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ON THE GAUSS MAP OF TRANSLATION SURFACES IN MINKOWSKI 3-SPACE

Dae Won Yoon

Abstract. In this article, we study translation surfaces in the 3-dimensional Minkowski space whose Gauss map G satisfies the condition $\Delta G = AG, A \in \mathrm{Mat}(3, \mathbb{R})$, where Δ denotes the Laplacian of the surface with respect to the induced metric and Mat(3, $\mathbb{R}$) the set of 3×3 real matrices, and also obtain the complete classification theorem for those.

1. Introduction

As is well-known, the theory of Gauss map is always one of interesting topics in Euclidean space and pseudo-Euclidean space and it has been investigated from the various viewpoints by many differential geometers [1, 2, 4, 7, 8, 10, 11].

F. Dillen, J. Pas and L. Verstraelen [10] studied surfaces of revolution in Euclidean 3-space $\mathbb{E}^3$ such that its Gauss map G satisfies the condition

$$\Delta G = AG, \qquad A = (a_{ij}) \in \mathrm{Mat}(3, \mathbb{R}), \tag{1.1}$$

where Δ denotes the Laplacian of the surface with respect to the induced metric and Mat$(3, \mathbb{R})$ the set of 3×3 real matrices. On the other hand, C. Baikoussis and D. E. Blair [3] investigated the ruled surfaces in $\mathbb{E}^3$ satisfying the condition (1.1). C. Baikoussis and L. Verstraelen [4, 5, 6] studied the helicoidal surfaces, the translation surfaces and the spiral surfaces in $\mathbb{E}^3$ satisfying the condition (1.1). Also, for the Lorentz version, S. M. Choi [8, 9] completely classified the surfaces of revolution and the ruled surfaces with non-null base curve satisfying the condition (1.1) in Minkowski 3-space $\mathbb{E}^3_1$. Furthermore, L. J. Alías, A. Ferrández, P. Lucas and M. A. Meroño [2] studied the ruled surfaces with null ruling satisfying the condition (1.1) in Minkowski 3-space $\mathbb{E}^3_1$. On the other hand, condition (1.1) is a

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special case of a finite type Gauss map introduced by B. Y. Chen [7]. Recently, Y. H. Kim and the author [13] studied the ruled surfaces with pointwise 1-type Gauss map in $\mathbb{E}^3_1$ and obtained a new characterization of minimal ruled surfaces. In [11], D.-S. Kim, Y. H. Kim and the author obtained the complete classification theorem of ruled surfaces with 1-type Gauss map in Minkowski m-space $\mathbb{E}^m_1$ and also characterized the extended B-scroll with Gauss map.

In this article, we investigate the Lorentz version of the translation surfaces satisfying condition (1.1) and prove the following theorem:

Theorem. *The only translation surfaces in Minkowski* 3*-space* $\mathbb{E}^3_1$ *whose Gauss map satisfies* (1.1) *are the Euclidean plane* $\mathbb{R}^2$*, the Minkowski plane* $\mathbb{R}^2_1$*, the Lorentz circular cylinder* $\mathbb{S}^1_1 \times \mathbb{R}$*, the hyperbolic cylinder* $\mathbb{H}^1 \times \mathbb{R}$ *and the circular cylinder of index* 1, $\mathbb{R}^1_1 \times \mathbb{S}^1$.

To prove this theorem, we use the reasoning first developed by C. Baikoussis and L. Verstraelen in [5], in which they classified translation surfaces satisfying the condition (1.1) in $\mathbb{E}^3$.

For the study of the translation surfaces in Minkowski 3-space $\mathbb{E}^3_1$, I. V. de Woestijne [15] studied minimal translation surfaces, and H. Liu [14] investigated the translation surfaces with constant mean curvature or constant Gauss curvature.

Throughout this paper, we assume that all objects are smooth and all surfaces are pseudo-Riemannian, unless otherwise specified.

2. Preliminaries

An m-dimensional vector space $L = L^m_1$ with scalar product $\langle , \rangle$ of index 1 is called a *Lorentz vector space*. In particular, if $L = \mathbb{E}^m_1$, $m \geq 2$, it is called a *Minkowski* m*-space*. A vector X of L^m_1 is said to be *space-like* if $\langle X, X\rangle > 0$ or $X = 0$, *time-like* if $\langle X, X\rangle < 0$ and *light-like* or *null* if $\langle X, X\rangle = 0$ and $X \neq 0$. A curve in L^m_1 is called *space-like* (*time-like* or *null*, respectively) if its tangent vector is space-like (time-like or null, respectively).

Let $X = (X_i)$ and $Y = (Y_i)$ be the vectors in a 3-dimensional Lorentz vector space L^3_1. Then the scalar product of X and Y is defined by

$$\langle X, Y\rangle = -X_1Y_1 + X_2Y_2 + X_3Y_3, \tag{2.1}$$

which is called *a Lorentz product*. Furthermore, a Lorentz cross product $X \times Y$ is given by

$$X \times Y = (-X_2Y_3 + X_3Y_2, X_3Y_1 - X_1Y_3, X_1Y_2 - X_2Y_1). \tag{2.2}$$

Let M^2 be a pseudo-Riemannian surface in Minkowski 3-space $\mathbb{E}^3_1$. The map $G : M^2 \longrightarrow Q^2(\varepsilon) \subset \mathbb{E}^3_1$ which sends each point of M^2 to the unit normal vector to M^2 at the point is called the *Gauss map* of surface M^2, where ε $(=\pm 1)$ denotes the sign of the vector field G and $Q^2(\varepsilon)$ is a 2-dimensional space form as follows:

$$Q^2(\epsilon) = \begin{cases} \mathbb{S}^2_1(1) = \{X \in \mathbb{E}^3_1 | \langle X, X\rangle = 1\} & \text{if} \quad \epsilon = 1; \\ \mathbb{H}^2(-1) = \{X \in \mathbb{E}^3_1 | \langle X, X\rangle = -1\} & \text{if} \quad \epsilon = -1. \end{cases}$$

$\mathbb{S}^2_1(1)$ is called the de Sitter space, $\mathbb{H}^2(-1)$ the hyperbolic space in $\mathbb{E}^3_1$. It is well-known that in terms of local coordinates $\{x_i\}$ of M^2, the Laplacian can be written as:

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x^i}\left(\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial x^j}\right), \tag{2.3}$$

where $\mathcal{G} = \det(g_{ij})$, $(g^{ij}) = (g_{ij})^{-1}$ and (g_{ij}) are the components of the metric of M^2 with respect to $\{x_i\}$.

3. Translation Surfaces in Minkowski 3-Spaces

Let $x : M^2 \longrightarrow \mathbb{E}^3_1$ be a translation surface in $\mathbb{E}^3_1$. Then M^2 is parametrized by

$$x(u, v) = (u, v, \tilde{f}(u) + \tilde{g}(v)), \tag{3.1}$$

$\tilde{f}$ and $\tilde{g}$ being smooth functions of the variables u and v, respectively, and we have the natural frame $\{x_u, x_v\}$ given by

$$x_u = \frac{\partial x}{\partial u} = (1, 0, f), \quad x_v = \frac{\partial x}{\partial v} = (0, 1, g),$$

where $f = d\tilde{f}/du$, $g = d\tilde{g}/dv$. Accordingly, the induced pseudo-Riemannian metric on M^2 is obtained by $g_{11} = \langle x_u, x_u\rangle = f^2 - 1$, $g_{12} = \langle x_u, x_v\rangle = fg$ and $g_{22} = \langle x_v, x_v\rangle = 1 + g^2$. Since the surface is non-degenerate, $\det(g_{ij}) = f^2 - g^2 - 1 \neq 0$. For later use, we define smooth function ω as:

$$\omega = \|x_u \times x_v\|^2 = \varepsilon\langle x_u \times x_v, x_u \times x_v\rangle = \varepsilon(-f^2 + g^2 + 1), \tag{3.2}$$

where ε denotes the sign of the vector $x_u \times x_v$ in $\mathbb{E}^3_1$. Then the Gauss map G of the surface M^2 is given by

$$G = (G_1, G_2, G_3) = \frac{1}{\|x_u \times x_v\|} x_u \times x_v = \frac{1}{\omega^{\frac{1}{2}}}(f, -g, 1). \tag{3.3}$$

If we make use of (2.3) together with such function ω, the Laplacian Δ of M^2 can be expressed as follows:

$$\Delta = \frac{1}{\omega}\Big\{\varepsilon(1+g^2)\frac{\partial^2}{\partial u^2} + \varepsilon(f^2-1)\frac{\partial^2}{\partial v^2} - 2\varepsilon fg\frac{\partial^2}{\partial u\partial v} + \frac{(f^2-1)g' + (1+g^2)f'}{\omega}\Big(f\frac{\partial}{\partial u} - g\frac{\partial}{\partial v}\Big)\Big\}. \tag{3.4}$$

By a straightforward computation, the Laplacian ΔG of the Gauss map G with the help of (3.3) turns out to be

$$\begin{aligned}
\Delta G_1 =& \frac{1}{\omega^{\frac{7}{2}}}\{(f^2+\varepsilon\omega)^2(4\varepsilon ff'^2+\omega f'') \\
&+\varepsilon f(g^2-\varepsilon\omega)[4g^2g'^2-\varepsilon\omega(g'^2+gg'')] \\
&+4\varepsilon ff'g^2g'(\varepsilon\omega+2f^2) - ff'g'\omega(f^2+\varepsilon\omega)\}, \\
\Delta G_2 =& \frac{1}{\omega^{\frac{7}{2}}}\{\varepsilon g(f^2+\varepsilon\omega)[-4f^2f'^2-\varepsilon\omega(f'^2+ff'')] \\
&+(g^2-\varepsilon\omega)^2(-4\varepsilon gg'^2+\omega g'') \\
&+4\varepsilon f^2f'gg'(\varepsilon\omega-2g^2) - f'gg'\omega(g^2-\varepsilon\omega)\}, \\
\Delta G_3 =& \frac{1}{\omega^{\frac{7}{2}}}\{\varepsilon(f^2+\varepsilon\omega)[3f^2+f'g^2g'+\varepsilon\omega ff''+(\varepsilon\omega+f^2)f'^2] \\
&+\varepsilon(g^2-\varepsilon\omega)[3g^2+f^2f'g'-\varepsilon\omega gg''+(g^2-\varepsilon\omega)g'^2]\}.
\end{aligned} \tag{3.5}$$

Before going into the study of translation surfaces with condition (1.1), let us examine some examples of surfaces in $\mathbb{E}^3_1$ satisfying that condition. They will be parts of our classifications of translation surfaces.

Example 3.1. Euclidean plane $\mathbb{R}^2$, or Minkowski plane $\mathbb{R}^2_1$.

In these cases the Gauss map is a constant normal time-like or space-like vector G, so $\Delta G = 0$. Thus, the Euclidean plane $\mathbb{R}^2$ or the Minkowski plane $\mathbb{R}^2_1$ satisfies (1.1) with $A = 0$.

Example 3.2. Lorentz circular cylinder $\mathbb{S}^1_1 \times \mathbb{R}$.

Let $-x_1^2 + x_3^2 = r^2$, $r > 0$, be the Lorentz circular cylinder. We consider this surface parametrized by $x(u,v) = (x_1 = u, x_2 = v, x_3 = \pm\sqrt{r^2+u^2})$. The Gauss map G is given by $G = (\pm u/r, 0, \sqrt{r^2+u^2}/r)$ and the Laplacian is $\Delta G = (1/r^2)G$. Thus, the Lorentz circular cylinder $\mathbb{S}^1_1 \times \mathbb{R}$ satisfies (1.1) with

$$A = \begin{bmatrix} \frac{1}{r^2} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & \frac{1}{r^2} \end{bmatrix}.$$

Example 3.3. Hyperbolic cylinder $\mathbb{H}^1 \times \mathbb{R}$.

Let $-x_1^2+x_3^2=-r^2$, $r>0$, be the hyperbolic cylinder and consider this surface parametrized by $x(u,v)=(x_1=u,\ x_2=v,\ x_3=\pm\sqrt{u^2-r^2})$. The Gauss map G is $G=(\pm u/r, 0, \sqrt{u^2-r^2}/r)$, and the Laplacian is $\Delta G=-(1/r^2)G$. Thus, the hyperbolic cylinder $\mathbb{H}^1 \times \mathbb{R}$ satisfies (1.1) with

$$A=\begin{bmatrix} -\frac{1}{r^2} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & -\frac{1}{r^2} \end{bmatrix}.$$

Example 3.4. Circular cylinder of index 1, $\mathbb{R}_1^1 \times \mathbb{S}^1$.

Let $x_2^2+x_3^2=r^2$, $r>0$, be the circular cylinder of index 1 and consider this surface parametrized by $x(u,v)=(x_1=u,\ x_2=v,\ x_3=\pm\sqrt{r^2-v^2})$. The Gauss map G is $G=(0,\pm v/r,\sqrt{r^2-v^2}/r)$, and the Laplacian ΔG of the Gauss map G can be expressed as $\Delta G=(1/r^2)G$. Thus, the circular cylinder of index 1, $\mathbb{R}_1^1\times\mathbb{S}$, satisfies (1.1) with

$$A=\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & \frac{1}{r^2} & 0 \\ a_{31} & 0 & \frac{1}{r^2} \end{bmatrix}.$$

4. Proof of the Theorem

We now assume that the surface M^2 satisfies condition (1.1). Then, combining (3.3) and (3.5), we have

$$\begin{aligned} &(f^2+\varepsilon\omega)^2(4\varepsilon f f'^2+\omega f'')+\varepsilon f(g^2-\varepsilon\omega)\{4g^2g'^2-\varepsilon\omega(g'^2+gg'')\} \\ &+4\varepsilon f f' g^2 g'(\varepsilon\omega+2f^2)-\omega(f^2+\varepsilon\omega)ff'g'=\omega^3(a_{11}f-a_{12}g+a_{13}), \end{aligned} \tag{4.1}$$

$$\begin{aligned} &(g^2-\varepsilon\omega)^2(-4\varepsilon g g'^2+\omega g'')-\varepsilon g(f^2+\varepsilon\omega)\{4f^2f'^2+\varepsilon\omega(f'^2+ff'')\} \\ &+4\varepsilon f^2 f' g g'(\varepsilon\omega-2g^2)-\omega(g^2-\varepsilon\omega)f'gg'=\omega^3(a_{21}f-a_{22}g+a_{23}), \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\varepsilon(f^2+\varepsilon\omega)\{3f^2+f'g^2g'+\varepsilon\omega ff''+(\varepsilon\omega+f^2)f'^2\} \\ &+\varepsilon(g^2-\varepsilon\omega)\{3g^2+f^2f'g'-\varepsilon\omega gg''+(g^2-\varepsilon\omega)g'^2\} \\ &=\omega^3(a_{31}f-a_{32}g+a_{33}). \end{aligned} \tag{4.3}$$

Furthermore, (4.3) can be rewritten in the form

$$\begin{aligned} &\omega(f^2+\varepsilon\omega)ff''=-\varepsilon(f^2+\varepsilon\omega)\{3f^2+f'g^2g'+(\varepsilon\omega+f^2)f'^2\} \\ &-\varepsilon(g^2-\varepsilon\omega)\{3g^2+f^2f'g'-\varepsilon\omega gg''+(g^2-\varepsilon\omega)g'^2\} \\ &+\omega^3(a_{31}f-a_{32}g+a_{33}), \end{aligned} \tag{4.4}$$

which implies from (4.1) and (4.4)

$$A_1 f'^2 + B_1 f' = \Gamma_1, \tag{4.5}$$

where we put

(4.6)
$$\begin{aligned} A_1 =& (f^2+\varepsilon\omega)^2(3\varepsilon f^2-\omega), \\ B_1 =& g^2g'\{4\varepsilon f^4-\varepsilon(g^2+1)(-3f^2+g^2+1)\}, \\ \Gamma_1 =& \omega^3 f(a_{11}f-a_{12}g+a_{13})-\omega^3(f^2+\varepsilon\omega)(a_{31}f-a_{32}g+a_{33}) \\ &+3\varepsilon f^2(f^2+\varepsilon\omega)^2+\varepsilon(g^2-\varepsilon\omega)\{(f^2+\varepsilon\omega)[3g^2-\varepsilon\omega gg''+(g^2-\varepsilon\omega)g'^2] \\ &+\varepsilon\omega f^2(g'^2+gg'')-4f^2g^2g'^2\}. \end{aligned}$$

Also, it follows using (4.2) and (4.4) that

$$A_2 f'^2 + B_2 f' = \Gamma_2, \tag{4.7}$$

where we set

$$\begin{aligned} A_2 =& -3\varepsilon f^2 g(f^2+\varepsilon\omega), \\ B_2 =& gg'(\omega(3f^2+\omega)-6\varepsilon f^2g^2), \\ \Gamma_2 =& \omega^3(a_{21}f-a_{22}g+a_{23})+\omega^3 g(a_{31}f-a_{32}g+a_{33}) \\ &+(g^2-\varepsilon\omega)^2(3\varepsilon gg'^2-\omega g'')-3\varepsilon f^2 g(f^2+\varepsilon\omega) \\ &-\varepsilon g^2(g^2-\varepsilon\omega)(3g-\varepsilon\omega g''). \end{aligned} \tag{4.8}$$

In case

$$A_1B_2 - A_2B_1 = 0, \tag{4.9}$$

from (4.6) and (4.8), we see that

$$\begin{aligned} &(f^2+\varepsilon\omega)^2(3\varepsilon f^2-\omega)(3f^2\omega+\omega^2-6\varepsilon f^2g^2)gg' \\ &+3\varepsilon f^2g^3g'(f^2+\varepsilon\omega)\{4\varepsilon f^4-\varepsilon(g^2+1)(-3f^2+g^2+1)\}=0. \end{aligned} \tag{4.10}$$

Thus, the function $f(u)$ satisfies a nontrivial polynomial whose coefficients depend exclusively on the function g and its derivative g'. Consequently, f must be constant. We will consider this situation further in the last step of the proof.
In case

$$A_1B_2 - A_2B_1 \neq 0, \tag{4.11}$$

from (4.5) and (4.9) we have

$$(A_1\Gamma_2 - A_2\Gamma_1)^2 = (A_1B_2 - A_2B_1)(B_2\Gamma_1 - B_1\Gamma_2). \tag{4.12}$$

Substituting (4.6) and (4.8) in (4.12), again we obtain a nontrivial polynomial in f whose coefficients now depend exclusively on the functions g, g' and g''. Hence, f must be constant. Now, we consider the situation that f is constant. If f is identically zero, then $\tilde{f}$ is constant, say, c. Thus, M^2 is a ruled surface in $\mathbb{E}^3_1$ and the position vector x can be written in the following form:

$$x(u,v) = (u, v, c + \tilde{g}(v)) = \alpha(v) + u\beta, \tag{4.13}$$

where $\alpha(v) = (0, v, c + \tilde{g}(v))$ is a space-like curve and $\beta = (1,0,0)$ is a time-like unit constant vector along α orthogonal to it. Consequently, the surface M^2 is locally the Minkowski plane $\mathbb{R}^2_1$ (Example 3.1) or the circular cylinder of index 1, $\mathbb{R}^1_1 \times \mathbb{S}^1$ (Example 3.4) according to Proposition 3.2 of [9]. Lastly, we assume that f is a nonzero constant. From (4.1) and (4.2) we obtain the following equations:

$$\begin{aligned} \varepsilon f(g^2 - \varepsilon\omega)\{4g^2g'^2 - \varepsilon\omega(g'^2 + gg'')\} &= \omega^3 f(a_{11}f - a_{12}g + a_{13}), \\ (g^2 - \varepsilon\omega)^2(\omega g'' - 4\varepsilon g g'^2) &= \omega^3(a_{21}f - a_{22}g + a_{23}). \end{aligned} \tag{4.14}$$

Considering (4.14) as a system of equations in g'^2 and g'', we observe that since $f \neq 0$, its unique solution is

$$\begin{aligned} g'^2 &= -\frac{\omega^2}{f(g^2 - \varepsilon\omega)^2}\{(g^2 - \varepsilon\omega)(a_{11}f - a_{12}g + a_{13}) \\ &\qquad + fg(a_{21}f - a_{22}g + a_{23})\}, \\ g'' &= -\frac{\varepsilon\omega}{f(g^2 - \varepsilon\omega)^2}\{f(4g^2 - \varepsilon\omega)(a_{21}f - a_{22}g + a_{23}) \\ &\qquad + 4g(g^2 - \varepsilon\omega)(a_{11}f - a_{12}g + a_{13})\}. \end{aligned} \tag{4.15}$$

Substituting (4.15) in (4.3) yields a nontrivial polynomial in g with constant coefficients. Hence, g must be constant, which gives $\Delta G = 0$. Consequently, M^2 is a nondegenerate plane, i.e., a Euclidean plane $\mathbb{R}^2$ or a Minkowski plane $\mathbb{R}^2_1$ (Example 3.1).

Now, we come back to relations (4.1), (4.2) and (4.3) and work as above to find from these the function g. Thus, we can rewrite (4.3) in the form

$$\begin{aligned} \omega(g^2 - \varepsilon\omega)gg'' &= \varepsilon(f^2 + \varepsilon\omega)\{3f^2 + f'g^2g' + \varepsilon\omega ff'' + (\varepsilon\omega + f^2)f'^2\} \\ &+ \varepsilon(g^2 - \varepsilon\omega)\{3g^2 + f^2f'g' + (g^2 - \varepsilon\omega)g'^2\} - \omega^3(a_{31}f - a_{32}g + a_{33}). \end{aligned} \tag{4.16}$$

Then, we combine (4.16) and (4.1) to obtain

$$A_3g'^2 + B_3g' = \Gamma_3, \tag{4.17}$$

where

$$A_3 = 3\varepsilon f g^2(g^2 - \varepsilon\omega), \quad B_3 = ff'\{6\varepsilon f^2 g^2 + \omega(3g^2 - \varepsilon\omega)\}, \tag{4.18}$$

and

$$\begin{aligned}\Gamma_3 =& \omega^3(a_{11}f - a_{12}g + a_{13}) - \omega^3 f(a_{31}f - a_{32}g + a_{33}) \\ & -(f^2 + \varepsilon\omega)^2(3\varepsilon f f'^2 + \omega f'') + 3\varepsilon f g^2(g^2 - \varepsilon\omega) \\ & +\varepsilon f^2(f^2 + \varepsilon\omega)(3f + \varepsilon\omega f'').\end{aligned} \tag{4.19}$$

Also, it follows from (4.16) and (4.2) that

$$A_4 g'^2 + B_4 g' = \Gamma_4, \tag{4.20}$$

where we put

$$A_4 = (g^2 - \varepsilon\omega)^2(-3\varepsilon g^2 - \omega), \quad B_4 = f^2 f'(\varepsilon\omega - 2g^2)(3\varepsilon g^2 + \omega), \tag{4.21}$$

and

(4.22)

$$\begin{aligned}\Gamma_4 =& \omega^3(a_{21}f - a_{22}g + a_{23}) + \omega^3(g^2 - \varepsilon\omega)(a_{31}f - a_{32}g + a_{33}) \\ & -3\varepsilon g^2(g^2 - \varepsilon\omega)^2 + \varepsilon(f^2 + \varepsilon\omega)\{(g^2 - \varepsilon\omega)[-3f^2 - \varepsilon\omega f f'' - (f^2 + \varepsilon\omega)f'^2] \\ & +\varepsilon\omega g^2(f'^2 + ff'') + 4f^2 g^2 f'^2\}.\end{aligned}$$

Now, by using (4.17) and (4.20), when

$$A_3 B_4 - A_4 B_3 = 0,$$

from (4.18) and (4.21) we have that

$$\begin{aligned}&(g^2 - \varepsilon\omega)^2(3\varepsilon g^2 + \varepsilon\omega)(3g^2\omega - \varepsilon\omega^2 + 6\varepsilon f^2 g^2) \\ &+3\varepsilon f^3 g^2 f'(g^2 - \varepsilon\omega)[-4\varepsilon g^4 + \varepsilon(f^2 - 1)(-3g^2 + f^2 - 1)] = 0.\end{aligned}$$

Thus, the function g satisfies a nontrivial polynomial whose coefficients depend exclusively on the function f and its derivative f'. Consequently, g must be constant. When

$$A_3 B_4 - A_4 B_3 \neq 0,$$

from (4.17) and (4.20) we have that

$$(A_3\Gamma_4 - A_4\Gamma_3)^2 = (A_3 B_4 - A_4 B_3)(B_4\Gamma_3 - B_3\Gamma_4). \tag{4.23}$$

Inserting (4.18), (4.19), (4.21) and (4.22) in (4.23), again we obtain a nontrivial polynomial in g whose coefficients now depend exclusively on the functions f, f' and f''. Hence, g must be constant.

If g is identically zero, then $\tilde{g}$ is constant, say, c. Thus, in this case M^2 is a ruled surface in $\mathbb{E}^3_1$ and the position vector field x takes the form

$$x(u,v) = (u, v, \tilde{f}(u) + c) = \alpha(u) + v\beta,$$

where $\alpha(u) = (u, 0, \tilde{f}(u) + c)$ is a space-like or time-like curve and $\beta = (0, 1, 0)$ is a space-like unit constant vector along α orthogonal to it. Consequently, the surface M^2 is locally the Euclidean plane $\mathbb{R}^2$, the Minkowski plane $\mathbb{R}^2_1$ (Example 3.1), the hyperbolic cylinder $\mathbb{H}^1 \times \mathbb{R}$ (Example 3.3) or the Lorentz circular cylinder $\mathbb{S}^1_1 \times \mathbb{R}$ (Example 3.2) according to Proposition 3.1 of [9].

Finally, if g is a nonzero constant, we obtain again, as above, that f is constant, and thus M^2 is a nondegenerate plane, i.e., a Euclidean plane $\mathbb{R}^2$ or a Minkowski plane $\mathbb{R}^2_1$ (Example 3.1). This completes the proof. ■

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References

1. R. Aiyama, On the Gauss map of complete space-like hypersurfaces of constant mean curvature in Minkowski space, *Tsukuba J. Math.* **16** (1992), 353-361.
2. L. J. Alías, A. Ferrández, P. Lucas and M. A. Meroño, On the Gauss map of B-scrolls, *Tsukuba J. Math.* **22** (1998), 371-377.
3. C. Baikoussis and D. E. Blair, On the Gauss map of ruled surfaces, *Glasgow Math. J.* **34** (1992), 355-359.
4. C. Baikoussis and L. Verstraelen, On the Gauss map of helicoidal surfaces, *Rend. Sem. Mat. Messina Ser. II* **16** (1993), 31-42.
5. C. Baikoussis and L. Verstraelen, On the Gauss map of translation surfaces, *Rend. Sem. Mat. Messina Ser II* (in press).
6. C. Baikoussis and L. Verstraelen, The Chen-type of the spiral surfaces, *Results Math.* **28** (1995), 214-223.
7. B. Y. Chen, A report on submanifolds of finite type, *Soochow J. Math.* **22** (1996), 117-337.
8. S. M. Choi, On the Gauss map of surfaces of revolution in a 3-dimensional Minkowski space, *Tsukuba J.Math.* **19** (1995), 351-367.

9. S. M. Choi, On the Gauss map of ruled surfaces in a 3-dimensional Minkowski space, *Tsukuba J. Math.* **19** (1995), 285-304.

10. F. Dillen, J. Pas and L. Verstralen, On the Gauss map of surfaces of revolution, *Bull. Inst. Math. Acad. Sinica* **18** (1990), 239-246.

11. D.-S. Kim, Y. H. Kim and D. W. Yoon, Extended B-scrolls and their Gauss map, to appear Indian J. Pure Appl. Math.

12. Y. H. Kim and D. W. Yoon, Ruled surfaces with finite type Gauss map in Minkowski spaces, *Soochow J. Math.* **26** (2000), 85-96.

13. Y. H. Kim and D. W. Yoon, Ruled surfaces with pointwise 1-type Gauss map, *J. Geom. Phys.* **34** (2000), 191-205.

14. H. Liu, Translation surfaces with constant mean curvature in 3-dimensional spaces, *J. Geom.* **64** (1999), 141-149.

15. I. V. de Woestijne, Minimal surfaces of 3-dimensional Minkowski space, in: *Geometry and Topology of Submanifolds II*, World Scientific Publ., 1990, pp. 344-369.

Dae Won Yoon
Department of Mathematics, Teachers College,
Kyungpook National University,
Taegu 702-701, Rep. of Korea
E-mail: dwyoon@gauss.knu.ac.kr