_{\mathbf{R}}, \mu_{\mathbf{R}}) := \left(\quad \times \mathbf{R}, \mathcal{A} \quad \mathcal{B}(\mathbf{R}), \mu \quad \frac{dx}{\pi(1+x^2)} \right). \tag{12}$$

We examine ergodic properties of the flow $\{\vartheta_t\}$ in detail and use them to prove our theorems. In particular, we prove in Section 3, motivated by [17] and [4], that the existence of a function f in $L_0(\ \ , \mu)$ such that $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$ is equivalent to the existence of a $\mu_{\mathbf{R}}$-equivalent finite invariant measure on $(K_{\mathbf{R}}, \mathcal{A}_{\mathbf{R}})$ with respect to the skew-product flow $\{\vartheta_t\}$. We also observe, as in Helson [7], that this condition is equivalent to the following condition:

For μ-a.e. $\omega \in$ there exists an integer $N \geq 1$ such that

$$\limsup_{b \to \infty} \frac{1}{b} \int_0^b \chi_{[-N,\, N]}(F_t(\omega))\, dt > 0,$$

where $\chi_{[-N,\, N]}$ stands for the indicator function of the interval $[-N,\, N]$.

From these results we deduce, for example, that if the flow $\{T_t\}$ is *ergodic* and if $0 < p \cdot \ \infty$, then the following conditions are equivalent:

(i) There exists a function f in $L_p(\ \ , \mu)$ such that $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$.

(ii) There exists a set A in $\mathcal{A}$,with $\mu(A) > 0$, such that

$$\liminf_{b \to \infty} \frac{1}{b} \int_0^b (\|\chi_A \cdot F_t\|_p)^\gamma \, dt < \infty, \quad \text{where } \gamma = \min\{p,\, 1\}.$$

(iii) The skew-product flow $\{\vartheta_t\}$ admits a $\mu_{\mathbf{R}}$-absolutely continuous invariant probability measure $\nu = P\, d\mu_{\mathbf{R}}$ with $0 \cdot \ P \in L_{1+(p/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$.

These may be regarded as continuous parameter refinements of the discrete parameter results of Alonso, Hong and Obaya [4] (see also [21]). For related topics we refer the reader to [5], [13], [16] and [22]. We also examine the relationship between these conditions and the validity of the pointwise ergodic theorem for the skew-product flow $\{\vartheta_t\}$. In Section 4 we restrict ourselves to considering the case where $\{F_t\} \subset L_p(\ \ , \mu)$ with $1 \cdot \ p < \infty$. It is then proved that the function $t \mapsto F_t$

from $\mathbf{R}$ to $L_p(\ \ ,\mu)$ is Bochner integrable over the unit interval [0, 1], whence we see that the process $\{F_t\}$ can be written as

$$F_t = -(\widehat{T}_t - I)\int_0^1 F_s\,ds + \int_0^t \widehat{T}_s F_1\,ds \quad \text{for every } t \in \mathbf{R}.$$

From this relation we deduce that the following conditions are equivalent:

(α) There exists a function f in $L_p(\ \ ,\mu)$ such that $F_t = \int_0^t \widehat{T}_s f\,ds$ for all $t \in \mathbf{R}$.

(β) There exists a function f in $L_p(\ \ ,\mu)$ such that $\lim_{t\downarrow 0} \left\| t^{-1}F_t - f \right\|_p = 0$.

(γ) The function $\int_0^1 F_s\,ds \in L_p(\ \ ,\mu)$ belongs to the domain of the infinitesimal generator $\widehat{A}_p$ of the one-parameter operator group $\{\widehat{T}_t : t \in \mathbf{R}\}$ in $L_p(\ \ ,\mu)$.

2. Preliminaries

In this section we prove some auxiliary results. First of all, for an additive process $\{F_t : t \in \mathbf{R}\}$ (with respect to the flow $\{T_t\}$) and an integer $n \geq 1$, we define a real-valued function H_n on $\ \ \times \mathbf{R}$ by

$$H_n(\omega, t) = F_{j/n}(\omega) \quad \text{if} \quad \frac{j}{n} \cdot \ t < \frac{j+1}{n}.$$

Then, using the continuity of the mapping $t \mapsto F_t$ from $\mathbf{R}$ to $L_0(\ \ ,\mu)$ (this continuity is equivalent to saying that the mapping is continuous in probability, i.e.,

$$\lim_{s\to t} \mu(\{\omega : |F_s(\omega) - F_t(\omega)| > \epsilon\}) = 0$$

for each $\epsilon > 0$ and $t \in \mathbf{R}$), we can choose a subsequence $(n(k))$ of (n) such that

$$d_0(F_t(\cdot),\ H_{n(k)}(\cdot, t)) < 2^{-(k+1)}$$

for all $t \in \mathbf{R}$ and $k \geq 1$. Then we have

$$\lim_{k\to\infty} H_{n(k)}(\omega, t) = F_t(\omega) \quad \mu\text{-a.e. } \omega \in \ \ , \text{ for every } t \in \mathbf{R}.$$

Taking this into account, we define

$$H(\omega, t) = \begin{cases} \lim_{k\to\infty} H_{n(k)}(\omega, t) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that H becomes a real-valued measurable function on $(\ \ \times \mathbf{R}, \mathcal{A}\ \ \mathcal{B}(\mathbf{R}), \mu\ \ dt)$ such that

$$F_t(\omega) = H(\omega,\, t) \quad \text{for } \mu\text{-a.e. } \omega \in \ \ , \text{ for every } t \in \mathbf{R}.$$

Hence we may assume below, without loss of generality, that

(13) $$F_t(\omega) = H(\omega,\, t) \quad \text{for } \omega \in \quad \text{and } t \in \mathbf{R}.$$

Next, we introduce a family $\{\vartheta_t : t \in \mathbf{R}\}$ of skew-product transformations in $(K_{\mathbf{R}}, \mathcal{A}_{\mathbf{R}}, \mu_{\mathbf{R}})$ as follows:

(14) $$\vartheta_t(\omega,\, x) = (T_t\omega,\, x + F_t(\omega)) \quad \text{for } (\omega,\, x) \in K_{\mathbf{R}} \text{ and } t \in \mathbf{R}.$$

It is easy to check that

(i) each ϑ_t is an invertible, null preserving transformation in $(K_{\mathbf{R}}, \mathcal{A}_{\mathbf{R}}, \mu_{\mathbf{R}})$,

(ii) $\vartheta_t(\vartheta_s(\omega,\, x)) = \vartheta_{t+s}(\omega,\, x)$ for $\mu_{\mathbf{R}}$-a.e. $(\omega,\, x) \in K_{\mathbf{R}}$, for every $t, s \in \mathbf{R}$,

(iii) the mapping $((\omega,\, x), t) \mapsto \vartheta_t(\omega,\, x)$ is a measurable transformation from $(K_{\mathbf{R}} \times \mathbf{R},\, \overline{\mathcal{A}_{\mathbf{R}} \quad \mathcal{B}(\mathbf{R})},\, \overline{\mu_{\mathbf{R}} \quad dt})$ to $(K_{\mathbf{R}}, \mathcal{A}_{\mathbf{R}}, \mu_{\mathbf{R}})$.

Thus, $\{\vartheta_t : t \in \mathbf{R}\}$ becomes a measurable flow of nonsingular transformations in $(K_{\mathbf{R}}, \mathcal{A}_{\mathbf{R}}, \mu_{\mathbf{R}})$. Hence it follows from Krengel [11] (see also [14], [19]) that if $\{U_t : t \in \mathbf{R}\}$ denotes the one-parameter group of positive linear isometries in $L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ defined by the relation

(15) $$\int_{K_{\mathbf{R}}} (U_t u) f \, d\mu_{\mathbf{R}} = \int_{K_{\mathbf{R}}} u \cdot (f \circ \vartheta_t) \; d\mu_{\mathbf{R}},$$

where $u \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ and $f \in L_\infty(K_{\mathbf{R}}, \mu_{\mathbf{R}})$, then $\{U_t\}$ becomes strongly continuous in $L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$, i.e., we have

(16) $$\lim_{t \to 0} \|U_t u - u\|_1 = 0 \quad \text{for } u \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}}).$$

Now, define a family $\{w_t : t \in \mathbf{R}\}$ in $L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ by the relation

(17) $$w_t = U_{-t} 1 \left(= \frac{d\mu_{\mathbf{R}} \circ \vartheta_t}{d\mu_{\mathbf{R}}} \right) \quad \text{for } t \in \mathbf{R},$$

where $\mu_{\mathbf{R}} \circ \vartheta_t$ stands for the probability measure on $(K_{\mathbf{R}}, \mathcal{A}_{\mathbf{R}})$ defined by $(\mu_{\mathbf{R}} \circ \vartheta_t)(E) = \mu_{\mathbf{R}}(\vartheta_t(E))$ for $E \in \mathcal{A}_{\mathbf{R}}$, and $d\mu_{\mathbf{R}} \circ \vartheta_t / d\mu_{\mathbf{R}}$ is the Radon-Nikodym derivative of the measure $\mu_{\mathbf{R}} \circ \vartheta_t$ with respect to $\mu_{\mathbf{R}}$. It follows that each w_t is a strictly positive function on $K_{\mathbf{R}}$ satisfying

(18) $$\int (u \circ \vartheta_t) \cdot w_t \, d\mu_{\mathbf{R}} = \int u \, d\mu_{\mathbf{R}} \quad \text{for } u \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}}).$$

From this and (15) we deduce without difficulty that U_{-t} has the form

(19) $$U_{-t} u = (u \circ \vartheta_t) \cdot w_t \quad \text{for } u \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}}).$$

Fact 1. *For every* $t, s \in \mathbf{R}$ *we have* $w_{t+s} = w_t \cdot (w_s \circ \vartheta_t)$ $\mu_{\mathbf{R}}$-a.e. *on* $K_{\mathbf{R}}$.

Proof. (19) yields

$$w_{t+s} = U_{-t-s}1 = U_{-t}U_{-s}1 = U_{-t}w_s = (w_s \circ \vartheta_t) \cdot w_t,$$

which establishes Fact 1.

Fact 2. *Let* $\nu = P d\mu_{\mathbf{R}}$, *where* $0 \cdot \;\; P \in L_0(K_{\mathbf{R}}, \mu_{\mathbf{R}})$. *Then* ν *becomes an invariant measure with respect to the skew-product flow* $\{\vartheta_t\}$ *if and only if* $(P \circ \vartheta_t) \cdot w_t = P$ $\mu_{\mathbf{R}}$-a.e. *on* $K_{\mathbf{R}}$, *for every* $t \in \mathbf{R}$.

Proof. If $A \in \mathcal{A}_{\mathbf{R}}$ and $t \in \mathbf{R}$, then (18) yields

$$\int_A (P \circ \vartheta_t) \cdot w_t \, d\mu_{\mathbf{R}} = \int [(P \cdot \chi_{\vartheta_t A}) \circ \vartheta_t] \cdot w_t \, d\mu_{\mathbf{R}} = \int_{\vartheta_t A} P \, d\mu_{\mathbf{R}} = \nu(\vartheta_t A).$$

Since $\nu(A) = \int_A P \, d\mu_{\mathbf{R}}$, it then follows that $\nu(\vartheta_t A) = \nu(A)$ for all $A \in \mathcal{A}_{\mathbf{R}}$ is equivalent to $(P \circ \vartheta_t) \cdot w_t = P$ $\mu_{\mathbf{R}}$-a.e. on $K_{\mathbf{R}}$, and this establishes Fact 2.

By a straightforward observation we know that w_t has the form

$$w_t(\xi) = \left(\frac{d\mu_{\mathbf{R}} \circ \vartheta_t}{d\mu_{\mathbf{R}}}\right)(\xi) = \frac{1 + x^2}{1 + (x + F_t(\omega))^2} \quad \text{for} \quad \mu_{\mathbf{R}}\text{-a.e. } \xi = (\omega, x) \in K_{\mathbf{R}},$$

so that we may assume below, without loss of generality, that

$$\text{(20)} \qquad w_t(\xi) = \frac{1 + x^2}{1 + (x + F_t(\omega))^2} \quad \text{for} \quad t \in \mathbf{R} \text{ and } \xi = (\omega, x) \in K_{\mathbf{R}}.$$

It follows that the function $(\xi, t) \mapsto w_t(\xi)$ is a strictly positive real-valued measurable function on $(K_{\mathbf{R}} \times \mathbf{R}, \mathcal{A}_{\mathbf{R}} \quad \mathcal{B}(\mathbf{R}), \mu_{\mathbf{R}} \quad dt)$.

Fact 3. *Suppose* $0 \cdot \;\; P \in L_0(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ *has the form*

$$P(\xi) = \frac{1 + x^2}{a(\omega)x^2 + b(\omega) + 2c(\omega)x} \quad \textit{for} \quad \xi = (\omega, x) \in K_{\mathbf{R}},$$

where a, b, c *are real-valued measurable functions on* $(\;\; , \mathcal{A}, \mu)$. *Then* $\nu = P \, d\mu_{\mathbf{R}}$ *becomes an invariant measure with respect to the skew-product flow* $\{\vartheta_t\}$ *if and only if*

$$\text{(21)} \qquad \begin{pmatrix} a(T_t\omega) \\ b(T_t\omega) \\ c(T_t\omega) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ F_t^2(\omega) & 1 & -2F_t(\omega) \\ -F_t(\omega) & 0 & 1 \end{pmatrix} \begin{pmatrix} a(\omega) \\ b(\omega) \\ c(\omega) \end{pmatrix}$$

for μ-a.e. $\omega \in \;$, *for every* $t \in \mathbf{R}$.

Proof. Since $\vartheta_t(\omega, x) = (T_t\omega, x + F_t(\omega))$ for $\xi = (\omega, x) \in K_{\mathbf{R}}$ by definition, we find

$$P \circ \vartheta_t(\omega, x) = \frac{1 + (x + F_t(\omega))^2}{a(T_t\omega) \cdot (x + F_t(\omega))^2 + b(T_t\omega) + 2c(T_t\omega) \cdot (x + F_t(\omega))}$$

and

$$\frac{P(\omega, x)}{w_t(\omega, x)} = \frac{1 + x^2}{a(\omega)x^2 + b(\omega) + 2c(\omega)x} \cdot \frac{1 + (x + F_t(\omega))^2}{1 + x^2}.$$

From this and Fact 2, the present Fact follows immediately through an elementary calculation.

We denote by $\mathcal{I}_{\mathbf{R}}$ the σ-field of subsets of $K_{\mathbf{R}}$ defined by

$$\mathcal{I}_{\mathbf{R}} = \{E \in \mathcal{A}_{\mathbf{R}} : \mu_{\mathbf{R}}(E \triangle \vartheta_t^{-1} E) = 0 \text{ for every } t \in \mathbf{R}\}, \tag{22}$$

and by $E\{\cdot\,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \mu_{\mathbf{R}})\}$ the conditional expectation operator with respect to the σ-field $\mathcal{I}_{\mathbf{R}}$ and the measure $\mu_{\mathbf{R}}$.

Fact 4. *The following conditions are equivalent*:

(I) *The skew-product flow* $\{\vartheta_t\}$ *admits a* $\mu_{\mathbf{R}}$*-equivalent finite invariant measure* ν.

(II) *The limit*

$$w(\xi) = \lim_{b \to \infty} \frac{1}{b} \int_0^b w_t(\xi)\, dt$$

exists and is a positive real number for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$.

Proof. (I) $\Rightarrow$ (II). Let $P = d\nu/d\mu_{\mathbf{R}}$. Thus $0 < P \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$. Since ν is invariant with respect to $\{\vartheta_t\}$, we have $U_t P = P$ for $t \in \mathbf{R}$. Then by the continuous parameter version of the Chacon-Ornstein ratio ergodic theorem (see e.g. [14]),

$$\begin{aligned} w(\xi) &= \lim_{b \to \infty} \frac{1}{b} \int_0^b w_t(\xi)\, dt \\ &= q\text{-}\lim_{b \to \infty} P(\xi) \cdot \frac{\left(\int_0^b U_{-t} 1\, dt\right)(\xi)}{\left(\int_0^b U_{-t} P\, dt\right)(\xi)} = P(\xi) \cdot \frac{E\{1\,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \mu_{\mathbf{R}})\}(\xi)}{E\{P\,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \mu_{\mathbf{R}})\}(\xi)} \end{aligned}$$

for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$, where the notation q-$\lim_{b \to \infty}$ means that the limit is taken as b tends to ∞ through the set of rational numbers. (We recall that, since

$(\int_0^b U_{-t}1\,dt)/(\int_0^b U_{-t}P\,dt)$, $b > 0$, are equivalence classes and not actual functions on , the a.e. convergence of $(\int_0^b U_{-t}1\,dt)/(\int_0^b U_{-t}P\,dt)$ as b tends to ∞ through the set $\mathbf{R}$ does not make sense, so that we must let b range through a countable dense subset of $\mathbf{R}$.) Thus (II) follows, because $E\{1\,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \mu_{\mathbf{R}})\}(\xi) = 1$ and $0 < E\{P\,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \mu_{\mathbf{R}})\}(\xi) < \infty$ on $K_{\mathbf{R}}$.

(II) $\Rightarrow$ (I). Since the function $\xi \mapsto (1/b)\int_0^b w_t(\xi)\,dt$ is a representative of the element $(1/b)\int_0^b U_{-t}1\,dt \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$, it follows from Fatou's lemma that

$$\int_{K_{\mathbf{R}}} w(\xi)\,d\mu_{\mathbf{R}} \cdot \;\; q\text{-}\liminf_{b\to\infty} \left\| \frac{1}{b}\int_0^b U_{-t}1\,dt \right\|_1 \cdot \;\; 1.$$

Hence we have $0 < w \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$, and for each $s \in \mathbf{R}$

$$\begin{aligned} 0 \cdot \;\; U_s w(\xi) \cdot \;\; & q\text{-}\liminf_{b\to\infty} U_s\left(\frac{1}{b}\int_0^b U_{-t}1\,dt\right)(\xi) \\ &= q\text{-}\liminf_{b\to\infty} \frac{1}{b}\left(\int_{-s}^{b-s} U_{-t}1\,dt\right)(\xi) = w(\xi) \end{aligned}$$

for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$. Since $\|U_s w\|_1 = \|w\|_1$, it follows that $U_s w = w$, and hence $\nu = w\,d\mu_{\mathbf{R}}$ is a $\mu_{\mathbf{R}}$-equivalent finite invariant measure with respect to $\{\vartheta_t\}$. This completes the proof.

Fact 5. *Let $\nu = P\,d\mu_{\mathbf{R}}$ be a $\mu_{\mathbf{R}}$-equivalent finite invariant measure with respect to the skew-product flow $\{\vartheta_t\}$. Suppose $0 < r < \infty$. Then the following hold*:

(I) *For $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$ the limit*

$$Q_r(\xi) = \lim_{b\to\infty} \frac{1}{b}\int_0^b \left(\frac{1}{w_t(\xi)}\right)^r dt$$

exists (*but it may be infinite*).

(II) *For $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$ we have $0 < Q_r(\xi) < \infty$ if and only if there exists a countable decomposition $\{E_n : n \geq 1\}$ of $K_{\mathbf{R}}$ such that $E_n \in \mathcal{I}_{\mathbf{R}}$ and $\int_{E_n} P^{1+r} d\mu_{\mathbf{R}} < \infty$ for every $n \geq 1$.*

Proof. The following argument is an adaptation of the proof of Theorem 5.5. in [17].

To prove (I), we may assume that $0 < P(\xi) < \infty$ for all $\xi \in K_{\mathbf{R}}$, by hypothesis. Furthermore, we may assume here that

$$w_t(\xi) = \frac{P(\xi)}{P(\vartheta_t \xi)} \quad \text{for } t \in \mathbf{R} \text{ and } \xi \in K_{\mathbf{R}}, \tag{23}$$

by Fact 2. Therefore, $(\xi, t) \mapsto w_t(\xi)$ becomes a measurable function on the measure space $(K_{\mathbf{R}} \times \mathbf{R}, \overline{\mathcal{A}_{\mathbf{R}} \quad \mathcal{B}(\mathbf{R})}, \overline{\mu_{\mathbf{R}} \quad dt})$. Then, as in the proof of Fact 4, we find that

$$(24) \qquad Q_r(\xi) = \frac{1}{P^r(\xi)} \cdot \lim_{b\to\infty} \frac{1}{b}\int_0^b P^r(\vartheta_t\xi)\,dt = \frac{E\{P^r\,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \nu)\}(\xi)}{P^r(\xi)}$$

for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$, where $E\{P^r\,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \nu)\}$ is the conditional expectation of the function P^r with respect to the σ-field $\mathcal{I}_{\mathbf{R}}$ and the measure $\nu = P\,d\mu_{\mathbf{R}}$. Thus (I) follows, because $0 < E\{P^r\,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \nu)\}(\xi) \cdot \ \infty$ for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$.

To prove (II), we first suppose $0 < Q_r(\xi) < \infty$ for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$. Then write

$$\widetilde{Q}_r(\xi) = \left(\frac{1}{Q_r(\xi)}\right)^{1/r} \quad \text{and} \quad \widetilde{Q}_{r,n}(\xi) = \left(\frac{1}{Q_{r,n}(\xi)}\right)^{1/r},$$

where

$$Q_{r,n}(\xi) := \frac{1}{n}\int_0^n \left(\frac{1}{w_t(\xi)}\right)^r dt.$$

In order to prove that $\widetilde{Q}_r \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$, we proceed as follows. By using the Hölder inequality, if $r \geq 1$, then

$$\widetilde{Q}_{r,n}(\xi) = 1\Big/\left(\frac{1}{n}\int_0^n \left(\frac{1}{w_t(\xi)}\right)^r dt\right)^{1/r} \cdot \ 1\Big/\left(\frac{1}{n}\int_0^n \frac{1}{w_t(\xi)}dt\right) \cdot \ \frac{1}{n}\int_0^n w_t(\xi)\,dt;$$

and if $0 < r < 1$, then

$$\widetilde{Q}_{r,n}(\xi) = 1\Big/\left(\frac{1}{n}\int_0^n \left(\frac{1}{w_t(\xi)}\right)^r dt\right)^{1/r} \cdot \ \left(\frac{1}{n}\int_0^n (w_t(\xi))^r dt\right)^{1/r} \cdot \ \frac{1}{n}\int_0^n w_t(\xi)dt.$$

Thus, in either case, we get

$$\int_{K_{\mathbf{R}}} \widetilde{Q}_{r,n}(\xi)\,d\mu_{\mathbf{R}}(\xi) \cdot \ \int_{K_{\mathbf{R}}} \frac{1}{n}\int_0^n w_t(\xi)\,dt\,d\mu_{\mathbf{R}}(\xi) = 1,$$

by Fubini's theorem. Since $\widetilde{Q}_r(\xi) = \lim_{n\to\infty} \widetilde{Q}_{r,n}(\xi)$ for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$, it then follows from Fatou's lemma that $\widetilde{Q}_r \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$.

Letting $\lambda = \widetilde{Q}_r\,d\mu_{\mathbf{R}}$, we next prove that λ is an invariant measure with respect to $\{\vartheta_t\}$. To do this, we use Facts 1 and 2 as follows. If $t \in \mathbf{R}$ is fixed arbitrarily, then, since $w_s(\vartheta_t\xi) = w_{t+s}(\xi)/w_t(\xi)$ for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$, for every $s \in \mathbf{R}$ by Fact 1, we can apply Fubini's theorem to infer that for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$ the equality

$$w_s(\vartheta_t\xi) = \frac{w_{t+s}(\xi)}{w_t(\xi)}$$

holds for ds-a.e. $s \in \mathbf{R}$; hence for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$ we have

$$\begin{aligned}\widetilde{Q}_r(\vartheta_t(\xi)) &= \lim_{b\to\infty} \left(\frac{1}{b} \int_0^b \left(\frac{1}{w_s(\vartheta_t\xi)} \right)^r ds \right)^{-1/r} \\ &= \lim_{b\to\infty} \frac{1}{w_t(\xi)} \cdot \left(\frac{1}{b} \int_0^b \left(\frac{1}{w_{t+s}(\xi)} \right)^r ds \right)^{-1/r} = \frac{1}{w_t(\xi)} \cdot \widetilde{Q}_r(\xi),\end{aligned}$$

so that $\lambda = \widetilde{Q}_r \, d\mu_{\mathbf{R}}$ is an invariant measure with respect to $\{\vartheta_t\}$, by Fact 2.

We then prove that $\widetilde{Q}_r \in L_{1+r}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$. To do this, we notice that if $t \in \mathbf{R}$ is fixed arbitrarily, then, since $Q_r(\vartheta_t\xi) = (w_t(\xi))^r Q_r(\xi)$ for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$, we have by Fubini's theorem that

$$\lim_{b\to\infty} \frac{1}{b} \int_0^b \frac{1}{Q_r(\vartheta_t\xi)} \, dt = \frac{1}{Q_r(\xi)} \cdot \lim_{b\to\infty} \frac{1}{b} \int_0^b \left(\frac{1}{w_t(\xi)} \right)^r dt = \frac{1}{Q_r(\xi)} \cdot Q_r(\xi) = 1$$

for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$. Since $\lambda = \widetilde{Q}_r \, d\mu_{\mathbf{R}}$ is a $\mu_{\mathbf{R}}$-equivalent finite invariant measure with respect to $\{\vartheta_t\}$, it now follows from the continuous parameter version of the Birkhoff pointwise ergodic theorem applied to the flow $\{\vartheta_t\}$ that

$$\int_{K_{\mathbf{R}}} \frac{1}{Q_r} \, d\lambda = \int_{K_{\mathbf{R}}} 1 \, d\lambda = \lambda(K_{\mathbf{R}}) < \infty,$$

whence we have

$$\int_{K_{\mathbf{R}}} \widetilde{Q}_r^{1+r} d\mu_{\mathbf{R}} = \int_{K_{\mathbf{R}}} \left(\frac{1}{Q_r} \right)^{(1/r)+1} d\mu_{\mathbf{R}} = \int_{K_{\mathbf{R}}} \frac{1}{Q_r} \cdot \widetilde{Q}_r \, d\mu_{\mathbf{R}} = \int_{K_{\mathbf{R}}} \frac{1}{Q_r} \, d\lambda < \infty.$$

Since $U_t P = P$ and $U_t \widetilde{Q}_r = \widetilde{Q}_r$ for $t \in \mathbf{R}$, it then follows that

$$\frac{P(\xi)}{\widetilde{Q}_r(\xi)} = q\text{-}\lim_{b\to\infty} \frac{\left(\int_0^b U_t P \, dt \right)(\xi)}{\left(\int_0^b U_t \widetilde{Q}_r \, dt \right)(\xi)} = \frac{E\{P \,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \mu_{\mathbf{R}})\}(\xi)}{E\{\widetilde{Q}_r \,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \mu_{\mathbf{R}})\}(\xi)}$$

for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$. Therefore, there exists an $\mathcal{I}_{\mathbf{R}}$-measurable positive real-valued function R on $K_{\mathbf{R}}$ such that

$$P(\xi) = \widetilde{Q}_r(\xi) \cdot R(\xi) \quad \text{for } \mu_{\mathbf{R}}\text{-a.e. } \xi \in K_{\mathbf{R}}.$$

Here, if $\{E_n : n \geq 1\}$ denotes the countable decomposition of $K_{\mathbf{R}}$ defined by

$$E_n = \{\xi \in K_{\mathbf{R}} : n - 1 \cdot \;\; R(\xi) < n\},$$

then, clearly, $E_n \in \mathcal{I}_{\mathbf{R}}$ and $\int_{E_n} P^{1+r} \, d\mu_{\mathbf{R}} \cdot \;\; n^{1+r} \int_{K_{\mathbf{R}}} \widetilde{Q}_r^{1+r} \, d\mu_{\mathbf{R}} < \infty$ for every $n \geq 1$.

To prove the converse implication of (II), let $\{E_n : n \geq 1\}$ be a countable decomposition of $K_{\mathbf{R}}$ such that $E_n \in \mathcal{I}_{\mathbf{R}}$ and $\int_{E_n} P^{1+r}\, d\mu_{\mathbf{R}} < \infty$ for every $n \geq 1$. Since $\nu = P\, d\mu_{\mathbf{R}}$ is an invariant measure with respect to $\{\vartheta_t\}$, and since $P^r(\xi) \cdot Q_r(\xi) = E\{P^r\, |(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \nu)\}(\xi)$ for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$ by (24), it follows that

$$\begin{aligned}\int_{E_n} P^{1+r}(\xi) \cdot Q_r(\xi)\, d\mu_{\mathbf{R}} &= \int_{E_n} P^r(\xi) \cdot Q_r(\xi) d\nu \\ &= \int_{E_n} E\{P^r|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \nu)\}(\xi) d\nu \\ &= \int_{E_n} P^r\, d\nu = \int_{E_n} P^{1+r} d\mu_{\mathbf{R}} < \infty,\end{aligned}$$

whence we have $Q_r(\xi) < \infty$ for $\mu_{\mathbf{R}}$-a.e. $\xi \in E_n$, and this completes the proof.

Remark 1. The above proof of (II) of Fact 5 shows that, without assuming the existence of a $\mu_{\mathbf{R}}$-equivalent finite invariant measure with respect to the skew-product flow $\{\vartheta_t\}$, if the limit $Q_r(\xi)$ exists and is a positive real number for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$, then the function $\widetilde{Q}_r(\xi) = (1/Q_r(\xi))^{1/r}$ is in $L_{1+r}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ and satisfies $U_t\widetilde{Q}_r = \widetilde{Q}_r$ for $t \in \mathbf{R}$; consequently the flow $\{\vartheta_t\}$ admits a $\mu_{\mathbf{R}}$-equivalent finite invariant measure $\nu = \widetilde{Q}_r\, d\mu_{\mathbf{R}}$.

Fact 6. *The following conditions are equivalent*:

(I) *For every* $F \in L_\infty(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ *the limit*

$$\widehat{F}(\xi) = \lim_{b\to\infty} \frac{1}{b} \int_0^b F(\vartheta_t\xi)\, dt$$

exists for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$.

(II) $\{U_t : t \in \mathbf{R}\}$ *satisfies the* L_1*-mean ergodic theorem, i.e., to every* $W \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ *there corresponds a function* $W^* \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ *such that*

$$\lim_{b\to\infty} \left\| \frac{1}{b} \int_0^b U_t W\, dt - W^* \right\|_1 = 0$$

Proof. (I) $\Rightarrow$ (II). If we write $\langle W, F\rangle = \int_{K_{\mathbf{R}}} WF\, d\mu_{\mathbf{R}}$ for $W \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ and $F \in L_\infty(K_{\mathbf{R}}, \mu_{\mathbf{R}})$, then, by (I) together with Fubini's theorem and Lebesgue's convergence theorem, we find

$$\begin{aligned}\left\langle \frac{1}{b} \int_0^b U_t W\, dt,\, F \right\rangle &= \int_{K_{\mathbf{R}}} W(\xi) \cdot \left(\frac{1}{b} \int_0^b F(\vartheta_t\xi)\, dt \right) d\mu_{\mathbf{R}}(\xi) \\ &\longrightarrow \int_{K_{\mathbf{R}}} W(\xi) \cdot \widehat{F}(\xi)\, d\mu_{\mathbf{R}}(\xi)\end{aligned}$$

as $b \to \infty$. Thus, the Vitali-Hahn-Saks theorem implies that there exists a function $W^* \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ such that $\lim_{b\to\infty} (1/b) \int_0^b U_t W\, dt = W^*$ in the weak topology of $L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$. It is then routine to check that W^* is a fixed point for $\{U_t\}$, and $W - W^*$ belongs to the closed linear subspace of $L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ generated by the set $\{U_t G - G : G \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}}),\ t > 0\}$. Thus (II) follows.

(II) $\Rightarrow$ (I). Let $W = 1$ $(\in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}}))$, and put $\nu = 1^*\, d\mu_{\mathbf{R}}$. Since 1^* is a fixed point for $\{U_t\}$ and $1^* > 0$ on $K_{\mathbf{R}}$, ν becomes a $\mu_{\mathbf{R}}$-equivalent finite invariant measure with respect to $\{\vartheta_t\}$, hence (I) follows from the continuous parameter version of the Birkhoff ergodic theorem. This completes the proof.

Remark 2. By the above proof, if $\{U_t\}$ satisfies the L_1-mean ergodic theorem, then there exists a strictly positive function e in $L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ with $U_t e = e$ for every $t \in \mathbf{R}$; it follows that for every $W \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ we have

$$\begin{aligned} W^*(\xi) &= q\text{-}\lim_{b\to\infty} \frac{1}{b}\left(\int_0^b U_t W\, dt\right)(\xi) = q\text{-}\lim_{b\to\infty} e(\xi) \cdot \frac{\left(\int_0^b U_t W\, dt\right)(\xi)}{\left(\int_0^b U_t e\, dt\right)(\xi)} \\ &= e(\xi) \cdot E\{W/e \,|\, (K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, e\, d\mu_{\mathbf{R}})\}(\xi) \end{aligned}$$

for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$, where $\{W/e \,|\, (K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, e\, d\mu_{\mathbf{R}})\}$ denotes the conditional expectation of the function $W/e \in L_1(K_{\mathbf{R}}, e\, d\mu_{\mathbf{R}})$ with respect to the σ-field $\mathcal{I}_{\mathbf{R}}$ and the measure $e\, d\mu_{\mathbf{R}}$. Similarly, W^* is also the $\mu_{\mathbf{R}}$-a.e. limit of the averages $b^{-1} \int_{-b}^0 U_t W\, dt$ as b tends to ∞ through the set of rational numbers.

Fact 7. *Let* $1 \cdot\ p_1, p_2 \cdot\ \infty$, *and* $1/p_i + 1/p_i' = 1$ *for* $i = 1, 2$. *Then the following conditions are equivalent*:

(I) *For every* $F \in L_{p_1}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ *the limit*

$$\widehat{F}(\xi) = \lim_{b\to\infty} \frac{1}{b} \int_0^b F(\vartheta_t \xi)\, dt$$

exists for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$, *and the limit function* $\widehat{F}$ *belongs to* $L_{p_2}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$.

(II) $\{U_t : t \in \mathbf{R}\}$ *satisfies the* L_1-*mean ergodic theorem*, *and furthermore if* $W \in L_{p_2'}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$, *then the limit function* W^* *in the condition* (II) *of Fact* 6 *belongs to* $L_{p_1'}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$.

Proof. (I) $\Rightarrow$ (II). Since $L_\infty(K_{\mathbf{R}}, \mu_{\mathbf{R}}) \subset L_{p_1}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$, (I) implies that $\{U_t\}$ satisfies the L_1-mean ergodic theorem, by Fact 6. Suppose $0 \cdot\ W \in L_{p_2'}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ and $0 \cdot\ F \in L_{p_1}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$. Then, by putting

$$F_N(\xi) = \min\{F(\xi),\ N\} \quad \text{for } \xi \in K_{\mathbf{R}},$$

we have

$$\begin{aligned}\int_{K_{\mathbf{R}}} W^*F\,d\mu_{\mathbf{R}} &= \lim_{N\to\infty}\int_{K_{\mathbf{R}}} W^*F_N\,d\mu_{\mathbf{R}}\\ &= \lim_{N\to\infty}\left(\lim_{b\to\infty}\int_{K_{\mathbf{R}}} W(\xi)\cdot\left(\frac{1}{b}\int_0^b F_N(\vartheta_t\xi)\,dt\right)\,d\mu_{\mathbf{R}}(\xi)\right)\\ &= \lim_{N\to\infty}\int_{K_{\mathbf{R}}} W(\xi)\cdot\widehat{F}_N(\xi)\,d\mu_{\mathbf{R}}(\xi)\cdot\quad\int_{K_{\mathbf{R}}} W(\xi)\cdot\widehat{F}(\xi)\,d\mu(\xi)\\ &\cdot\quad \|W\|_{p_2'}\|\widehat{F}\|_{p_2}<\infty.\end{aligned}$$

This proves that $W^*\in L_{p_1'}(K_{\mathbf{R}},\mu_{\mathbf{R}})$, and hence (II) follows.

(II) $\Rightarrow$ (I). Suppose $0\cdot\quad F\in L_{p_1}(K_{\mathbf{R}},\mu_{\mathbf{R}})$. Since the first condition of (II) implies the existence of a $\mu_{\mathbf{R}}$-equivalent finite invariant measure ν with respect to $\{\vartheta_t\}$, it follows from the continuous parameter version of the Birkhoff ergodic theorem that the limit

$$\widehat{F}(\xi)=\lim_{b\to\infty}\frac{1}{b}\int_0^b F(\vartheta_t\xi)\,dt$$

exists (but it may be infinite) for $\mu_{\mathbf{R}}$-a.e. $\xi\in K_{\mathbf{R}}$. Since $0\cdot\quad F_N(\xi)\uparrow F(\xi)$ as $N\to\infty$ on $K_{\mathbf{R}}$, it then follows that

$$0\cdot\quad \widehat{F}_N(\xi)\uparrow\widehat{F}(\xi)\quad\text{as}\quad N\to\infty$$

for $\mu_{\mathbf{R}}$-a.e. $\xi\in K_{\mathbf{R}}$. Thus, if $0\cdot\quad W\in L_{p_2'}(K_{\mathbf{R}},\mu_{\mathbf{R}})$, then, using the relation $W^*\in L_{p_1'}(K_{\mathbf{R}},\mu_{\mathbf{R}})$ which comes from the second condition of (II), we obtain

$$\begin{aligned}\int_{K_{\mathbf{R}}} W\widehat{F}\,d\mu_{\mathbf{R}} &= \lim_{N\to\infty}\int_{K_{\mathbf{R}}} W\widehat{F}_N\,d\mu_{\mathbf{R}} = \lim_{N\to\infty}\int_{K_{\mathbf{R}}} W^*F_N\,d\mu_{\mathbf{R}}\\ &= \int_{K_{\mathbf{R}}} W^*F\,d\mu_{\mathbf{R}}\cdot\quad \|W^*\|_{p_1'}\|F\|_{p_1}<\infty.\end{aligned}$$

Therefore we have $\widehat{F}\in L_{p_2}(K_{\mathbf{R}},\mu_{\mathbf{R}})$, and hence the proof is complete.

3. Solvability of the Equation $F_t=\widehat{T}_t f-f$ in $L_p(\ \ ,\mu)$, with $0\cdot\quad p\cdot\quad\infty$

From the results of Section 2 we first observe that this problem is strongly connected with the problem of the existence of a $\mu_{\mathbf{R}}$-equivalent finite invariant measure with respect to the skew-product flow $\{\vartheta_t : t\in\mathbf{R}\}$.

Proposition 1. *Let* $\{F_t : t\in\mathbf{R}\}$ *be an additive process* (*with respect to* $\{T_t\}$). *Then the following conditions are equivalent*:

(I) *There exists a function* f *in* $L_0(\ \ , \mu)$ *such that* $F_t = \widehat{T}_t f - f$ *for all* $t \in \mathbf{R}$.

(II) *The skew-product flow* $\{\vartheta_t\}$ *admits a* $\mu_{\mathbf{R}}$*-equivalent finite invariant measure.*

Here if $\{T_t\}$ *is assumed to be ergodic*, *then the above conditions are also equivalent to the condition*:

(II)$'$ *The skew-product flow* $\{\vartheta_t\}$ *admits a* $\mu_{\mathbf{R}}$*-absolutely continuous nontrivial finite invariant measure.*

Proof. (I) $\Rightarrow$ (II). If f is a function in $L_0(\ \ , \mu)$ such that $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$, define

$$P(\xi) = \frac{1 + x^2}{1 + (x - f(\omega))^2} \quad \text{for} \quad \xi = (\omega, x) \in K_{\mathbf{R}}. \tag{24}$$

Then it follows that $0 < P \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$, and if we set $\nu = P\, d\mu_{\mathbf{R}}$, then ν becomes a $\mu_{\mathbf{R}}$-equivalent finite invariant measure with respect to $\{\vartheta_t\}$, by Fact 3.

(II) $\Rightarrow$ (I). Let $\nu = P\, d\mu_{\mathbf{R}}$ be a $\mu_{\mathbf{R}}$-equivalent finite invariant measure with respect to $\{\vartheta_t\}$. By Fubini's theorem we see that for μ-a.e. $\omega \in\ \ $, $P(\omega, x)$ belongs, as a function of $x \in \mathbf{R}$, to $L_1(\mathbf{R}, \mathcal{B}(\mathbf{R}), dx/\pi(1 + x^2))$; and since P is a strictly positive function on $K_{\mathbf{R}}$, we can define for μ-a.e. $\omega \in\ \ $ the probability measure λ_ω on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, absolutely continuous with respect to dx, by the relation

$$\lambda_\omega = \Box \int_{\mathbf{R}} P(\omega, x) \frac{dx}{\pi(1 + x^2)} \Big]^{-1} \cdot \frac{P(\omega, x)}{\pi(1 + x^2)}\, dx.$$

We will identify the measure λ_ω and the Radon-Nikodym derivative $d(\lambda_\omega)/dx \in L_1(\mathbf{R}, \mathcal{B}(\mathbf{R}), dx)$. Since there exists a sequence (P_n) of nonnegative simple functions in $L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ such that $\lim_{n\to\infty} \|P - P_n\|_1 = 0$, it follows that the mapping $\omega \mapsto \lambda_\omega = d(\lambda_\omega)/dx$ from $(\ \ , \mathcal{A}, \mu)$ to $L_1(\mathbf{R}, \mathcal{B}(\mathbf{R}), dx)$ is strongly measurable.

We next prove that if $t \in \mathbf{R}$ is fixed arbitrarily, then, for μ-a.e. $\omega \in\ \ $, the mapping $F_{(\omega, t)} : \mathbf{R} \to \mathbf{R}$ defined by

$$F_{(\omega, t)}(x) = x + F_t(\omega) \quad \text{for} \quad x \in \mathbf{R}$$

becomes a measure preserving transformation from $(\mathbf{R}, \mathcal{B}(\mathbf{R}), \lambda_\omega)$ to $(\mathbf{R}, \mathcal{B}(\mathbf{R}), \lambda_{T_t\omega})$.

To do this, let $\mathcal{D}$ denote the set of all intervals of the form $[a, b)$, where $a,\ b$ are rational numbers with $a < b$. Since $\nu = P\, d\mu_{\mathbf{R}}$ is invariant with respect to ϑ_t, it follows that

$$\nu(A \times B) = \nu(\vartheta_t(A \times B)) \quad \text{for every } A \in \mathcal{A} \text{ and } B \in \mathcal{D},$$

so that if ν denotes the μ-equivalent finite measure on $(\ , \mathcal{A})$ defined by

$$\text{(26)} \qquad \nu \ (A) = \nu(A \times \mathbf{R}) \quad \text{for} \quad A \in \mathcal{A},$$

then, by using Fubini's theorem and the relation

$$\left(\frac{d\nu}{d\mu}\right)(\omega) = \int_{\mathbf{R}} P(\omega,\, x) \cdot \frac{1}{\pi(1+x^2)}\, dx \quad \text{for } \mu\text{-a.e. } \omega \in \ ,$$

we have that for every $A \in \mathcal{A}$ and $B \in \mathcal{D}$

$$\begin{aligned} \int_A \lambda_\omega(B)\, d\nu \ (\omega) &= \nu(A \times B) = \nu(\vartheta_t(A \times B)) \\ &= \int_{T_t A} \lambda_\omega(B + F_t(T_t^{-1}\omega))\, d\nu \ (\omega) \\ &= \int_A \lambda_{T_t\omega}(B + F_t(\omega))\, d\nu \ (\omega), \end{aligned}$$

where the last equality comes from the T_t-invariance of the measure ν . Since $A \in \mathcal{A}$ is arbitrary, this implies that

$$\text{(27)} \qquad \lambda_\omega(B) = \lambda_{T_t\omega}(B + F_t(\omega))$$

for μ-a.e. $\omega \in$. Then, since $\mathcal{D}$ is countable and generates the σ-field $\mathcal{B}(\mathbf{R})$, we conclude that the equality (27) holds for all $B \in \mathcal{B}(\mathbf{R})$, for μ-a.e. $\omega \in$. This establishes the desired result.

We now define, for μ-a.e. $\omega \in$,

$$\text{(28)} \qquad f(\omega) = \sup\left\{a \in \mathbf{R} : \lambda_\omega((-\infty,\, a]) = 2^{-1}\right\}.$$

Then, f is a real-valued measurable function on $(\ , \mathcal{A}, \mu)$, and by (27) we have

$$F_t(\omega) = f(T_t\omega) - f(\omega) \quad \text{for } \mu\text{-a.e. } \omega \in \ ,$$

for each fixed $t \in \mathbf{R}$. Thus (I) follows.

Lastly, suppose $\{T_t\}$ is ergodic and (II)$'$ holds. We will prove that (I) follows. To do this, let $\nu = P\, d\mu_{\mathbf{R}}$ be a $\mu_{\mathbf{R}}$-absolutely continuous nontrivial finite invariant measure with respect to $\{\vartheta_t\}$. Then, the measure ν on $(\ , \mathcal{A})$ defined in (26) becomes a μ-absolutely continuous finite invariant measure with respect to $\{T_t\}$. Since $\{T_t\}$ is ergodic and ν is nontrivial, it follows that ν is μ-equivalent. Therefore we get

$$\left(\frac{d\nu}{d\mu}\right)(\omega) = \int_{\mathbf{R}} P(\omega,\, x) \cdot \frac{dx}{\pi(1+x^2)} > 0$$

for μ-a.e. $\omega \in$, whence (I) follows as in the proof of (II) $\Rightarrow$ (I).
This completes the proof of Proposition 1.

Example. (c) An additive process $\{F_t\}$ need not have the form $F_t = \widehat{T}_t f - f$ for some f in $L_0(\ , \mu)$. To see this, we give an example of an ergodic measure preserving flow $\{T_t\}$ and an additive process $\{F_t\} \subset L_\infty(\ , \mu)$ such that $\{F_t\}$ cannot have the form $F_t = \widehat{T}_t f - f$ for any $f \in L_0(\ , \mu)$. For this purpose, let $(\ , \mathcal{A}, \mu) = ([0, 1), \mathcal{B}([0, 1)), dx)$, where $\mathcal{B}([0, 1))$ and dx stand for the σ-field of all Borel subsets of the interval [0, 1) and the Lebesgue measure on [0, 1), respectively. Consider the ergodic measure preserving flow $\{T_t\}$ in $(\ , \mathcal{A}, \mu)$ defined by $T_t x = t + x$ (mod 1) for $x \in [0, 1)$ and $t \in \mathbf{R}$. If h is a nonnegative function in $L_\infty([0, 1))$ such that $\|h\|_\infty > 0$, then define

$$F_t(x) = \int_0^t h(T_s x)\, ds$$

for $x \in [0, 1) = $ and $t \in \mathbf{R}$. Clearly, $\{F_t\}$ becomes an additive process in $L_\infty([0, 1))$. It has the desired property, because if the process $\{F_t\}$ had the form $F_t = \widehat{T}_t f - f$ for some $f \in L_0(\ , \mu)$, then, since

$$\widehat{T}_t f = f + F_t \quad \text{and} \quad \lim_{t\to\infty} F_t(x) = \infty \quad \text{for all } x \in [0, 1) = \ ,$$

we must have $\lim_{t\to\infty} \|\widehat{T}_t f\|_0 = 1$, which contradicts $\|\widehat{T}_t f\|_0 = \|f\|_0 < 1$ for all $t \in \mathbf{R}$.

Theorem 1. *Let $\{F_t\}$ be an additive process (with respect to $\{T_t\}$). Suppose $0 < r \cdot \infty$. Then the following conditions are equivalent:*

(I) *The skew-product flow $\{\vartheta_t\}$ admits a $\mu_{\mathbf{R}}$-equivalent finite invariant measure $\nu = P\, d\mu_{\mathbf{R}}$ such that $0 < P \in L_{1+(r/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$.*

(II) *There exists a function f in $L_0(\ , \mu)$, with $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$, and a countable decomposition $\{A_n : n \geq 1\}$ of , with $A_n \in \mathcal{I}$ for all $n \geq 1$, such that the restriction $f|_{A_n}$ belongs to $L_r(A_n, \mu)$, for every $n \geq 1$.*

(III) *There exists a countable decomposition $\{A_n : n \geq 1\}$ of , with $A_n \in \mathcal{I}$ for all $n \geq 1$, such that if $F \in L_{1+(2/r)}(A_n \times \mathbf{R}, \mu_{\mathbf{R}})$, then the limit*

$$\widehat{F}(\xi) = \lim_{b\to\infty} \frac{1}{b} \int_0^b F(\vartheta_t \xi)\, dt \tag{29}$$

exists for $\mu_{\mathbf{R}}$-a.e. $\xi \in A_n \times \mathbf{R}$, and the limit function $\widehat{F}$ belongs to $L_1(A_n \times \mathbf{R}, \mu_{\mathbf{R}})$ for every $n \geq 1$.

Here if $\{T_t\}$ is assumed to be ergodic, then the above conditions are also equivalent to the condition:

(I)$'$ *The skew-product flow* $\{\vartheta_t\}$ *admits a* $\mu_{\mathbf{R}}$*-absolutely continuous nontrivial finite invariant measure* $\nu = P\,d\mu_{\mathbf{R}}$ *such that* $0 \cdot \quad P \in L_{1+(r/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$.

Proof. (I) $\Rightarrow$ (II). By Proposition 1 there exists a function f in $L_0(\quad, \mu)$ such that $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$. Here we may assume below, without loss of generality, that

$$F_t(\omega) = f(T_t\omega) - f(\omega) \quad \text{for all } \omega \in \quad \text{ and } t \in \mathbf{R}. \tag{30}$$

Case 1 : Suppose $r = \infty$. By the proof of (I) $\Rightarrow$ (II) of Proposition 1, we see that if P_f denotes the function on $K_{\mathbf{R}}$ defined by

$$P_f(\xi) = \frac{1 + x^2}{1 + (x - f(\omega))^2} \quad \text{for} \quad \xi = (\omega, x) \in K_{\mathbf{R}}, \tag{31}$$

then it satisfies $0 < P_f \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ and $P_f = U_t P_f$ for all $t \in \mathbf{R}$. On the other hand, since the measure $\nu = P\,d\mu_{\mathbf{R}}$ is invariant with respect to $\{\vartheta_t\}$ by hypothesis, we also see that $U_t P = P$ for all $t \in \mathbf{R}$. Thus, as in the proof of Fact 5, we find that

$$\frac{P_f(\xi)}{P(\xi)} = \frac{E\{P_f\,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \mu_{\mathbf{R}})\}(\xi)}{E\{P\,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, \mu_{\mathbf{R}})\}(\xi)} \quad \text{for} \quad \mu_{\mathbf{R}}\text{-a.e. } \xi \in K_{\mathbf{R}}; \tag{32}$$

and hence P_f can be written as $P_f(\xi) = R(\xi) \cdot P(\xi)$ on $K_{\mathbf{R}}$, where the function $R(\xi) = P_f(\xi)/P(\xi)$ is measurable with respect to the σ-field $\mathcal{I}_{\mathbf{R}}$, and thus it satisfies $R \circ \vartheta_t = R$ $\mu_{\mathbf{R}}$-a.e. on $K_{\mathbf{R}}$ for all $t \in \mathbf{R}$. Since $\|P\|_\infty < \infty$ in this case, if $\mathbf{Q}$ denotes the set of all rational numbers, then, by (31) and (30), we have for $\mu_{\mathbf{R}}$-a.e. $\xi = (\omega, x) \in K_{\mathbf{R}}$

$$\begin{aligned}
\infty > R(\omega, x) \cdot \sup_{t \in \mathbf{Q}} P(\vartheta_t(\omega, x)) &= \sup_{t \in \mathbf{Q}} P_f(\vartheta_t(\omega, x)) \\
= \sup_{t \in \mathbf{Q}} P_f(T_t\omega, x + F_t(\omega)) &= \sup_{t \in \mathbf{Q}} \frac{1 + (x + F_t(\omega))^2}{1 + \{(x + F_t(\omega)) - f(T_t\omega)\}^2} \\
= \sup_{t \in \mathbf{Q}} \frac{1 + (x + f(T_f\omega) - f(\omega))^2}{1 + (x - f(\omega))^2}
\end{aligned}$$

Hence, Fubini's theorem implies that for μ-a.e. $\omega \in \quad$ the inequality

$$\sup_{t \in \mathbf{Q}} \frac{1 + (x + f(T_t\omega) - f(\omega))^2}{1 + (x - f(\omega))^2} < \infty$$

holds for dx-a.e. $x \in \mathbf{R}$, and thus the function

$$h_\infty(\omega) = \sup_{t \in \mathbf{Q}} |f(T_t\omega)|$$

satisfies $0 \cdot \ \ h_\infty(\omega) < \infty$ for μ-a.e. $\omega \in \ \ $. Letting $A_n = \{\omega \in \ \ : n-1 \cdot \ h_\infty(\omega) < n\}$ for $n \geq 1$, we then obtain a countable decomposition $\{A_n : n \geq 1\}$ of . Here, by the definition of h_∞, we get $\widehat{T}_t h_\infty = h_\infty$ for all $t \in \mathbf{Q}$. Then, since $\lim_{t \to s} \|\widehat{T}_t h_\infty - \widehat{T}_s h_\infty\|_0 = 0$ for every $s \in \mathbf{R}$ by (10), $\widehat{T}_t h_\infty = h_\infty$ holds for all $t \in \mathbf{R}$. Consequently, we find that $A_n \in \mathcal{I}$ and $|f| \cdot \ \ n$ on A_n for every $n \geq 1$.

Case 2: Suppose $0 < r < \infty$. Then, by Fact 5 together with (20), the limit

$$\lim_{b \to \infty} \frac{1}{b} \int_0^b \left(\frac{1 + (x + F_t(\omega))^2}{1 + x^2} \right)^{r/2} dt$$

exists and is a positive real number for $\mu_{\mathbf{R}}$-a.e. $\xi = (\omega, x) \in K_{\mathbf{R}}$. It then follows from Fubini's theorem that for μ-a.e. $\omega \in \ \ $ the inequality

$$\limsup_{b \to \infty} \frac{1}{b} \int_0^b |x + F_t(\omega)|^r \, dt < \infty \tag{33}$$

holds for dx-a.e. $x \in \mathbf{R}$; but this is obviously equivalent to the validity of the inequality (33) for a given real number x. Thus, using (30), we get for μ-a.e. $\omega \in$

$$\limsup_{b \to \infty} \frac{1}{b} \int_0^b |f(T_t \omega)|^r \, dt = \limsup_{b \to \infty} \frac{1}{b} \int_0^b |f(\omega) + F_t(\omega)|^r \, dt < \infty.$$

By this and the pointwise ergodic theorem for the measure preserving flow $\{T_t\}$, the limit

$$g_r(\omega) = \lim_{b \to \infty} \frac{1}{b} \int_0^b |f|^r (T_t \omega) \, dt \tag{34}$$

exists and satisfies $0 \cdot \ \ g_r(\omega) < \infty$ for μ-a.e. $\omega \in \ \ $. Letting $A_n = \{\omega \in \ \ : n-1 \cdot \ \ g_r(\omega) < n\}$ for $n \geq 1$, we obtain a countable decomposition $\{A_n : n \geq 1\}$ of , and since $\widehat{T}_t g_r = g_r$ for all $t \in \mathbf{R}$, it follows that $A_n \in \mathcal{I}$ and $\int_{A_n} |f|^r \, d\mu = \int_{A_n} g_r \, d\mu < n$ for every $n \geq 1$.

(II) $\Rightarrow$ (I). Let $f \in L_0(\ \ , \mu)$ be the function given in (II). We first notice that $P_f \, d\mu_{\mathbf{R}}$ is a $\mu_{\mathbf{R}}$-equivalent finite invariant measure with respect to $\{\vartheta_t\}$ and, by an elementary calculation, the inequality

$$P_f(\xi) \left(= \frac{1 + x^2}{1 + (x - f(\omega))^2} \right) < 2 + f^2(\omega) \quad (\text{see}(31)) \tag{35}$$

holds for every $\xi = (\omega, x) \in K_{\mathbf{R}}$.

Case 1 : Suppose $r = \infty$. Then, since $f|_{A_n} \in L_\infty(A_n, \mu)$ in this case, there exists a constant α_n such that $|f(\omega)| \cdot \ \alpha_n$ on the set A_n. Thus, if we define a function P on $K_{\mathbf{R}}$ by

$$P(\xi) = \sum_{n=1}^{\infty} \frac{1}{2+\alpha_n^2} P_f(\xi) \cdot \chi_{A_n \times \mathbf{R}}(\xi) \quad \text{for} \quad \xi \in K_{\mathbf{R}},$$

then it satisfies $0 < P(\xi) < 1$ on $K_{\mathbf{R}}$ by (35), and the measure $\nu = P\, d\mu_{\mathbf{R}}$ becomes an invariant measure with respect to $\{\vartheta_t\}$, because $A_n \in \mathcal{I}$ for every $n \geq 1$, by hypothesis.

Case 2 : Suppose $0 < r < \infty$. Then, since $f|_{A_n} \in L_r(A_n, \mu)$ by hypothesis, it follows from (35) and Fubini's theorem that

$$\begin{aligned}\int_{A_n \times \mathbf{R}} |P_f(\xi)|^{1+(r/2)}\, d\mu_{\mathbf{R}} \cdot \ \int_{A_n \times \mathbf{R}} &\{2 + f^2(\omega)\}^{r/2} \cdot P_f(\omega, x)\ \ d\mu_{\mathbf{R}}(\omega, x) \\ &= \int_{A_n} \int_{-\infty}^{\infty} \frac{\{2+f^2(\omega)\}^{r/2}}{\{1+(x-f(\omega))^2\}\,\pi}\, dx\, d\mu(\omega) \quad \text{(by (31))} \\ &= \int_{A_n} \{2+f^2(\omega)\}^{r/2}\, d\mu(\omega) < \infty.\end{aligned}$$

Therefore, we can define a function P on $K_{\mathbf{R}}$ by

$$P(\xi) = \sum_{n=1}^{\infty} \frac{1}{2^n \beta_n} P_f(\xi) \cdot \chi_{A_n \times \mathbf{R}}(\xi) \quad \text{for} \quad \xi \in K_{\mathbf{R}},$$

where $\beta_n := \|P_f \cdot \chi_{A_n \times \mathbf{R}}\|_{1+(r/2)} < \infty$ for $n \geq 1$, to obtain a $\mu_{\mathbf{R}}$-equivalent finite invariant measure $\nu = P\, d\mu_{\mathbf{R}}$ with respect to $\{\vartheta_t\}$ such that $P \in L_{1+(r/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$.

(II) $\Rightarrow$ (III). Since (II) implies the existence of $\mu_{\mathbf{R}}$-equivalent finite invariant measure with respect to $\{\vartheta_t\}$, the condition (I) of Fact 6 holds, so that $\{U_t\}$ satisfies the L_1-mean ergodic theorem. Since every A_n in the condition (II) of Theorem 1 is an invariant subset of with respect to $\{T_t\}$, it may be assumed for the proof, without loss of generality, that $= A_n$. Then, Fact 7 implies that it suffices to show that for every $W \in L_\infty(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ the limit function W^* in the condition (II) of Fact 6 belongs to $L_{1+(r/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$. But, for this purpose it is clearly enough to show that $1^* \in L_{1+(r/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$. And to do this, we observe that

$$(36) \qquad 1^*(\xi) = \lim_{b\to\infty} \frac{1}{b} \int_0^b w_t(\xi)\, dt = \lim_{b\to\infty} \frac{1}{b} \int_0^b \frac{1+x^2}{1+(x+F_t(\omega))^2}\, dt$$

for $\mu_{\mathbf{R}}$-a.e. $\xi = (\omega, x) \in K_{\mathbf{R}}$, by (17), (20) and Remark 2.

Case 1: Suppose $r = \infty$. Then, since

$$0 \cdot \quad \frac{1+x^2}{1+(x+F_t(\omega))^2} < 2 + F_t^2(\omega) \tag{37}$$
$$= 2 + (f(T_t\omega) - f(\omega))^2 \cdot \quad 2 + 4\|f\|_\infty^2 < \infty$$

for $\mu_{\mathbf{R}}$-a.e. $\xi = (\omega, x) \in K_{\mathbf{R}}$, it follows from (36) that $0 \cdot \quad 1^*(\xi) \cdot \quad 2 + 4\|f\|_\infty^2 < \infty$ for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$. Hence, we have $1^* \in L_\infty(K_{\mathbf{R}}, \mu_{\mathbf{R}})$.

Case 2: Suppose $0 < r < \infty$. Then, since (37) implies

$$w_t(\xi) = \frac{1+x^2}{1+(x+F_t(\omega))^2} \cdot \quad 2 + 2\left(|f|^2(T_t\omega) + |f|^2(\omega)\right)$$

for $\xi = (\omega, x) \in K_{\mathbf{R}}$, the function

$$G_b(\xi) = \frac{1}{b}\int_0^b w_t(\xi)\,dt \quad \text{for} \quad \xi = (\omega, x) \in K_{\mathbf{R}} \tag{38}$$

satisfies, by Hölder's inequality,

$$|G_b(\xi)|^{1+(r/2)} \cdot \quad \frac{1}{b}\int_0^b |w_t(\xi)|^{1+(r/2)}\,dt$$
$$\cdot \quad \frac{1}{b}\int_0^b \left\{2 + 2|f|^2(T_t\omega) + 2|f|^2(\omega)\right\}^{r/2} \cdot w_t(\xi)\,dt$$
$$\cdot \quad \frac{C_r}{b}\int_0^b \{1 + |f|^r(T_t\omega) + |f|^r(\omega)\} \cdot w_t(\xi)\,dt,$$

where C_r is an absolute constant depending only on r. Thus, by Fubini's theorem,

$$\int_{K_{\mathbf{R}}} |G_b(\omega, x)|^{1+(r/2)}\,d\mu_{\mathbf{R}}(\omega, x)$$
$$\cdot \quad \frac{C_r}{b}\int_0^b \int_{K_{\mathbf{R}}} \{1 + |f|^r(T_t\omega) + |f|^r(\omega)\} \cdot w_t(\omega, x)\,d\mu_{\mathbf{R}}(\omega, x)\,dt$$
$$= \frac{C_r}{b}\int_0^b \int \{1 + |f|^r(T_t\omega) + |f|^r(\omega)\}\,d\mu(\omega)\,dt \qquad \text{(by (20))}$$
$$= C_r\{1 + 2\|f\|_r^r\} < \infty$$

for all $b > 0$. Since $1^*(\xi) = \lim_{b\to\infty} G_b(\xi)$ for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$, it follows from Fatou's lemma that the limit function 1^* belongs to $L_{1+(r/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$.

(III) $\Rightarrow$ (I). If $F \in L_\infty(K_{\mathbf{R}}, \mu_{\mathbf{R}})$, then (III) implies that the limit

$$\widehat{F}(\xi) = \lim_{b\to\infty} \frac{1}{b}\int_0^b F(\vartheta_t\xi)\,dt$$

exists for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$. Thus, $\{U_t\}$ satisfies the L_1-mean ergodic theorem by Fact 6. Since every A_n in (III) is an invariant subset of   with respect to $\{T_t\}$, we can apply Fact 7, with $p_1 = 1 + (2/r)$ and $p_2 = 1$, to infer that the restriction of the limit function 1^* to the set $A_n \times \mathbf{R}$ belongs to $L_{1+(r/2)}(A_n \times \mathbf{R}, \mu_{\mathbf{R}})$. Then the function

$$P(\xi) = \sum_{n=1}^{\infty} \frac{1}{2^n \|1^* \cdot \chi_{A_n \times \mathbf{R}}\|_{1+(r/2)}} 1^*(\xi) \cdot \chi_{A_n \times \mathbf{R}}(\xi) \quad \text{for } \xi \in K_{\mathbf{R}}$$

satisfies $P(\xi) > 0$ for $\mu_{\mathbf{R}}$-a.e. $\xi \in K_{\mathbf{R}}$, $P \in L_{1+(r/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ and $U_t P = P$ for every $t \in \mathbf{R}$. Thus, by putting $\nu = P\, d\mu_{\mathbf{R}}$, (I) follows.

Lastly, suppose $\{T_t\}$ is ergodic and (I)$'$ holds. We will prove that (II) follows. To do this, let $\nu = P\, d\mu_{\mathbf{R}}$ be a $\mu_{\mathbf{R}}$-absolutely continuous nontrivial finite invariant measure with respect to $\{\vartheta_t\}$ such that $0 \cdot \ P \in L_{1+(r/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$. By the second part of Proposition 1 there exists a function f in $L_0(\ , \mu)$ such that $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$. Then the function P_f satisfies, as before, that $0 < P_f \in L_1(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ and $U_t P_f = P_f$ for all $t \in \mathbf{R}$. On the other hand, since $0 \cdot \ P = U_t P \in L_{1+(r/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$ for all $t \in \mathbf{R}$, it also follows, as in the proof of (I) $\Rightarrow$ (II), that there exists an $\mathcal{I}_{\mathbf{R}}$-measurable real-valued function R on $K_{\mathbf{R}}$ such that

$$P(\xi) = P_f(\xi) \cdot R(\xi) \quad \text{on} \quad K_{\mathbf{R}}.$$

Therefore, the relation $\{\xi \in K_{\mathbf{R}} : P(\xi) > 0\} = \{\xi \in K_{\mathbf{R}} : R(\xi) > 0\}$ (mod $\mu_{\mathbf{R}}$) holds, and hence the set

$$E = \{\xi \in K_{\mathbf{R}} : P(\xi) > 0\}$$

belongs to $\mathcal{I}_{\mathbf{R}}$. Here we notice that $\mu_{\mathbf{R}}(E) > 0$, because ν $(= P\, d\mu_{\mathbf{R}})$ is a nontrivial measure, by hypothesis.

Case 1: Suppose $r = \infty$. Then $\|P\|_\infty < \infty$ holds, and thus for $\mu_{\mathbf{R}}$-a.e. $\xi = (\omega, x) \in E$ we have

$$\begin{aligned} \infty > \sup_{t \in \mathbf{Q}} P(\vartheta_t(\omega, x)) &= R(\omega, x) \cdot \sup_{t \in \mathbf{Q}} P_f(\vartheta_t(\omega, x)) \\ &= R(\omega, x) \cdot \sup_{t \in \mathbf{Q}} \frac{1 + (x + f(T_t\omega) - f(\omega))^2}{1 + (x - f(\omega))^2}, \end{aligned}$$

so that the function

$$h_\infty(\omega) = \sup_{t \in \mathbf{Q}} |f(T_t \omega)| \qquad (\omega \in \ \ )$$

satisfies $h_\infty(\omega) < \infty$ for $\mu_{\mathbf{R}}$-a.e. $(\omega, x) \in E$, and $h_\infty(T_t\omega) = h_\infty(\omega)$ for μ-a.e. $\omega \in$, for every $t \in \mathbf{Q}$ (and hence for every $t \in \mathbf{R}$ as before). By the fact that $\mu_{\mathbf{R}}(E) > 0$, Fubini's theorem implies that $h_\infty(\omega) < \infty$ on a set of positive μ-measure. This, together with the ergodicity of $\{T_t\}$, shows that h_∞ is a constant function in $L_\infty(\ ,\mu)$, and consequently f is a function in $L_\infty(\ ,\mu)$.

Case 2: Suppose $0 < r < \infty$. Since $\nu = P\,d\mu_{\mathbf{R}}$ is invariant with respect to $\{\vartheta_t\}$, it follows from Fact 2 that $(P \circ \vartheta_t) \cdot w_t = P$ $\mu_{\mathbf{R}}$-a.e. on $K_{\mathbf{R}}$, for every $t \in \mathbf{R}$. By this and (20), we have

$$\sqrt{1 + (x + F_t(\omega))^2} \cdot P^{1/2}(\omega, x) \cdot \quad P^{1/2} \circ \vartheta_t(\omega, x)$$

for $\mu_{\mathbf{R}}$-a.e. $(\omega, x) \in K_{\mathbf{R}}$. Since $F_t(\omega) = f(T_t\omega) - f(\omega)$ for μ-a.e. $\omega \in\ $, for every $t \in \mathbf{R}$, it follows that

$$|x + f(T_t\omega) - f(\omega)| \cdot P^{1/2}(\omega, x) \cdot \quad P^{1/2} \circ \vartheta_t(\omega, x)$$

for $\mu_{\mathbf{R}}$-a.e. $(\omega, x) \in K_{\mathbf{R}}$, for every $t \in \mathbf{R}$. Then we find, by Fubini's theorem and the pointwise ergodic theorem applied to the flow $\{\vartheta_t\}$, that

$$\begin{aligned} P^{r/2}(\omega, x) \cdot \limsup_{b \to \infty} \frac{1}{b} \int_0^b |x + f(T_t\omega) - f(\omega)|^r\,dt \\ \cdot \lim_{b \to \infty} \frac{1}{b} \int_0^b P^{r/2} \circ \vartheta_t(\omega, x)\,dt \\ = E\{P^{r/2}\,|(K_{\mathbf{R}}, \mathcal{I}_{\mathbf{R}}, P\,d\mu_{\mathbf{R}})\}(\omega, x) < \infty \end{aligned}$$

for ν-a.e. $(\omega, x) \in K_{\mathbf{R}}$, where the last inequality comes from the hypothesis that $P \in L_{1+(r/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$. Hence the pointwise ergodic theorem for the flow $\{T_t\}$ implies that the almost everywhere limit function

$$g_r(\omega) = \lim_{b \to \infty} \frac{1}{b} \int_0^b |f|^r(T_t\omega)\,dt \qquad (\omega \in\ )$$

satisfies $g_r(\omega) < \infty$ for $\mu_{\mathbf{R}}$-a.e. $(\omega, x) \in E$. Then, using the invariance of the function g_r with respect to $\{T_t\}$ and the ergodicity of $\{T_t\}$, we see, as in Case 1, that $g_r(\omega) = \int\ |f|^r\,d\mu < \infty$ for μ-a.e. $\omega \in\ $. Consequently, we find that $f \in L_r(\ ,\mu)$.

This completes the proof of Theorem 1.

For further studies of the ergodic properties of the process $\{F_t\}$ we need to introduce another probability measure space. Denote

(39) $$(K_{\partial\mathbf{D}}, \mathcal{A}_{\partial\mathbf{D}}, \mu_{\partial\mathbf{D}}) := \left(\ \times \partial\mathbf{D}, \mathcal{A}\quad \mathcal{B}(\partial\mathbf{D}), \mu\quad \frac{dx}{2\pi}\right),$$

where $\partial\mathbf{D} = \{e^{ix} : 0 \cdot \ \ x < 2\pi\}$, and where $\mathcal{B}(\partial\mathbf{D})$ and dx stand for the σ-field of all Borel subsets of $\partial\mathbf{D}$ and the Lebesgue measure on $\partial\mathbf{D}$, respectively.

If $s \in \mathbf{R}$ is fixed arbitrarily, then we introduce a family $\{\widetilde{\vartheta}(s)_t : t \in \mathbf{R}\}$ of skew-product transformations as follows:

$$(40) \qquad \widetilde{\vartheta}(s)_t(\omega, e^{ix}) = \left(T_t\omega, e^{ix}e^{-sF_t(\omega)}\right) \quad \text{for } (\omega, e^{ix}) \in K_{\partial\mathbf{D}} \text{ and } t \in \mathbf{R}.$$

It is easy to check that

(i) each $\widetilde{\vartheta}(s)_t$ is an invertible measure preserving transformation in $(K_{\partial\mathbf{D}}, \mathcal{A}_{\partial\mathbf{D}}, \mu_{\partial\mathbf{D}})$,

(ii)′ $\widetilde{\vartheta}(s)_t \circ \widetilde{\vartheta}(s)_u(\omega, e^{ix}) = \widetilde{\vartheta}(s)_{t+u}(\omega, e^{ix})$ for $\mu_{\partial\mathbf{D}}$-a.e. $(\omega, e^{ix}) \in K_{\partial\mathbf{D}}$, for every $t, u \in \mathbf{R}$,

(iii) the mapping $((\omega, e^{ix}), t) \mapsto \widetilde{\vartheta}(s)_t(\omega, e^{ix})$ is a measurable transformation from $(K_{\partial\mathbf{D}} \times \mathbf{R}, \overline{\mathcal{A}_{\partial\mathbf{D}} \quad \mathcal{B}(\mathbf{R})}, \overline{\mu_{\partial\mathbf{D}} \quad dt})$ to $(K_{\partial\mathbf{D}}, \mathcal{A}_{\partial\mathbf{D}}, \mu_{\partial\mathbf{D}})$.

Thus, $\{\widetilde{\vartheta}(s)_t : t \in \mathbf{R}\}$ becomes a measure preserving flow in $(K_{\partial\mathbf{D}}, \mathcal{A}_{\partial\mathbf{D}}, \mu_{\partial\mathbf{D}})$. Since the function $g(\omega, e^{ix}) = e^{ix}$ for $(\omega, e^{ix}) \in K_{\partial\mathbf{D}}$ belongs to $L_1(K_{\partial\mathbf{D}}, \mu_{\partial\mathbf{D}})$, it then follows from the pointwise ergodic theorem, applied to the flow $\{\widetilde{\vartheta}(s)_t : t \in \mathbf{R}\}$ with the function g, and Fubini's theorem that the limit

$$\lim_{b\to\infty} \frac{1}{b}\int_0^b e^{-isF_t(\omega)}\,dt$$

exists for μ-a.e. $\omega \in \quad$. Taking this into account, we define a real-valued function J on $\quad \times \mathbf{R}$ by

$$(41) \qquad J(\omega, s) = \begin{cases} \lim_{b\to\infty} \frac{1}{b}\int_0^b e^{-isF_t(\omega)}\,dt & \text{if the limit exists,} \\ 2 & \text{otherwise.} \end{cases}$$

Then J becomes a measurable function on $(\quad \times \mathbf{R}, \mathcal{A} \quad \mathcal{B}(\mathbf{R}), \mu \quad ds)$ (cf. (13)), so that, by Fubini's theorem, there exists $\ _1 \in \mathcal{A}$, with $\mu(\ _1) = 1$, such that

$$_1 = \{\omega \in \quad : J(\omega, s) \neq 2 \text{ for } ds\text{-a.e. } s \in \mathbf{R}\}.$$

If $\omega \in \ _1$, then $J(\omega, s)$ is, as a function of $s \in \mathbf{R}$, the ds-a.e. limit of the continuous positive definite functions $b^{-1}\int_0^b e^{-isF_t(\omega)}\,dt$ as $b \to \infty$, whence there exists a finite measure μ_ω on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ such that

$$(42) \qquad J(\omega, s) = \int_{\mathbf{R}} e^{ist}\,d\mu_\omega(t) \quad \text{for } ds\text{-a.e. } s \in \mathbf{R}.$$

(This is a well-known fact on harmonic analysis. See e.g. §32 and §33 of [8].) Here we notice that $0 \cdot \ \mu_\omega(\mathbf{R}) \cdot \ 1$ for $\omega \in \ {}_1$, because $|J(\omega, s)| \cdot \ 1$ for ds-a.e. $s \in \mathbf{R}$, by (41).

Since $F_u(\omega) + F_t(T_u\omega) = F_{u+t}(\omega)$ for μ-a.e. $\omega \in \ $, for every $u, t \in \mathbf{R}$, we can choose another set ${}_2 \in \mathcal{A}$, with $\mu({}_2) = 1$, such that

$$(43) \qquad {}_2 = \{\omega \in \ : F_u(\omega) + F_t(T_u\omega) = F_{u+t}(\omega) \text{ for } dt\text{-a.e. } t \in \mathbf{R}, \text{ for all } u \in \mathbf{Q}\}.$$

If $\omega \in \ {}_1 \cap \ {}_2$ and $u \in \mathbf{Q}$, then we get, by (41), that

$$(44) \qquad T_u\omega \in \ {}_1 \quad \text{and} \quad J(T_u\omega, s) = e^{isF_u(\omega)} J(\omega, s)$$

for every $s \in \mathbf{R}$ with $J(\omega, s) \neq 2$. Therefore, (42) implies that

$$(45) \qquad \int_{\mathbf{R}} e^{ist}\, d\mu_{T_u\omega} = e^{isF_u(\omega)} \int_{\mathbf{R}} e^{ist}\, d\mu_\omega(t) \quad \text{for } ds\text{-a.e. } s \in \mathbf{R}.$$

But, since both sides of (45) are continuous functions of $s \in \mathbf{R}$, it follows that the equality in (45) holds for all $s \in \mathbf{R}$. This proves that if $\omega \in \ {}_1 \cap \ {}_2$ and $u \in \mathbf{Q}$, then

$$(46) \qquad \mu_{T_u\omega}(B) = \mu_\omega(B - F_u(\omega)) \quad \text{for every } B \in \mathcal{B}(\mathbf{R}).$$

We next characterize the set $\{\omega \in \ {}_1 : \mu_\omega \neq 0\}$ as follows. For an integer $N \geq 1$, we introduce a function α_N on by

$$(47) \qquad \alpha_N(\omega) = \limsup_{b \to \infty} \frac{1}{b} \int_0^b \chi_{[-N, N]}(F_t(\omega))\, dt \qquad (\omega \in \ ),$$

and then put

$$(48) \qquad \alpha_\infty(\omega) = \lim_{N \to \infty} \alpha_N(\omega) \qquad (\omega \in \ ).$$

Since each α_N is a measurable function on $(\ , \mathcal{A}, \mu)$ by Fubini's theorem (cf. (13)), their limit function α_∞ is also a measurable function on $(\ , \mathcal{A}, \mu)$. Furthermore, if $\omega \in \ {}_2$ and $u \in \mathbf{Q}$, then we have, by (43), that

$$(49) \qquad \alpha_\infty(\omega) > 0 \quad \text{if and only if} \quad \alpha_\infty(T_u\omega) > 0.$$

Taking this into account, we introduce a set ${}_+$ in $\mathcal{A}$ by

$$(50) \qquad {}_+ := \{\omega \in \ : \alpha_\infty(\omega) > 0\} \cap \ {}_1 \cap \ {}_2.$$

Since $\mu_\omega = 0$ ($\omega \in \ _1$) is equivalent to

$$\int_{\mathbf{R}} \widehat{v}(t)\, d\mu_\omega(t) = 0 \quad \text{for every } \ v \in L_1(\mathbf{R}, ds),$$

where $\widehat{v}$ is the Fourier transform of v, i.e., $\widehat{v}(t) = \int_{\mathbf{R}} v(s)e^{-its}\, ds$ for $t \in \mathbf{R}$, and since

$$\begin{aligned} (51) \quad \int_{\mathbf{R}} \widehat{v}(t) d\mu_\omega(t) &= \int_{\mathbf{R}} \int_{\mathbf{R}} v(s)e^{-ist} ds\, d\mu_\omega(t) = \int_{\mathbf{R}} \left(v(s) \int_{\mathbf{R}} e^{-ist} d\mu_\omega(t) \right) ds \\ &= \int_{\mathbf{R}} v(s) J(\omega, -s)\, ds = \int_{\mathbf{R}} v(s) \left(\lim_{b\to\infty} \frac{1}{b} \int_0^b e^{isF_t(\omega)} dt \right) ds \\ &= \lim_{b\to\infty} \frac{1}{b} \int_0^b \widehat{v}(-F_t(\omega)) dt, \end{aligned}$$

it follows that $\mu_\omega = 0$ is equivalent to

$$\lim_{b\to\infty} \frac{1}{b} \int_0^b \widehat{v}(-F_t(\omega))\, dt = 0$$

for every $v \in L_1(\mathbf{R}, ds)$. Consequently, we see that $\mu_\omega = 0$ is equivalent to $\alpha_\infty(\omega) = 0$. Thus, the equality

$$(52) \qquad \ _+ = \{\omega \in \ _1 : \mu_\omega > 0\} \cap \ _2$$

holds and, by (46) and/or (49), we have

$$(53) \qquad \mu(\ _+ \triangle T_u \ _+) = 0 \quad \text{for all } \ u \in \mathbf{Q}.$$

It follows that $\mu(\ _+ \triangle T_u \ _+) = 0$ for all $u \in \mathbf{R}$, i.e., $\ _+ \in \mathcal{I}$.

We now define a real-valued function f on   by

$$(54) \qquad f(\omega) = \begin{cases} \sup \left\{ a \in \mathbf{R} : \mu_\omega((-\infty, a]) \cdot \ \ 2^{-1}\mu_\omega(\mathbf{R}) \right\} & \text{if } \omega \in \ _1, \\ 0 & \text{otherwise.} \end{cases}$$

By an easy approximation argument, together with Fubini's theorem and (51), we see that f is an extended real-valued measurable functin on $(\ , \mathcal{A}, \mu)$ such that $-\infty < f(\omega) < \infty$ for every $\omega \in \ _+$. We also see, by (46), that the equality

$$F_u(\omega) = f(T_u\omega) - f(\omega)$$

holds for all $\omega \in \ _+$ and $u \in \mathbf{Q}$. Then, since the mappings $u \to F_u$ and $u \to f \circ T_u - f$ are continuous from $\mathbf{R}$ to $L_0(\ , \mu)$ with respect to the metric d_0, the equality $F_u(\omega) = f(T_u\omega) - f(\omega)$ holds for μ-a.e. $\omega \in \ _+$, for every $u \in \mathbf{R}$.

Lastly, let $A \in \mathcal{I}$ be such that there exists a real-valued measurable function f_A on A with $F_t(\omega) = f_A(T_t\omega) - f_A(\omega)$ for μ-a.e. $\omega \in A$, for every $t \in \mathbf{R}$. (Here, it may be assumed without loss of generality that $A \subset \ _1 \cap \ _2$, because $\mu(\ _1 \cap \ _2) = 1$.) By Fubini's theorem there exists $B \in \mathcal{A}$, with $B \subset A$ and $\mu(A \setminus B) = 0$, such that if $\omega \in B$, then the function $t \to f_A(T_t\omega)$ is Lebesgue measurable on $\mathbf{R}$, and the equality

$$F_t(\omega) = f_A(T_t\omega) - f_A(\omega) \tag{55}$$

holds for dt-a.e. $t \in \mathbf{R}$. Let

$$B_N = \{\omega \in B : |f_A(\omega)| \cdot \ N\} \qquad (N \geq 1).$$

Since $B_1 \subset B_2 \subset \ldots \uparrow B$ and $B \in \mathcal{I}$, the functions

$$\beta_N(\omega) = \lim_{b\to\infty} \frac{1}{b}\int_0^b \chi_{B_N}(T_t\omega)\, dt \qquad (\omega \in \ )$$

satisfy $\lim_{N\to\infty} \beta_N(\omega) = 1$ for μ-a.e. $\omega \in B$. Thus, by putting

$$A_1 = \left\{\omega \in B : \lim_{N\to\infty} \beta_N(\omega) = 1\right\},$$

we have $A_1 \subset A$ and $\mu(A \setminus A_1) = 0$. Suppose $\omega \in A_1$. Then there exists an integer $N \geq 1$ such that $\omega \in B_N$ and $\beta_N(\omega) > 0$. Then, by using the inequalities

$$|F_t(\omega)| \cdot \ |f_A(T_t\omega)| + |f_A(\omega)| \cdot \ |f_A(T_t\omega)| + N \cdot \ 2N$$

for dt-a.e. $t \in \mathbf{R}$ with $T_t\omega \in B_N$, we have

$$\begin{aligned} 0 < \beta_N(\omega) &= \lim_{b\to\infty} \frac{1}{b}\int_0^b \chi_{B_N}(T_t\omega)\, dt \\ &\cdot \ \limsup_{b\to\infty} \frac{1}{b}\int_0^b \chi_{[-2N,\,2N]}(F_t(\omega))\, dt = \alpha_{2N}(\omega). \end{aligned}$$

Consequently, $\alpha_\infty(\omega) > 0$ for all $\omega \in A_1$, and we find that $\mu(A \setminus \ _+) = 0$.

We have thus established the following

Proposition 2. *Let* $\{F_t\}$ *be an additive process* (*with respect to* $\{T_t\}$). *Then the following conditions are equivalent*:

(I) *There exists a function* f *in* $L_0(\ , \mu)$ *such that* $F_t = \widehat{T}_t f - f$ *for all* $t \in \mathbf{R}$.

(II) *The inequality* $\alpha_\infty(\omega) > 0$ *holds for* μ-a.e. $\omega \in \ $.

Here if $\{T_t\}$ *is assumed to be ergodic, then the condition* (II) *can be replaced with the following weaker condition*:

(II)′ *The inequality* $\alpha_\infty(\omega) > 0$ *holds on a set of positive* μ*-measure.*

By using Theorem 1 and Proposition 2 we next prove the following

Theorem 2. *Let* $\{F_t\}$ *be an additive process* (*with respect to* $\{T_t\}$). *Suppose* $0 < p < \infty$. *Then, among the following conditions, the implications*

$$\text{(I)} \;\Rightarrow\; \text{(II)} \;\Rightarrow\; \text{(III)} \;\Rightarrow\; \text{(II)}'$$

hold. Here if $\{T_t\}$ *is assumed to be ergodic*, *then the implication*

$$\text{(II)}' \;\Rightarrow\; \text{(I)}$$

also holds, *so that all the conditions are equivalent.*

(I) *There exists a function* f *in* $L_p(\;, \mu)$ *such that* $F_t = \widehat{T}_t f - f$ *for all* $t \in \mathbf{R}$.

(II) *The inequality* $\liminf_{b\to\infty} (1/b) \int_0^b \|F_t\|_p^p \, dt < \infty$ *holds.*

(III) *The skew-product flow* $\{\vartheta_t\}$ *admits a* $\mu_{\mathbf{R}}$*-equivalent finite invariant measure* $\nu = P \, d\mu_{\mathbf{R}}$ *such that* $0 < P \in L_{1+(p/2)}(K_{\mathbf{R}}, \mu_{\mathbf{R}})$.

(II)′ *The inequality* $\liminf_{b\to\infty}(1/b) \int_0^b \|\chi_A \cdot F_t\|_p^p \, dt < \infty$ *holds for some* $A \in \mathcal{A}$ *with* $\mu(A) > 0$.

Proof. (I) ⇒ (II). Suppose $F_t = \widehat{T}_t f - f$ for some $f \in L_p(\;, \mu)$ and all $t \in \mathbf{R}$. Then, from the relations

$$\|F_t\|_p^p = \int |f \circ T_t - f|^p \, d\mu \cdot \; 2^{p+1} \|f\|_p^p,$$

(II) follows at once.

(II) ⇒ (III). Define an extended real-valued measurable function η on $(\;, \mathcal{A}, \mu)$ by

$$\eta(\omega) = \liminf_{b\to\infty} \frac{1}{b} \int_0^b |F_t(\omega)|^p \, dt \quad (\omega \in \;). \tag{56}$$

Then, (II) implies that

$$0 \cdot \int \eta(\omega) d\mu \cdot \; \liminf_{b\to\infty} \frac{1}{b} \int_0^b \int |F_t(\omega)|^p d\mu(\omega) dt < \infty \tag{57}$$

by Fubini's theorem and Fatou's lemma, so that we have $0 \cdot \; \eta(\omega) < \infty$ for μ-a.e. $\omega \in \;$. This implies easily that $\alpha_\infty(\omega) > 0$ for μ-a.e. $\omega \in \;$. Thus, by Proposition

2, there exists a function f in $L_0(\ ,\mu)$ such that $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$. Then we have

$$\eta(\omega) = \liminf_{b\to\infty} \frac{1}{b}\int_0^b |f(T_t\omega) - f(\omega)|^p\, dt < \infty \tag{58}$$

for μ-a.e. $\omega \in\ $. By this and the pointwise ergodic theorem for the flow $\{T_t\}$, we find that the limit function

$$g_p(\omega) = \lim_{b\to\infty} \frac{1}{b}\int_0^b |f|^p(T_t\omega)\, dt \qquad (\omega \in\ ) \tag{59}$$

satisfies $0 \cdot\ g_p(\omega) < \infty$ for μ-a.e. $\omega \in\ $. Thus, as in the proof of (I) $\Rightarrow$ (II) of Theorem 1, letting $A_n = \{\omega \in\ : n - 1 \cdot\ g_p(\omega) < n\}$ for $n \geq 1$, we get a countable decomposition $\{A_n : n \geq 1\}$ of such that $A_n \in \mathcal{I}$ and $\int_{A_n} |f|^p\, d\mu = \int_{A_n} g_p\, d\mu < n$ for every $n \geq 1$, which is equivalent to (III), by Theorem 1.

(III) $\Rightarrow$ (II)$'$. By Theorem 1, (III) implies the existence of a function f in $L_0(\ ,\mu)$, with $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$, and a countable decomposition $\{A_n : n \geq 1\}$ of such that $A_n \in \mathcal{I}$ and $f|_{A_n} \in L_p(A_n,\mu)$ for every $n \geq 1$. Thus (II)$'$ follows by setting $A = A_n$ for some $n \geq 1$, as before.

Lastly, suppose $\{T_t\}$ is ergodic and (II)$'$ holds. We will prove that (I) follows. As in the proof of (II) $\Rightarrow$ (III), we find that $\eta(\omega) < \infty$ and hence $\alpha_\infty > 0$ for μ-a.e. $\omega \in A$, so that by Proposition 2 there exists a function f in $L_0(\ ,\mu)$ with $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$. It follows that the function g_p in (59) satisfies $g_p(\omega) < \infty$ for μ-a.e. $\omega \in A$. Since $\widehat{T}_t g_p = g_p$ for all $t \in \mathbf{R}$, the ergodicity of $\{T_t\}$ implies that g_p is a constant real-valued function on , whence we have $|f|^p \in L_1(\ ,\mu)$ by the pointwise ergodic theorem for the flow $\{T_t\}$, and thus (I) follows.

This completes the proof of Theorem 2.

Example. (d) We give here an example of ergodic measure preserving flow $\{T_t\}$ and an additive process $\{F_t\}$ in $L_p(\ ,\mu)$, with $0 < p \cdot\ \infty$, such that the process $\{F_t\}$ has the form $F_t = \widehat{T}_t f - f$ for some $f \in \cap_{r<p} L_r(\ ,\mu)$, but this f cannot be a function in $L_p(\ ,\mu)$. Since $\{T_t\}$ is ergodic, this means that the additive process $\{F_t\} \subset L_p(\ ,\mu)$ induces a $\mu_{\mathbf{R}}$-equivalent finite invariant measasure $\nu = P d\mu_{\mathbf{R}}$ (with respect to the skew-product flow $\{\vartheta_t\}$) such that $0 < P \in L_r(K_{\mathbf{R}},\mu_{\mathbf{R}})$ for every $r < 1 + (p/2)$, but there does not exist a nontrivial finite measure $\widetilde{\nu} = \widetilde{P} d\mu_{\mathbf{R}}$, with $0 \cdot\ \widetilde{P} \in L_{1+(p/2)}(K_{\mathbf{R}},\mu_{\mathbf{R}})$, which is invariant with respect to the flow $\{\vartheta_t\}$. We also note that this skew-product flow $\{\vartheta_t\}$ is not ergodic, because the flow is, at the same time, a measure preserving flow in the σ-finite product measure space $(\ \times \mathbf{R}, \mathcal{A}\ \ \mathcal{B}(\mathbf{R}), \mu\ \ dx)$.

In order to construct such an example, we first consider a single transformation T. It is known (cf. [4]) that if $p = \infty$, then there exists an ergodic invertible measure preserving transformation T in a probability measure space $(\Omega_0, \mathcal{A}_0, \mu_0)$ and a function $f_0 \notin L_\infty(\Omega_0, \mu_0)$, with $f_0 \in L_r(\Omega_0, \mu_0)$ for all $r < \infty$, such that $\widehat{T} f_0 - f_0 \in L_\infty(\Omega_0, \mu_0)$. A similar result holds for every p, with $0 < p < \infty$. That is, there exists an ergodic invertible measure preserving transformation T in a probability measure space $(\Omega_0, \mathcal{A}_0, \mu_0)$ and a function $f_0 \notin L_p(\Omega_0, \mu_0)$, with $f_0 \in L_r(\Omega_0, \mu_0)$ for all $r < p$ $(< \infty)$, such that $\widehat{T} f_0 - f_0 \in L_p(\Omega_0, \mu_0)$. To see this, let $(b_n)_{n=0}^\infty$ be a strictly decreasing sequence of positive real numbers such that $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^\infty b_n = 1$. Putting $\alpha_n = b_{n-1} - b_n$ for $n \ge 1$, we get a sequence $(\alpha_n)_{n=1}^\infty$ of positive real numbers such that

$$\sum_{n=1}^\infty \alpha_n = b_0 \qquad \text{and} \qquad \sum_{n=1}^\infty n\alpha_n = \sum_{n=0}^\infty b_n = 1.$$

By an elementary argument (we may omit here the details) we see that the sequence $(b_n)_{n=0}^\infty$ can be modified so that it has the property that there exist two sequences $(p_n)_{n=1}^\infty$ and $(d_n)_{n=1}^\infty$ of positive real numbers such that $p_n < p$ for $n \ge 1$ and $\lim_{n\to\infty} p_n = p$, and also such that

$$\sum_{n=1}^\infty n\alpha_n \cdot (d_n)^{p_n} < \infty, \quad \sum_{n=1}^\infty \alpha_n \cdot (d_n)^p < \infty \quad \text{and} \quad \sum_{n=1}^\infty n\alpha_n \cdot (d_n)^p = \infty.$$

Putting $A_n = \{(n, x) : x \in [0, b_n)\}$ for $n \ge 0$, we then set

$$\Omega_0 = \bigcup_{n=0}^\infty A_n,$$

and

$$\mu_0(B) = \sum_{n=0}^\infty \int_0^{b_n} \chi_B(n, x)\, dx \qquad \text{for } B \in \mathcal{A}_0,$$

where $\mathcal{A}_0$ is the σ-field of all subsets B of Ω_0 with the property that for each $n \ge 0$, the set $B_n = \{x : (n, x) \in B\}$ is a Lebesgue measurable subset of $[0, b_n)$. Thus, $(\Omega_0, \mathcal{A}_0, \mu_0)$ becomes a probability measure space. Let S be an ergodic invertible measure preserving transformation in the interval $[0, b_0)$ with respect to the Lebesgue measure on $[0, b_0)$. By Kakutani's skyscraper construction (cf. e.g. p. 21 of [12]), we can obtain an ergodic invertible measure preserving transformation T in $(\Omega_0, \mathcal{A}_0, \mu_0)$ as follows. For $(n, x) \in A_n$ we define

$$T(n, x) = \begin{cases} (n+1, x) & \text{if } 0 \le x < b_{n+1}, \\ (0, Sx) & \text{if } b_{n+1} \le x < b_n. \end{cases}$$

We next define a function f_0 on Ω_0 as follows. If $\omega_0 \in \Omega_0$, then $\omega_0 = (n, x) \in A_n$ and $b_k \le x < b_{k-1}$ for some $n \geq 0$ and $k \geq n+1$, unless $x = 0$. Since these n, k are uniquely determined, we can define

$$f_0(\omega_0) = d_k$$

to obtain a function f_0 on Ω_0. It is easy to check that $f_0 \notin L_p(\Omega_0, \mu_0)$, that $f_0 \in L_r(\Omega_0, \mu_0)$ for all $r < p$, and that $\widehat{T} f_0 - f_0 \in L_p(\Omega_0, \mu_0)$. Hence the desired result holds.

Now, using the above T and f_0, we define an ergodic measure preserving flow $\{T_t\}$ in the product probability measure space

$$(\Omega, \mathcal{A}, \mu) = (\Omega_0 \times [0, 1), \mathcal{A} \otimes \mathcal{B}([0, 1)), \mu \otimes dx),$$

where $\mathcal{B}([0, 1))$ is the σ-field of all Borel subsets of $[0, 1)$, by the relation

$$T_t(\omega_0, x) = \left(T^{[t+x]}\omega_0,\, t + x - [t+x]\right) \quad \text{for } (\omega_0, x) \in \Omega \quad \text{and } t \in \mathbf{R},$$

where $[t+x]$ denotes the greatest integer not exceeding $t + x$, and a function f on Ω by the relation

$$f(\omega_0, x) = f_0(\omega_0) \quad \text{for } (\omega_0, x) \in \Omega.$$

Then we find that $f \notin L_p(\Omega, \mu)$ and that $f \in L_r(\Omega, \mu)$ for all $r < p$. Nevertherless, the additive process $F_t = \widehat{T}_t f - f$ $(t \in \mathbf{R})$ satisfies $\{F_t\} \subset L_p(\Omega, \mu)$.

Theorem 3. *Let* $\{F_t\}$ *be an additive process* (*with respect to* $\{T_t\}$). *Suppose* $1 \le p \le \infty$. *Then the following conditions are equivalent*:

(I) *There exists a function f in $L_p(\Omega, \mu)$ such that $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$.*

$(\mathrm{II})_p$ *The inequality* $\liminf_{b\to\infty} (1/b) \int_0^b \|F_t\|_p \, dt < \infty$ *holds.*

Here if $\{T_t\}$ *is assumed to be ergodic, then the condition* $(\mathrm{II})_p$ *can be replaced with the following weaker condition*:

$(\mathrm{II})'_p$ *The inequality* $\liminf_{b\to\infty}(1/b) \int_0^b \|\chi_A \cdot F_t\|_p \, dt < \infty$ *holds for some* $A \in \mathcal{A}$ *with* $\mu(A) > 0$.

Proof. (I) $\Rightarrow$ $(\mathrm{II})_p$. This is obvious.

$(\mathrm{II})_p \Rightarrow$ (I). Since $\|F_t\|_1 \le \|F_t\|_p$, $(\mathrm{II})_p$ implies that

$$\liminf_{b\to\infty} \frac{1}{b} \int_0^b \|F_t\|_1 \, dt < \infty. \tag{60}$$

Thus, by Theorem 2 and Proposition 1, there exists a function g in $L_0(\ ,\mu)$ such that $F_t = \widehat{T}_t g - g$ for all $t \in \mathbf{R}$. Then, using the inequality $|g \circ T_t| \cdot \ |F_t| + |g|$ on , we see from (60), together with the pointwise ergodic theorem for the flow $\{T_t\}$ and Fatou's lemma, that

$$\lim_{b\to\infty} \frac{1}{b}\int_0^b |g|(T_t\omega)\,dt \cdot \ \liminf_{b\to\infty} \frac{1}{b}\int_0^b |F_t(\omega)|\,dt + |g|(\omega) < \infty \tag{61}$$

for μ-a.e. $\omega \in$. Hence

$$\lim_{b\to\infty} \frac{1}{b}\int_0^b F_t(\omega)\,dt = \lim_{b\to\infty} \frac{1}{b}\int_0^b g(T_t\omega)\,dt - g(\omega) \tag{62}$$

for μ-a.e $\omega \in$. Let G denote the almost everywhere limit function on defined by

$$G(\omega) = \lim_{b\to\infty} \frac{1}{b}\int_0^b g(T_t\omega)\,dt \qquad (\omega \in \ ).$$

Then, clearly, it is invariant with respect to the flow $\{T_t\}$, and so if f denotes the almost everywhere limit function on defined by

$$f(\omega) = \lim_{b\to\infty} \frac{-1}{b}\int_0^b F_t(\omega)\,dt \qquad (\omega \in \ ), \tag{63}$$

then $\widehat{T}_t f - f = \widehat{T}_t g - g = F_t$ for all $t \in \mathbf{R}$. Furthermore, by $(\mathrm{II})_p$ and Fatou's lemma,

$$\|f\|_p \cdot \ \liminf_{b\to\infty} \left\| \frac{1}{b}\int_0^b F_t(\omega)\,dt \right\|_{L_p(\ ,\mu)} \cdot \ \liminf_{b\to\infty} \frac{1}{b}\int_0^b \|F_t\|_p\,dt < \infty, \tag{64}$$

so that $f \in L_p(\ ,\mu)$, and hence (I) follows.

Next, suppose $\{T_t\}$ is ergodic and $(\mathrm{II})'_p$ holds. We will prove that (I) follows. To do this, we first notice that there exists a function g in $L_1(\ ,\mu)$ such that $F_t = \widehat{T}_t g - g$ for all $t \in \mathbf{R}$ (cf. the condition $(\mathrm{II})'$ of Theorem 2, with $p = 1$). Here, modifying the set A slightly, if necessary, we may assume without loss of generality that $\chi_A \cdot g \in L_p(\ ,\mu)$. Then, since $\chi_A \cdot (g \circ T_t) = \chi_A \cdot (F_t + g)$, it follows that

$$\begin{aligned} &\frac{1}{b}\int_0^b \|\chi_A \cdot F_t\|_p\,dt + \|\chi_A \cdot g\|_p \geq \frac{1}{b}\int_0^b \|\chi_A \cdot (g\circ T_t)\|_p\,dt \\ &\quad = \frac{1}{b}\int_0^b \|(\chi_A \circ T_{-t})\cdot g\|_p\,dt \geq \left\| \frac{1}{b}\int_0^b \chi_A(T_{-t}\omega)\cdot g(\omega)\,dt \right\|_{L_p(\ ,\mu)}. \end{aligned} \tag{65}$$

Here if we set, for $b > 0$,

$$\gamma_b(\omega) = \inf_{a \geq b} \frac{1}{a} \int_0^a \chi_A(T_{-t}\omega)\, dt \qquad (\omega \in \Omega), \tag{65}$$

then, by the pointwise ergodic theorem for the flow $\{T_t\}$ and the ergodicity of $\{T_t\}$, we see that

$$0 \leq \gamma_b(\omega) \uparrow \mu(A) > 0 \quad \text{for } \mu\text{-a.e. } \omega \in \Omega \quad \text{as } b \to \infty.$$

Thus, by Lebesgue's convergence theorem,

$$\infty > \liminf_{b\to\infty} \frac{1}{b} \int_0^b \|\chi_A \cdot F_t\|_p\, dt + \|\chi_A \cdot g\|_p \geq \lim_{b\to\infty} \|\gamma_b \cdot g\|_p = \mu(A)\|g\|_p, \tag{67}$$

whence $g \in L_p(\Omega, \mu)$, and (I) follows.

This completes the proof of Theorem 3.

Remark 3. If $1 \leq p < \infty$, then, by the inequality $a^p \geq a$ for $a \geq 1$, it is clear that

$$\liminf_{b\to\infty} \frac{1}{b} \int_0^b \|F_t\|_p^p\, dt < \infty \quad \text{implies} \quad \liminf_{b\to\infty} \frac{1}{b} \int_0^b \|F_t\|_p\, dt < \infty;$$

Theorems 2 and 3 show, on the other hand, that the converse implication is also true.

4. Ergodic Properties of $\{F_t\}$ in $L_p(\Omega, \mu)$, with $1 \leq p < \infty$

First of all we prove the following lemma, which is stated in a more general setting than needed.

Lemma 1. *Let $\{F_t\}$ be an additive process (with respect to $\{T_t\}$). If $\{F_t\} \subset L_r(\Omega, \mu)$, where $0 < r \leq \infty$, then we have*

$$M(r) := \sup\{\|F_t\|_r : 0 \leq t \leq 1\} < \infty. \tag{68}$$

Proof. If $0 < r < 1$, then, by Fubini's theorem, the function

$$\phi_r(t) = \int |F_t|^r\, d\mu \qquad (t \in \mathbf{R}) \tag{69}$$

is Lebesgue measurable on $\mathbf{R}$, and since $\{T_t\}$ is a measure preserving flow, it follows that

$$0 \le \phi_r(t+s) = \int |F_t + F_s \circ T_t|^r \, d\mu \le \int (|F_t|^r + |F_s \circ T_t|^r) \, d\mu$$
$$= \phi_r(t) + \phi_r(s) < \infty$$

for $t, s \in \mathbf{R}$. Thus, by Theorem 7.4.1 of [9], ϕ_r is bounded on the interval [0, 1]. This proves (68), when $0 < r < 1$. A similar argument is sufficient to prove (68), when $1 \le r < \infty$. Hence we omit the details. If $r = \infty$, then, since $\|F_t\|_\infty = \lim_{r\to\infty} \|F_t\|_r$, the function

(70) $$\phi_\infty(t) = \|F_t\|_\infty \qquad (t \in \mathbf{R})$$

is also Lebesgue measurable on $\mathbf{R}$, and satisfies

$$0 \le \phi_\infty(t+s) \le \phi_\infty(t) + \phi_\infty(s) < \infty \quad \text{for } t, s \in \mathbf{R}.$$

Thus, as before, we have (68) for $r = \infty$, and this completes the proof.

From now on, we assume that $\{F_t\} \subset L_p(\Omega, \mu)$, where $1 \le p < \infty$, unless the contrary is explained explicitly. Since the function $(\omega, t) \mapsto F_t(\omega)$ on $\Omega \times \mathbf{R}$ is measurable with respect to the product σ-field $\mathcal{A} \otimes \mathcal{B}(\mathbf{R})$ (cf. (13)), it then follows from Lemma III.11.16 of [6] that the mapping $t \mapsto F_t$ becomes a strongly measurable function from $\mathbf{R}$ to $L_p(\Omega, \mu)$, so that it is Bochner integrable over every bounded interval, by Lemma 1. Thus, using the additivity of the process $\{F_t\}$ with respect to the flow $\{T_t\}$, we can deduce, through a standard calculation, the fundamental relation

(71) $$F_t = (I - \widehat{T}_t) \int_0^1 F_s \, ds + \int_0^t \widehat{T}_s F_1 \, ds \quad \text{for all } t \in \mathbf{R},$$

where $\int_0^t \widehat{T}_s F_1 \, ds := -\int_t^0 \widehat{T}_s F_1 \, ds$ if $t < 0$.

By the pointwise and mean ergodic theorems for the measure preserving flow $\{T_t\}$, we then see that the limit

(72) $$f_\infty(\omega) = \lim_{t\to\infty} \frac{1}{t} \int_0^t F_1(T_s\omega) \, ds$$

exists for μ-a.e. $\omega \in \Omega$, the limit function f_∞ is a function in $L_p(\Omega, \mu)$ such that

(73) $$\lim_{t\to\infty} \left\| \frac{1}{t} \int_0^t \widehat{T}_s F_1 \, ds - f_\infty \right\|_p = 0,$$

and f_∞ can be written as $f_\infty = E\{F_1\,|(\ ,\mathcal{I},\mu)\}$.

Let $\widehat{A}_p$ denote the infinitesimal generator of the one-parameter group $\{\widehat{T}_t : t \in \mathbf{R}\}$ in $L_p(\ ,\mu)$. Thus, if $f \in \mathcal{D}(\widehat{A}_p)$, then

$$\widehat{A}_p f = \text{ strong-}\lim_{t\to 0} \frac{\widehat{T}_t f - f}{t} \qquad (\text{in } L_p(\ ,\mu)); \tag{74}$$

and the domain $\mathcal{D}(\widehat{A}_p)$ of $\widehat{A}_p$ is the set of all f in $L_p(\ ,\mu)$ for which the limit of the right-hand side of (74) exists in $L_p(\ ,\mu)$. We denote by $\mathcal{R}(\widehat{A}_p)$ and $\overline{\mathcal{R}(\widehat{A}_p)}$ the range of $\widehat{A}_p$ and the closure of the range of $\widehat{A}_p$ in $L_p(\ ,\mu)$, respectively. The following are known results from semigroup theory and mean ergodic theory (cf. e.g. Chapter VIII of [6]):

(i) If f and g are functions in $L_p(\ ,\mu)$ such that

$$\liminf_{t\to 0+0} \left\| \frac{\widehat{T}_t g - g}{t} - f \right\|_p = 0,$$

then $f = \widehat{A}_p g$.

(ii) $f = \widehat{A}_p g$ if and only if $\int_0^t \widehat{T}_s f\,ds = \widehat{T}_t g - g$ in $L_p(\ ,\mu)$, for all $t > 0$.

(iii) The set $\{\widehat{T}_t f - f : f \in L_p(\ ,\mu), t > 0\}$ is a dense subset of $\mathcal{R}(\widehat{A}_p)$.

(iv) $E\{f\,|(\ ,\mathcal{I},\mu)\} = 0$ in $L_p(\ ,\mu)$ if and only if $f \in \overline{\mathcal{R}(\widehat{A}_p)}$.

It is also known (see [15]) that

(v) $\overline{\mathcal{R}(\widehat{A}_p)} = \mathcal{R}(\widehat{A}_p)$ is equivalent to the validity of the uniform mean ergodic theorem for $\{\widehat{T}_t\}$ in $L_p(\ ,\mu)$, i.e.,

$$\lim_{t\to\infty} \left\| E\{\cdot|(\ ,\mathcal{I},\mu)\} - \frac{1}{t}\int_0^t \widehat{T}_s(\cdot)ds \right\| = 0, \tag{75}$$

where $\|\ \|$ denotes the operator norm in $L_p(\ ,\mu)$.

From these results, together with Theorem 3 in Section 3 and the fundamental relation (71), we can obtain immediately the next theorem; we may omit the details.

Theorem 4. *Let $\{F_t\}$ be an additive process (with respect to $\{T_t\}$). Assume that $\{F_t\} \subset L_p(\ ,\mu)$, where $1 \cdot\ p < \infty$. Then:*

(I) *$F_1 \in \mathcal{R}(\widehat{A}_p)$ is equivalent to the existence of a function f in $L_p(\ ,\mu)$ such that $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$ (which is also equivalent to $\liminf_{t\to\infty} (1/t)\int_0^t \|F_s\|_p\,ds < \infty$).*

(II) $F_1 \in \overline{\mathcal{R}(\widehat{A}_p)} \setminus \mathcal{R}(\widehat{A}_p)$ *is equivalent to*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|F_s\|_p \, ds = \infty \ \ \textit{and} \ \ \lim_{t\to\infty} \frac{1}{t}\|F_t\|_p = 0. \tag{76}$$

(III) $F_1 \notin \overline{\mathcal{R}(\widehat{A}_p)}$ *is equivalent to*

$$\lim_{t\to\infty} \frac{1}{t}\|F_t\|_p = \|E\{F_1 \,|\, (\ \ , \mathcal{I}, \mu)\}\|_p > 0. \tag{77}$$

(IV) $\int_0^1 F_s \, ds \in \mathcal{D}(\widehat{A}_p)$ *is equivalent to the existence of a function* f *in* $L_p(\ \ , \mu)$ *such that* $F_t = \int_0^t \widehat{T}_s f \, ds$ *for all* $t \in \mathbf{R}$, *which is also equivalent to the existence of a function* f *in* $L_p(\ \ , \mu)$ *such that*

$$\liminf_{t\to 0+0} \left\| \frac{1}{t} F_t - f \right\|_p = 0. \tag{78}$$

Corollary (cf. [13]). *Let* $f \in L_p(\ \ , \mu)$, *where* $1 \cdot \ \ p < \infty$. *Then* $f \in \mathcal{R}(\widehat{A}_p)$ *is equivalent to*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \left\| \int_0^s \widehat{T}_u f \, du \right\|_p ds < \infty. \tag{79}$$

In particular, *if* $\{T_t\}$ *is assumed to be ergodic*, *then* $f \in \mathcal{R}(\widehat{A}_p)$ *is equivalent to the existence of a set* A *in* $\mathcal{A}$, *with* $\mu(A) > 0$, *such that*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \left\| \chi_A \cdot \left(\int_0^s \widehat{T}_u f \, du \right) \right\|_p ds < \infty. \tag{80}$$

Remark 4. Let $\{F_t\}$ be an additive process (with respect to $\{T_t\}$). Assume that $\{F_t\} \subset L_p(\ \ , \mu)$, where $1 \cdot \ \ p \cdot \ \ \infty$, and that $F_1 = \widehat{T}_1 g - g$ for some $g \in L_p(\ \ , \mu)$. Then there exists a function f in $L_p(\ \ , \mu)$ such that $F_t = \widehat{T}_t f - f$ for all $t \in \mathbf{R}$. In fact, we have for an integer $n \geq 0$

$$F_{n+s} = F_n + \widehat{T}_n F_s = \left(\widehat{T}_n g - g\right) + \widehat{T}_n F_s,$$

so that Lemma 1 implies that

$$\sup_{0 \ \ s \ \ 1} \|F_{n+s}\|_p \cdot \ \ 2\|g\|_p + \sup_{0 \ \ s \ \ 1} \|F_s\|_p \cdot \ \ 2\|g\|_p + M(p) < \infty,$$

whence Theorem 3 can be applied to obtain the desired conclusion. (Incidentally, we remark that (i) if $\{F_t\} \subset L_0(\ ,\mu)$ and $F_1 = \widehat{T}_1 g - g$ for some $g \in L_0(\ ,\mu)$, then $\{F_t\}$ has the form $F_t = \widehat{T}_t f - f$ $(t \in \mathbf{R})$ for some $f \in L_0(\ ,\mu)$ (cf. Proposition 1); (ii) if $\{F_t\} \subset L_r(\ ,\mu)$, where $0 < r < 1$, and $F_1 = \widehat{T}_1 g - g$ for some $g \in L_r(\ ,\mu)$, then the skew-product flow $\{\vartheta_t\}$ admits a $\mu_{\mathbf{R}}$-equivalent finite invariant measure $\nu = P\, d\mu_{\mathbf{R}}$ such that $0 < P \in L_{1+(r/2)}(\ ,\mu_{\mathbf{R}})$, thus $\{F_t\}$ has the form $F_t = \widehat{T}_t f - f$ $(t \in \mathbf{R})$ for some $f \in L_0(\ ,\mu)$ with $f|_{A_n} \in L_{1+r}(A_n,\mu)$ for every $n \geq 1$, where $\{A_n : n \geq 1\}$ is some countable decomposition of such that $A_n \in \mathcal{I}$ for every $n \geq 1$ (cf. Theorems 2 and 1).)

If $\{F_t\}$ is an additive process in $L_p(\ ,\mu)$, with $1 \cdot\ p < \infty$, then we have

$$\lim_{t\to\infty} \left\| \frac{1}{t} F_t - f_\infty \right\|_p = 0, \tag{81}$$

by (71) and (73). But we cannot expect in general the following pointwise convergence result (cf. the example below):

$$q\text{-}\lim_{t\to\infty} \frac{1}{t} F_t(\omega) = f_\infty(\omega) \quad \text{ for } \mu\text{-a.e. } \omega \in\ . \tag{82}$$

Example. (e) We give a simple example of an ergodic measure preserving flow $\{T_t\}$ and an additive process $\{F_t\} \subset \bigcap_{1\ \ p<\infty} L_p(\ ,\mu)$ such that the limit q-$\lim_{t\to\infty} t^{-1}F_t(\omega)$ fails to exist for μ-a.e. $\omega \in$. For this purpose, let $(\ ,\mathcal{A},\mu)$ and $\{T_t\}$ be the same as in Example (c). Take a nonnegative increasing continuous function f on $[0,\,1)$ such that $\lim_{x\to 1-0} f(x) = \infty$, and also such that $f \in L_p([0,\,1))$ for all p with $1 \cdot\ p < \infty$. Then, define an additive process $\{F_t\} \subset \bigcap_{1\ \ p<\infty} L_p([0,\,1))$ by $F_t = f \circ T_t - f$ for $t \in \mathbf{R}$. It is clear that for every $x \in [0,\,1) =$

$$0 = q\text{-}\liminf_{t\to\infty} \frac{1}{t} F_t(x) < q\text{-}\limsup_{t\to\infty} \frac{1}{t} F_t(x) = \infty.$$

By the above example, studying the a.e. convergence of the averages $(1/t)F_t$ as $t \to \infty$ (or $t \to 0+0$) becomes interesting. We will examine in the rest this a.e. convergence problem.

(A) First, it follows from Kingman [10] (see also [3]) that in order to obtain (82) it suffices to assume the following condition:

The function

$$G^\sharp(\omega) = \sup\{|F_t(\omega) - F_s(\omega)| : t,s \in \mathbf{Q},\ 0 \cdot\ t < s \cdot\ 1\} \qquad (\omega \in\ ) \tag{83}$$

belongs to $L_1(\ ,\mu)$.

The process $\{F_t\} \subset L_p(\Omega, \mu)$, where $1 \le p \le \infty$, is called *linearly bounded* [resp. *bounded*] in $L_p(\Omega, \mu)$ if

$$\sup_{t>0} \frac{1}{t}\|F_t\|_p < \infty \quad \bigl[resp.\ \sup_{t>0} \|F_t\|_p < \infty\bigr].$$

It is immediate that the linear boundedness of $\{F_t\}$ in $L_p(\Omega, \mu)$, where $1 < p \le \infty$, implies the linear boundedness of the process in $L_1(\Omega, \mu)$, and the latter condition implies $G^\sharp \in L_1(\Omega, \mu)$.

Here it is interesting to note that the condition $G^\sharp \in L_1(\Omega, \mu)$ need not imply the a.e. convergence of $(1/t)F_t$ as $t \to 0+0$ (through the set $\mathbf{Q}$). To see this, we give the following

Example. (f) Let $(\Omega, \mathcal{A}, \mu)$ and $\{T_t\}$ be the same as in Example (c). Take a real-valued continuous function f on $[0, 1) = \Omega$ such that for every $x \in [0, 1)$ the limit

$$\lim_{t\to 0+0} \frac{f(x+t) - f(x)}{t}$$

fails to exist (existence of such a function is well-known), and let $F_t = \widehat{T}_t f - f$ for $t \in \mathbf{R}$. Then, since $\|F_t\|_\infty \le 2\|f\|_\infty < \infty$, $G^\sharp$ is a function in $L_\infty(\Omega, \mu)$, and hence $G^\sharp \in L_1(\Omega, \mu)$. On the other hand, the limit

$$q\text{-}\lim_{t\to 0+0} \frac{1}{t} F_t(x) \left(= q\text{-}\lim_{t\to 0+0} \frac{f(x+t) - f(x)}{t}\right)$$

cannot exist for any $x \in [0, 1) = \Omega$.

(B) It follows from [1] (see also [20]), together with the next lemma, that if $\{F_t\}$ is linearly bounded in $L_1(\Omega, \mu)$, then the limit

$$f_0(\omega) = q\text{-}\lim_{t\to 0} \frac{1}{t} F_t(\omega) \tag{84}$$

exists for μ-a.e. $\omega \in \Omega$.

Lemma 2. *Let $\{F_t\}$ be an additive process in $L_0(\Omega, \mu)$ (with respect to $\{T_t\}$). If the two local limits*

$$f_{0+}(\omega) = q\text{-}\lim_{t\to 0+0} \frac{1}{t}F_t(\omega) \quad \textit{and} \quad f_{0-}(\omega) = q\text{-}\lim_{t\to 0-0} \frac{1}{t}F_t(\omega) \tag{85}$$

exist and are finite for μ-a.e. $\omega \in \Omega$, then $f_{0+} = f_{0-}$ on Ω.

Proof of Lemma 2. If $t \neq 0$, then we write

$$f_t = \frac{1}{t}F_t.$$

Suppose $\{t_k\}$ is a sequence of positive rational numbers satisfying $\lim_{k\to\infty} t_k = 0$. Then, since

$$f_{0+}(\omega) = \lim_{k\to\infty} f_{t_k}(\omega) \quad \text{and} \quad f_{0-}(\omega) = \lim_{k\to\infty} f_{-t_k}(\omega)$$

for μ-a.e. $\omega \in \ $ by hypothesis, it suffices to show that f_{-t_k} converges to f_{0+} in probability.

To do this, we first notice that the relation $F_{-t_k} + F_{t_k} \circ T_{-t_k} = 0$ implies

(86) $$f_{-t_k} = f_{t_k} \circ T_{-t_k},$$

so that

(87) $$f_{-t_k} - f_{0+} = \{f_{t_k} \circ T_{-t_k} - f_{t_k}\} + \{f_{t_k} - f_{0+}\} =: \mathrm{I}_k + \mathrm{II}_k\,,$$

and

(88) $$\lim_{k\to\infty} \mathrm{II}_k(\omega) = \lim_{k\to\infty} \{f_{t_k}(\omega) - f_{0+}(\omega)\} = 0$$

for μ-a.e. $\omega \in \ $.

To estimate I_k we use the relations

(89)
$$\begin{aligned} f_{t_k} \circ T_{-t_k} - f_{t_k} &= \{(f_{t_k} - f_{0+}) + f_{0+}\} \circ T_{-t_k} - \{(f_{t_k} - f_{0+}) + f_{0+}\} \\ &= \{f_{t_k} - f_{0+}\} \circ T_{-t_k} - \{f_{t_k} - f_{0+}\} + \{f_{0+} \circ T_{-t_k} - f_{0+}\}, \end{aligned}$$

where (88) implies that

(90) $$\lim_{k\to\infty} \{f_{t_k} - f_{0+}\} \circ T_{-t_k} = 0 \qquad \text{(in probability)},$$

and (9) implies that

(91) $$\lim_{k\to\infty} \{f_{0+} \circ T_{-t_k} - f_{0+}\} = 0 \qquad \text{(in probability)}.$$

Hence, $\lim_{k\to\infty} \mathrm{I}_k = \lim_{k\to\infty}\{f_{t_k} \circ T_{-t_k} - f_{t_k}\} = \lim_{k\to\infty} \{f_{t_k} - f_{0+}\} = 0$ (in probability), and this completes the proof of Lemma 2.

Remark 5. Let $1 < p < \infty$, and suppose $\{F_t\} \subset L_p(\ ,\mu)$. Then, $\{F_t\}$ is linearly bounded in $L_p(\ ,\mu)$ if and only if there exists a function f in $L_p(\ ,\mu)$ such that $F_t = \int_0^t \widehat{T}_s f\,ds$ for all $t \in \mathbf{R}$. In fact, if $\{F_t\}$ is linearly bounded in $L_p(\ ,\mu)$, then, since $L_p(\ ,\mu)$ is a reflexive Banach space, we can choose a sequence $\{t_k\}$ of positive real numbers, with $\lim_{k\to\infty} t_k = 0$, for which there exists a function

$f \in L_p(\ \ , \mu)$ such that $f =$ weak- $\lim_{k\to\infty} (1/t_k)F_{t_k}$ in $L_p(\ \ , \mu)$. By using (71) and (10), we then see that

$$f - F_1 = \text{ weak-}\lim_{k\to\infty} \frac{1}{t_k}(I - \widehat{T}_{t_k}) \int_0^1 F_s\, ds, \tag{92}$$

so that

$$\begin{aligned} \int_0^t \widehat{T}_s f\, ds &= \int_0^t \widehat{T}_s(f - F_1)\, ds + \int_0^t \widehat{T}_s F_1\, ds \\ &= (I - \widehat{T}_t) \int_0^1 F_s\, ds + \int_0^t \widehat{T}_s F_1\, ds = F_t \end{aligned} \tag{93}$$

for all $t > 0$ (and hence for all $t \in \mathbf{R}$). The converse implication is obvious. (This can be shown in a more general setting. See e.g. Theorem 10 of [2].)

It is interesting to note that the condition $1 < p < \infty$ cannot be replaced with $p = 1$, in Remark 5. To see this, we give the following

Example. (g) Let $(\ \ , \mathcal{A}, \mu)$ and $\{T_t\}$ be the same as in Example (c). Take a nonnegative function g in $L_1([0,\ 1))$ with $\|g\|_1 = 1$. If $t \ge 0$, then define a function G_t in $L_1([0,\ 1))$ by

$$G_t(x) = \int_x^{x+t} \widetilde{g}(u)\, du \quad \text{ for } x \in [0,\ 1) = \quad ,$$

where $\widetilde{g}$ denotes the periodic function on $\mathbf{R}$, with period 1, such that $\widetilde{g} = g$ on $[0,\ 1)$. Next, if $t \ge 0$ and $x \in [0,\ 1)$, then let $H_t(x)$ be the number of integers k satisfying $x \cdot\ \ k + 2^{-1} < x + t$. By putting

$$F_t = G_t - H_t \quad \text{if } t \ge 0, \text{ and } \quad F_t = -F_{-t} \circ T_t \quad \text{if } t < 0,$$

we obtain a bounded and linearly bounded additive process $\{F_t\}$ in $L_1([0,\ 1))$. This $\{F_t\}$ cannot have the form $F_t = \int_0^t \widehat{T}_s f\, ds$ for any $f \in L_1(\ \ , \mu)$. In fact, by Theorem 3, the process $\{F_t\}$ has the form $F_t = \widehat{T}_t h - h$ for some $h \in L_1([0,\ 1))$. Hence (or directly), we find

$$\int_{[0,\,1)} F_t(x)\, dx = 0 \quad \text{ for all } t \in \mathbf{R}.$$

Thus, if $\{F_t\}$ had the form $F_t = \int_0^t \widehat{T}_s f\, ds$ for some $f \in L_1([0,\ 1))$, then we must have from (10) that $\int_{[0,\,1)} f(x)\, dx = 0$. But, this is a contradiction, because

$$f(x) = q\text{-}\lim_{t\to 0} \frac{1}{t} F_t(x) = g(x) \quad \text{for } dx\text{-a.e. } x \in [0,\ 1).$$

(C) From now on we will restrict ourselves to considering the case where $\{F_t\} \subset L_1(\ ,\mu)$. The process $\{F_t\}$ is called *positive* if $F_t \geq 0$ for all $t \geq 0$, and *absolutely continuous* if there exists a function f in $L_1(\ ,\mu)$ such that $F_t = \int_0^t \widehat{T}_s f\, ds$ for all $t \in \mathbf{R}$. It is also called *singular* if there exists a positive additive process $\{G_t\}$ in $L_1(\ ,\mu)$ with the properties that $|F_t| \cdot\ \ G_t$ for all $t \geq 0$ and that $0 \cdot\ \ g \in L_1(\ ,\mu)$ and $\int_0^t \widehat{T}_s g\, ds \cdot\ \ G_t$ for all $t \geq 0$ imply $g = 0$. Assume $\{F_t\}$ is linearly bounded in $L_1(\ ,\mu)$, and let $f_0 \in L_1(\ ,\mu)$ be the local limit function in (84). Since $f_0 \in L_1(\ ,\mu)$ by Fatou's lemma, we can define

$$X_t = F_t - \int_0^t \widehat{T}_s f_0\, ds \qquad (t \in \mathbf{R}) \tag{94}$$

to obtain a linearly bounded additive process $\{X_t\}$ in $L_1(\ ,\mu)$ such that

$$q\text{-}\lim_{t\to 0} \frac{1}{t} X_t(\omega) = 0 \quad \text{ for } \mu\text{-a.e. } \omega \in\ \ . \tag{95}$$

It is clear that $X_t = 0$ for all $t \in \mathbf{R}$ if and only if $\{F_t\}$ is absolutely continuous; and the former condition is equivalent to $\liminf_{t\to 0+0} \|t^{-1}X_t\|_1 = 0$ by the fundamental relation (71) and the statements (i) and (ii) over Theorem 4. Furthermore, we notice the following

Remark 6. A necessary and sufficient condition for a linearly bounded additive process $\{F_t\}$ in $L_1(\ ,\mu)$ to be singular is that $f_0 = 0$. For this proof we use Theorem (3.2) of [1], by which $\{F_t\}$ can be written as $F_t = F_t^{(1)} - F_t^{(2)}$, where $\{F_t^{(j)}\}$, $j = 1, 2$, are two positive linearly bounded additive processes in $L_1(\ ,\mu)$. Then, by putting

$$f_0^{(j)}(\omega) = q\text{-}\lim_{t\to 0} \frac{1}{t} F_t^{(j)}(\omega) \qquad (\omega \in\ \ ), \tag{96}$$

and

$$H_t^{(j)} = \int_0^t \widehat{T}_s f_0^{(j)}\, ds \qquad (t \in \mathbf{R}), \tag{97}$$

we obtain two positive linearly bounded additive processes $\{H_t^{(j)}\}$ in $L_1(\ ,\mu)$. Since $f_0^{(1)}(\omega) = \liminf_{n\to\infty} nF_{n^{-1}}^{(1)}(\omega)$ for μ-a.e. $\omega \in\ \ $, if we put $h_n(\omega) = \inf_{m\geq n} mF_{m^{-1}}^{(1)}(\omega)$ for $n \geq 1$, then for every $t \geq 0$

$$\begin{aligned} 0 \cdot\ \ H_t^{(1)} &= \int_0^t \widehat{T}_s f_0^{(1)}\, ds = \lim_{n\to\infty} \int_0^t \widehat{T}_s h_n\, ds \\ &\cdot\ \lim_{n\to\infty} \int_0^t \widehat{T}_s \left(nF_{n^{-1}}^{(1)}\right) ds = \lim_{n\to\infty} \int_0^t n\left(F_{s+n^{-1}}^{(1)} - F_s^{(1)}\right) ds \\ &= \lim_{n\to\infty} n\left(\int_t^{t+n^{-1}} F_s^{(1)}\, ds - \int_0^{n^{-1}} F_s^{(1)}\, ds\right) = F_t^{(1)}, \end{aligned}$$

where the last equality comes from the strong continuity of the function $s \mapsto F_s^{(1)}$ in $L_1(\ \ ,\mu)$, together with the relation

$$\left\| n \int_0^{n^{-1}} F_s^{(1)}\, ds \right\|_1 \cdot\ \ n \int_0^{n^{-1}} \left\| F_s^{(1)} \right\|_1 ds \to 0, \quad \text{as } n \to \infty.$$

Similarly, we find that $0 \cdot\ \ H_t^{(2)} \cdot\ \ F_t^{(2)}$ for $t \geq 0$.

Now, assume the process $\{F_t\}$ is singular. By definition there exists a positive additive process $\{G_t\}$ in $L_1(\ \ ,\mu)$, with $|F_t| \cdot\ \ G_t$ for all $t \geq 0$, such that $0 \cdot\ \ g \in L_1(\ \ ,\mu)$ and $\int_0^t \widehat{T}_s g\, ds \cdot\ \ G_t$ for all $t \geq 0$ imply $g = 0$. Then, from Akcoglu and Krengel's construction of the processes $\{F_t^{(j)}\}$ $(j = 1, 2)$ it follows (cf. [1]) that $0 \cdot\ \ F_t^{(j)} \cdot\ \ G_t$ for all $t > 0$, whence

$$0 \cdot\ \ \int_0^t \widehat{T}_s f_0^{(j)}\, ds = H_t^{(j)} \cdot\ \ F_t^{(j)} \cdot\ \ G_t \quad \text{for all } t \geq 0,$$

and this implies that $f_0^{(j)} = 0$, and hence $f_0 = f_0^{(1)} - f_0^{(2)} = 0$.

Conversely, assume $f_0 = 0$. Then, since $f_0 = f_0^{(1)} - f_0^{(2)} = 0$, it follow that $\{H_t^{(1)}\} = \{H_t^{(2)}\}$, which implies that

$$F_t = (F_t^{(1)} - H_t^{(1)}) - (F_t^{(2} - H_t^{(2)}) \quad \text{for all } t \in \mathbf{R}$$

Furthermore, we see that $\{(F_t^{(1)} - H_t^{(1)}) + (F_t^{(2)} - H_t^{(2)}) : t \in \mathbf{R}\}$ is a positive (linearly bounded) additive process in $L_1(\ \ ,\mu)$ such that $|F_t| \cdot\ \ (F_t^{(1)} - H_t^{(1)}) + (F_t^{(2} - H_t^{(2)})$ for all $t \geq 0$, and also such that

$$\begin{aligned} q\text{-}\lim_{t \to 0} \frac{1}{t} &\ \left\{ \left(F_t^{(1)} - H_t^{(1)}\right)(\omega) + \left(F_t^{(2)} - H_t^{(2)}\right)(\omega) \right\} \\ &= \left(f_0^{(1)}(\omega) - f_0^{(1)}(\omega)\right) + \left(f_0^{(2)}(\omega) - f_0^{(2)}(\omega)\right) = 0 \end{aligned}$$

for μ-a.e. $\omega \in\ \ $. Consequently, we find that the process $\{F_t\}$ is singular.

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