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A NOTE ON SIX-DIMENSIONAL G1-SUBMANIFOLDS OF THE OCTAVE ALGEBRA

M. Banaru

Abstract. It is proved that six-dimensional $G1$-submanifolds of the octave algebra are $W_1 \oplus W_3$ - manifolds.

Dedicated to Professor Vadim F. Kirichenko on his 55-th birthday

1. Introduction

The classification of the almost Hermitian structures on first-order differential-geometrical invariants can be rightfully attributed to the most significant results obtained by the outstanding American mathematician Alfred Gray in collaboration with his Spanish colleague Luis M. Hervella. According to this classification, all the almost Hermitian structures are divided into 16 classes. Analytical criteria for every concrete structure to belong to one or another class have been obtained [7].

One of the least studied Gray-Hervella classes is the $G1$ (or $W_1 \oplus W_3 \oplus W_4$) class. Almost the only significant work devoted to the $G1$-manifolds is the original and interesting article by L. M. Hervella and E. Vidal [8]. This situation is at least strange because the $G1$-manifolds class includes the Kählerian (K), the nearly-Kählerian (NK), the Hermitian (H), the special Hermitian (SH) and the locally conformal Kählerian (LCK) manifolds as well as the Vaisman-Gray manifolds. And what is more, the $G1$-manifolds class forms a minimal class that includes two of the most interesting classes of almost Hermitian manifolds – the nearly-Kählerian and Hermitian manifolds classes, to the study of which a great number of publications have been devoted. Without going into details on such an extensive subject, we can mark out the classical works [5, 6, 9, 10, 11, 13].

In the present article, we give some results obtained in this direction by using the Cartan structural equations of the octave algebra submanifold.

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2. Preliminaries

Let $\mathbf{O} \equiv \mathbb{R}^{\mathbf{8}}$ be the Cayley algebra. As is well-known [5], two nonisomorphic 3-vector cross products are defined on it by

$$P_1(X,Y,Z) = -X(\overline{Y}Z) + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$
$$P_2(X,Y,Z) = -(X\overline{Y})Z + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

where $X, Y, Z \in \mathbf{O}$, $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbf{O}$, $X \mapsto \overline{X}$ is the operator of conjugation [4]. Moreover, any other 3-vector cross product in the octave algebra is isomorphic to one of the above-mentioned [5].

If $M^6 \subset \mathbf{O}$ is a six-dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_\alpha, g = \langle\cdot,\cdot\rangle\}$ is determined by the relation

$$J_\alpha(X) = P_\alpha(X, e_1, e_2), \quad \alpha = 1, 2,$$

where $\{e_1, e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at a point p, $X \in T_p(M^6)$. The submanifold $M^6 \subset \mathbf{O}$ is called a G1-submanifold if for arbitrary vector fields $X, Y \in \aleph(M^6)$ the following condition is fulfilled:

$$\nabla_x(F)(X,Y) - \nabla_{JX}(F)(JX,Y) = 0. \tag{1}$$

Here $F(X,Y) = \langle X, JY\rangle$ is the fundamental (or Kählerian [11]) form of M^6, and ∇ is the Riemannian connection of the metric. We note that A. Gray and L. M. Hervella [7] have proved that condition (1) is equivalent to the following relation imposed on the Nijenhuis tensor S:

$$\langle S(X,Y), X\rangle = 0.$$

We recall that the point $p \in M^6$ is called general [6] if

$$e_0 \notin T_p(M^6) \ \text{ and } \ T_p(M^6) \subseteq L(e_0)^\perp,$$

where e_0 is the unit of Cayley algebra and $L(e_0)^\perp$ is its orthogonal supplement. A submanifold $M^6 \subset O$ consisting only of general points is called a general-type submanifold. In what follows all the considered M^6 are meant as general-type submanifolds.

3. The Main Results

We use the Cartan structural equations of an almost Hermitian $M^6 \subset \mathbf{O}$ obtained in [10]:

$$d\omega^a = \omega^a_b \wedge^b + \frac{1}{\sqrt{2}}\varepsilon^{ah[b}D_h{}^{c]}\omega_b \wedge \omega_c + \frac{1}{\sqrt{2}}\varepsilon^{abh}D_{hc}\omega^c \wedge \omega_b;$$
$$d\omega_a = -\omega^b_a \wedge \omega_b + \frac{1}{\sqrt{2}}\varepsilon_{ah[b}D^h{}_{c]}\omega^b \wedge \omega^c + \frac{1}{\sqrt{2}}\varepsilon_{abh}D^{hc}\omega_c \wedge \omega^b;$$
$$d\omega^a_b = \omega^a_c \wedge \omega^c_b - \left(\frac{1}{2}\delta^{ah}_{bg}D_{h[k}D^g{}_{j]} + \sum_\varphi T^\varphi_{\hat{a}[k}T^\varphi_{j]b}\right)\omega^k \wedge \omega^j.$$

Here $\varepsilon_{abc} = \varepsilon^{123}_{abc}$, $\varepsilon^{abc} = \varepsilon^{abc}_{123}$ are components of the Kronecker tensor of the third order [12], $\delta^{ah}_{bg} = \delta^a_b\delta^h_g - \delta^a_g\delta^h_b$,

$$D^{ch} = D_{\hat{c}\hat{h}}, \quad D_h{}^c = D_{h\hat{c}}, \quad D^h{}_c = D_{\hat{h}c},$$
$$D_{cj} = \pm T^8_{cj} + iT^7_{cj}, \quad D_{\hat{c}j} = \pm T^8_{\hat{c}j} - iT^7_{\hat{c}j},$$

where T^φ_{kj} are components of the configuration tensor (in A. Gray's notation [5], or of the Euler curvature tensor [3]), $i = \sqrt{-1}$. Here and further: $a, b, c, d, g, h = 1, 2, 3$; $\hat{a} = a + 3$; $\varphi = 7, 8$; $k, j = 1, 2, 3, 4, 5, 6$.

The Gray-Hervella condition (1) on almost Hermitian structures belonging to the $G1$ $(= W_1 \oplus W_3 \oplus W_4)$ class is equivalent to the skew-symmetry of the Kirichenko structure tensor on the first pair of indexes [1]:

$$B^{abc} = -B^{bac}, \qquad B_{abc} = -B_{bac}.$$

We use the Kirichenko tensor relations for six-dimensional almost Hermitian submanifolds of Cayley algebra [1, 2]:

$$B^{abc} = \frac{1}{\sqrt{2}}\varepsilon^{ah[b}D_h{}^{c]}, \; B_{abc} = \frac{1}{\sqrt{2}}\varepsilon_{ah[b}D^h{}_{c]}.$$

Hence, the following equalities are a criterion for an arbitrary six-dimensional submanifold of Cayley algebra to belong to the $G1$-class:

$$\varepsilon^{ah[b}D_h{}^{c]} = \varepsilon^{bh[a}D_h{}^{c]}, \qquad \varepsilon_{ah[b}D^h{}_{c]} = \varepsilon_{bh[a}D^h{}_{c]}. \tag{2}$$

Lemma. *Condition* (2) *is equivalent to the fact that the matrix* $(D_{d\hat{a}})$ *is scalar*, *i.e.*,

$$D_{d\hat{a}} = \frac{1}{3}\,\delta^a_d\; tr\; D_{d\hat{a}}.$$

Proof. First we note that each equality from (2) can be obtained by the complex conjugation of the other.

1. Let (2) be fulfilled. Then

$$\varepsilon^{ah[b}D_h{}^{c]} = \varepsilon^{bh[a}D_h{}^{c]} \Leftrightarrow$$
$$\varepsilon^{ah[b}D_h{}^{c]} - \varepsilon^{bh[a}D_h{}^{c]} = 0 \Leftrightarrow$$
$$\varepsilon^{ahb}D_h{}^c - \varepsilon^{ahc}D_h{}^b + \varepsilon^{bha}D_h{}^c - \varepsilon^{bhc}D_h{}^a = 0 \Leftrightarrow$$
$$\varepsilon^{ahc}D_{h\hat{b}} + \varepsilon^{bhc}D_{h\hat{a}} = 0.$$

We reduce this equality (ε_{cfd}):

$$\delta^{ah}_{fd}D_{h\hat{b}} + \delta^{bh}_{fd}D_{h\hat{a}} = 0 \Leftrightarrow$$
$$(\delta^a_f\delta^h_d - \delta^a_d\delta^h_f)D_{h\hat{b}} + (\delta^b_f\delta^h_d - \delta^b_d\delta^h_f)D_{h\hat{a}} = 0 \Leftrightarrow$$
$$\delta^a_f D_{d\hat{b}} - \delta^a_d D_{f\hat{b}} + \delta^b_f D_{d\hat{a}} - \delta^b_d D_{f\hat{a}} = 0,$$
$$(bf) : D_{d\hat{a}} - \delta^a_d \; tr \; D_{d\hat{a}} + 3D_{d\hat{a}} - D_{d\hat{a}} = 0 \Leftrightarrow$$
$$D_{d\hat{a}} = \frac{1}{3}\,\delta^a_d \; tr \; D_{d\hat{a}}.$$

Thus, (2) $\Rightarrow D_{d\hat{a}} = \frac{1}{3}\delta^a_d \operatorname{tr} D_{d\hat{a}}$.

2. We prove the converse statement. Let $D_{d\hat{a}} = \frac{1}{3}\delta^a_d \operatorname{tr} D_{d\hat{a}}$. Then

$$\begin{aligned}\varepsilon^{ah[b}D_h{}^{c]} - \varepsilon^{bh[a}D_h{}^{c]} &= \varepsilon^{ahc}D_{h\hat{b}} + \varepsilon^{bhc}D_{h\hat{a}} \\ &= \frac{1}{3}(\varepsilon^{ahc}\delta^b_h \operatorname{tr} D_{d\hat{a}} + \varepsilon^{bhc}\delta^a_h \operatorname{tr} D_{d\hat{a}}) \\ &= \frac{1}{3}(\varepsilon^{abc} \operatorname{tr} D_{d\hat{a}} + \varepsilon^{bac} \operatorname{tr} D_{d\hat{a}}) \\ &= \frac{1}{3} \operatorname{tr} D_{d\hat{a}}(\varepsilon^{abc} + \varepsilon^{bac}) \\ &= \frac{1}{3} \operatorname{tr} D_{d\hat{a}}(\varepsilon^{abc} - \varepsilon^{abc}) = 0.\end{aligned}$$

Taking into account the remark at the beginning of the proof, we come to the conclusion:

$$D_{d\hat{a}} = \frac{1}{3}\delta^a_d \operatorname{tr} D_{d\hat{a}} \Rightarrow (2).$$

As a result we have that conditions (2) and $D_{d\hat{a}} = \frac{1}{3}\delta^a_d \operatorname{tr} D_{d\hat{a}}$ are equivalent. ■

It is interesting to note that the condition for the matrix $(D_{d\hat{a}})$ to be scalar is a criterion for an almost Hermitian submanifold of Cayley algebra to belong to the $W_1 \oplus W_3$-manifolds class [1, 2].

Thus is proved

Theorem. *Every six-dimensional $G1$-submanifold of Cayley algebra is a manifold of the $W_1 \oplus W_3$-class.*

As the almost Hermitian structures of the $W_1 \oplus W_3$-class are semi-Kählerian (SK-structures) [1, 7], we get an important.

Corollary 1. *Every six-dimensional $G1$-submanifold of Cayley algebra is a semi-Kählerian manifold.*

We remark that the Hermitian (H, or $W_3 \oplus W_4$) structures on the six-dimensional submanifolds of Cayley algebra can be represented only by the structures belonging to the $H \cap SK$-class:

$$H = W_3 \oplus W_4, \quad SK = W_1 \oplus W_2 \oplus W_3,$$
$$H \cap SK = W_3.$$

Corollary 2. *Every six-dimensional Hermitian submanifold of Cayley algebra is a special Hermitian (W_3) manifold.*

Similar considerations will lead us to the correctness of two other statements.

Corollary 3. *Every six-dimensional Vaisman-Gray submanifold of Cayley algebra is a nearly-Kählerian manifold.*

Corollary 4. *Every six-dimensional locally conformal Kählerian submanifold of Cayley algebra is a Kählerian manifold.*

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M. Banaru
Smolensk University of Humanities
Gertsen str., 2
Smolensk, 214014
Russia