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ON AVERAGE CONVERGENCE OF THE ITERATIVE PROJECTION METHODS

Ya. I. Alber

Abstract. We study the iterative subgradient methods for nonsmooth convex constrained optimization problems in a uniformly convex and uniformly smooth Banach space, followed by metric and generalized projections onto the feasible sets. The normalized stepsizes α_n are chosen *apriori*, satisfying the conditions $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$. We prove that the every sequence generated in this way is weakly convergent to a minimizer in the average if the problem has solutions. In addition, we show that the perturbed ϵ_n-subgradient method is stable when $\epsilon_n \to 0$. More general case of variational inequalities with monotone (possibly) nonpotential operators is also considered.

1. Introduction

We investigate the following optimization problem:

$$f(x) \to \min, \tag{1.1}$$

$$\text{s.t. } x \in \quad , \tag{1.2}$$

where $f(x) : B \to \mathbb{R}$ is a convex in general, nondifferential functional, and is a nonempty convex closed subset of Banach space B. Denote $\partial f(x)$ the subdifferential of $f(x)$ at $x \in B$, that is,

$$\partial f(x) = \{u \in B^* : f(y) - f(x) \geq \langle u, y - x\rangle \quad \text{for all } y \in B.\}$$

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and $u \in \partial f(x)$ an arbitrary subgradient of $f(x)$ at $x \in B$. Here, $\langle \phi, x \rangle$ denotes the dual product (the bilinear functional of duality) between $x \in B$ and $\phi \in B^*$, where B^* is the dual space of B.

Let $\delta_B(\epsilon)$ and $\rho_B(\tau)$ be the modulus of convexity and modulus of smoothness of the Banach space B, respectively (see, for instance, [2, 15]). Suppose that

(i) B is a uniformly convex and uniformly smooth (reflexive) Banach space;

(ii) $\partial f(x)$ is a bounded operator from B to B^*, i.e. it carries bounded sets from B into bounded sets of B^*;

(iii) the set M of solutions of the problem (1.1) and (1.2) is not empty.

Let us note that B^* is also a uniformly convex and uniformly smooth (reflexive) Banach space [12].

In this paper, the normalized iterative method

$$x^{n+1} = \pi \left[Jx^n - \alpha_n \frac{u^n}{\|u^n\|_*} \right], \quad u^n \in \partial f(x^n), \;\; n = 1, 2, \dots , \tag{1.3}$$

is studied, where π is *a generalized* projection operator, J is a duality mapping (see Section 2) and the stepsizes $\{\alpha_n\}$ are chosen according to the rule of divergent series:

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \;\; \alpha_n > 0, \;\; \alpha_n \to 0. \tag{1.4}$$

In Hilbert spaces, this method is transformed into:

$$x^{n+1} = P \left[x^n - \alpha_n \frac{u^n}{\|u^n\|_H} \right], \quad u^n \in \partial f(x^n), \;\; n = 1, 2, ..., \tag{1.5}$$

where P is a metric projection operator.

A convergence nature and asymptotical behaviour of the approximations $\{x^n\}$ are defined, basically, by a structure of the functional $f(x)$. Its smoothness has an influence, mainly, on the estimates of the convergence rate and on the chose rule of the stepsizes α_n. So, if the functional $f(x)$ is uniformly convex, i.e.,

$$\langle \partial f(x) - \partial f(y), x - y \rangle \geq \psi(\|x - y\|_B), \;\; \forall x, y \in B,$$

where $\psi(t)$ is a continuous positive function and $\psi(0) = 0$, then M is a singleton, i.e., $M = \{x^*\}$, and any sequence of iterates generated by (1.3), (1.4) and started from any initial point $x^1 \in B$ converges *strongly* to $\{x^*\}$ [2].

In the more general situation of arbitrary convex functionals $f(x)$, when the inequality

$$\langle \partial f(x) - \partial f(y), x - y \rangle \geq 0, \quad \forall x, y \in B,$$

takes place, M is not necessarily a singleton and the strong convergence of $\{x^n\}$ can not be ensured. However, weak convergence is supported by the following important inequality:

$$\langle \partial f(x), x - x^* \rangle \geq f(x) - f^*, \ \ \forall x \in B, \ \ \forall x^* \in M.$$

First the weak convergence in Hilbert space (= strong convergence in $\mathbb{R}^n$) of the sequence $\{x^n\}$ generated by

$$x^{n+1} = x^n - \alpha_n f'(x^n), \ \ a_1 \cdot \ \ \alpha_n \cdot \ \ a_2, \ \ a_1, a_2 = \text{const.}, \ \ n = 1, 2, ..., \tag{1.6}$$

has been proved in [3] for the convex functionals of the class $C^{1,1}$ (the gradient $f'(x)$ of the functional $f(x)$ satisfies the Lipschitz condition). For the functionals of the class $C^{1,\mu}, \ 0 < \mu < 1$ (the gradient $f'(x)$ satisfies the Hölder condition), it was considered the modification of (1.6) in the form

$$\begin{aligned} x^{n+1} =& x^n - \alpha_n \|f'(x^n)\|^{\mu/(\mu+1)} f'(x^n), \ 0 < a_1 \cdot \ \ \alpha_n \cdot \ \ a_2, \\ & a_1, a_2 = \text{const.}, \ n = 1, 2, ... \ . \end{aligned}$$

Recently in [6] and [7], weak convergence of the iterations (1.3), (1.4) was investigated for nonsmooth convex functionals in Hilbert spaces and Banach spaces with modulus of convexity $\delta_B(\epsilon) \geq D\epsilon^2$, $D =$ const. > 0. In addition to divergence condition of the series (1.4), we suggested that

$$\sum_{n=0}^{\infty} \alpha_n^2 < \infty.$$

Besides, we have obtained in [7] the similar result in arbitrary uniformly convex and uniformly smooth Banach spaces for the Cesàro averages $\{v^m\}$ of $\{x^n\}$ which are defined by the formula

$$v^m := \left(\sum_{n=0}^{m} \alpha_n \right)^{-1} \sum_{n=0}^{m} \alpha_n x^n,$$

under the condition

$$\sum_{n=0}^{\infty} \rho_{B^*}(\alpha_n) < \infty. \tag{1.7}$$

The aim of this paper is to get rid of the condition (1.7), i.e., to consider a convergence of the Cesàro average for the method (1.3), (1.4). We also establish convergence of this kind for the iterative method with the metric projection P in a Banach space. This algorithm is of the following form: having x^n, a current approximation to the solution of (1.1), (1.2), compute

$$(1.8) \quad x^{n+1} = J^* y^n, \ y^n = Jx^n - \alpha_n \left(\frac{\bar{u}^n}{\|\bar{u}^n\|_{B^*}} + 2\frac{J(x^n - \bar{x}^n)}{\|x^n - \bar{x}^n\|_B} \right), \ n = 1, 2, ...,$$

where

$$\bar{x}^n = P \ [x^n], \quad \bar{u}^n \in \partial f(\bar{x}^n),$$

and J is a normalized duality mapping. If B is the Hilbert space H, then

$$x^{n+1} = x^n - \alpha_n \left(\frac{\bar{u}^n}{\|\bar{u}^n\|_H} + 2\frac{x^n - \bar{x}^n}{\|x^n - \bar{x}^n\|_H} \right), \quad n = 1, 2, ...,$$

if $\quad = H$, then

$$x^{n+1} = x^n - \alpha_n \frac{u^n}{\|u^n\|_H}, \quad u^n \in \partial f(x^n), \quad n = 1, 2, ... \, .$$

We also study a stability of the weak convergence for the iterations (1.3) and (1.8) under perturbations of the functional and its subgradient. The methods constructed in this paper are applied not only to the problems with potential (subgradient) maps: more general case of variational inequalities with arbitrary nonsmooth monotone operators is investigated in Section 4.

2. Metric and Generalized Projections and Their Properties

Let $J : B \to B^*$ be the normalized duality mapping determined by the equalities:

$$\langle Jx, x \rangle = \|Jx\|_* \|x\| = \|x\|^2,$$

where $\|\cdot\|$ and $\|\cdot\|_*$ denote norms in B and B^*, respectively. Similarly, J^* will denote the normalized duality mapping from B^* to B. From (i) it follows that $J^* = J^{-1}$, where J^{-1} is the inverse operator to J. Therefore, $JJ^* = I_{B^*}$ and $J^*J = I_B$. In a Hilbert space H, J is the identity operator, i.e., $J = I_H$. Other properties of the mapping J are summarized in [2]. In particular, the following statement holds:

Theorem 2.1. *For all $x \in B$ and $\xi \in B$,*

$$2C^2 \delta_B(\|x - \xi\|/2C) \cdot \quad \langle Jx - J\xi, x - \xi \rangle \cdot \quad 2C^2 \rho_B(4\|x - \xi\|/C),$$

where

$$C = \sqrt{(\|x\|^2 + \|\xi\|^2)/2}.$$

If $\|x\| \cdot \; R$ *and* $\|\xi\| \cdot \; R$, *then*

$$(2L)^{-1}R^2\delta_B(\|x-\xi\|/2R)\cdot \;\; \langle Jx - J\xi, x-\xi\rangle \qquad (2.1)$$
$$\cdot \;\; 2LR^2\rho_B(4\|x-\xi\|/R),$$

where $1 < L < 3.18$ *is the constant from the Figiel inequalities* [13].

This is an analytical formulation of the well-known fact that a normalized duality mapping is a uniformly monotone (resp. uniformly continuous) operator on each bounded set in a uniformly convex (resp. uniformly smooth) Banach space.

The construction of the generalized projection operators π in Banach spaces was introduced in [2] by analog to metric projections in Hilbert space.

Definition 2.2. *The operator* $P \; : B \to \;\; \subset B$ *is called a metric projection operator if it assigns to each* $x \in B$ *its nearest point* $\bar{x} \in$, *i.e., the solution* $\bar{x}$ *for the minimization problem*

$$\|x - \bar{x}\| = \inf_{\xi\in} \|x-\xi\|. \qquad (2.2)$$

In our conditions, the metric projection operator is well-defined, i.e., there exists a unique *projection* $\bar{x}$ for each $x \in B$ called the best approximation.

It is obvious that

$$P \; = I, \quad \text{if} \quad \; = H.$$

The following properties make metric projection operator P essentially effective in Hilbert spaces.

a) P is fixed at each point ξ, i.e., $P \; \xi = \xi$.

b) P is monotone (accretive) in H, i.e.,

$$(\bar{x} - \bar{y}, x - y) \geq 0.$$

c) The point $\bar{x}$ is the metric projection of x on $\;\; \subset H$ if and only if the following inequality is satisfied:

$$(x - \bar{x}, \bar{x} - \xi) \geq 0, \qquad \forall \xi \in \;\; .$$

We call the property c) *the basic variational principle* for P in H.

d) The operator P produces an absolutely best approximation of each $x \in H$ relative to the functional $V_1(x,\xi) = \|x-\xi\|_H^2$, in the sense that

$$\|\bar{x} - \xi\|_H^2 \cdot \;\; \|x-\xi\|_H^2 - \|x - \bar{x}\|_H^2, \qquad \forall \xi \in \;\; .$$

Consequently, P is *the conditionally nonexpansive* operator relative to the functional $V_1(x,\xi) = \|x-\xi\|^2$ in Hilbert space, i.e.,

e) $\|\bar{x}-\xi\|_H \cdot \ \ \|x-\xi\|_H$.

In reality, the metric projection operator P in Hilbert (and only in Hilbert) spaces has a stronger property of nonexpansiveness:

$$\|\bar{x}-\bar{y}\|_H \cdot \ \ \|x-y\|_H, \qquad \forall x,y \in H.$$

It is important to emphasize that the metric projection operator P has no the properties b), d) and e) in Banach spaces.

The minimization problem (2.2) is equivalent to

$$\|x-\bar{x}\|^2 = \inf_{\xi\in} V_1(x,\xi), \ \ V_1(x,\xi) = \|x-\xi\|^2.$$

Now we notice that $V_1(x,\xi)$ can be considered not only as the square of the distance between points x and ξ but also as the Lyapunov functional with respect to ξ with fixed x. Therefore, we can rewrite (2.2) in the form

$$P\ x = \bar{x}; \quad \bar{x} : V_1(x,\bar{x}) = \inf_{\xi\in} V_1(x,\xi).$$

In Hilbert (and only in Hilbert) spaces,

$$V_1(x,\xi) = \|x\|_H^2 - 2(x,\xi) + \|\xi\|_H^2.$$

We have shown in [2] that one can construct similar functionals in Banach spaces using the Young-Fenchel transformation.

Let $f(\xi) : B \to R$ be a given functional and $\varphi \in B^*$. Recall that the Young-Fenchel transformation is defined by the relation

$$f^*(\varphi) = \sup_{\xi\in B}\{\langle\varphi,\xi\rangle - f(\xi)\}.$$

Under that the functional $f^*(\varphi)$ is called conjugate to $f(\xi)$. Obviously,

$$V^f(\varphi,\xi) = f^*(\varphi) - \langle\varphi,\xi\rangle + f(\xi) \geq 0$$

The functional $V^f(\varphi,\xi) : B^* \times B \to \mathbb{R}^+$ is nonstandard because it is defined on both the elements ξ from the primary space B and elements φ from the dual space B^*.

Let us introduce the functional $V(\varphi,\xi) : B^* \times B \to \mathbb{R}$ by the formula (see [1, 2]):

$$V(\varphi,\xi) = \|\varphi\|_*^2 - 2\langle\varphi,\xi\rangle + \|\xi\|^2. \tag{2.3}$$

It is easy to see that

$$V(\varphi,\xi) \geq (\|\varphi\|_* - \|\xi\|)^2 \geq 0,$$

i.e., the functional $V(\varphi,\xi) : B^* \times B \to \mathbb{R}^+$ is nonnegative. Moreover, setting $\xi = J^*\varphi$ and using the definition of normalized duality mapping, we obtain the equality

$$\|\varphi\|_*^2 - 2\langle\varphi, J^*\varphi\rangle + \|J^*\varphi\|_*^2 = 0.$$

Thus

$$\|\varphi\|_*^2 = \sup_{\xi\in B}\{2\langle\varphi,\xi\rangle - \|\xi\|^2\}$$

is the Young-Fenchel transformation and $f^*(\varphi) = 4^{-1}\|\varphi\|_*^2$ is conjugate to $f(\xi) = \|\xi\|^2$. We describe main properties of the functional $V(\varphi,\xi)$:

1. $V(\varphi,\xi)$ is continuous and differentiable with respect to φ and ξ.
2. $\mathrm{grad}_\varphi V(\varphi,\xi) = 2(J^*\varphi - \xi)$ and $\mathrm{grad}_\xi V(\varphi,\xi) = 2(J\xi - \varphi)$.
3. $V(\varphi,\xi)$ is convex with respect to φ (resp. ξ) when ξ (resp. φ) is fixed.
4. $(\|\varphi\|_* - \|\xi\|)^2 \cdot \;\; V(\varphi,\xi) \cdot \;\; (\|\varphi\|_* + \|\xi\|)^2$.
5. $V(\varphi,\xi) \geq 0, \;\; \forall x,\xi \in B$, and $V(\varphi,\xi) = 0$ if and only if $\varphi = J\xi$.
6. $V(\varphi,\xi) \to \infty$ if $\|\xi\| \to \infty$ or/and $\|\varphi\| \to \infty$, and vice versa.

Now we can present the generalized projection operator in Banach spaces [1, 2].

Definition 2.3. *Operator $\pi \;\; : B^* \to \;\; \subset B$ is called the generalized projection operator if it associates to an arbitrary fixed point $\varphi \in B^*$ the minimum point of the functional $V(\varphi,\xi)$, i.e., solution to the minimization problem*

$$\pi \;\; \varphi = \tilde\varphi; \quad \tilde\varphi : V(\varphi,\tilde\varphi) = \inf_{\xi\in} V(\varphi,\xi).$$

$\tilde\varphi \in \;\; \subset B$ is then called a generalized projection of the point φ.

Existence and uniqueness of the operator π follow from the properties of the functional $V(\varphi,\xi)$ and strict monotonicity of the map J.

It is not difficult to prove that

$$\pi \;\; = J^*, \quad \text{if} \quad = B.$$

Denote $\tilde\varphi_1 = \pi \;\; \varphi_1$, $\tilde\varphi_2 = \pi \;\; \varphi_2$ and let ξ be any point in the set $\;\; \subset B$. Next, we describe the properties of the operator π similar to a)-e) in a uniformly convex and uniformly smooth Banach space.

f) The operator π is J-fixed in each point $\xi \in$, i.e., $\pi \;\; J\xi = \xi$.

g) π is monotone in B^*, i.e., for all $\varphi_1, \varphi_2 \in B^*$,

$$\langle\varphi_1 - \varphi_2, \tilde\varphi_1 - \tilde\varphi_2\rangle \geq 0.$$

h) The point $\tilde\varphi \in$ is a generalized projection of φ on $\;\; \subset B$ if and only if the following inequality is satisfied:

$$\langle\varphi - J\tilde\varphi, \tilde\varphi - \xi\rangle \geq 0, \qquad \forall\xi \in \;\; .$$

The property h) is *the basic variational principle* for π in the dual couple (B, B^*).

j) The operator π produces an absolutely best approximation of $\varphi \in B^*$ relative to functional $V(\varphi, \xi)$, that is,

$$V(J\tilde{\varphi}, \xi) \cdot \quad V(\varphi, \xi) - V(\varphi, \tilde{\varphi}).$$

Consequently, π is *the conditionally nonexpansive* operator relative to the functional $V(\varphi, \xi)$ in Banach spaces, i.e.,

k) $V(J\tilde{\varphi}, \xi) \cdot \quad V(\varphi, \xi)$.

Namely, these properties make generalized projection operators essentially effective in uniformly convex and uniformly smooth Banach spaces.

3. Convergence and Stability Analysis for Optimization Problems

First of all, we consider the case when the modulus of convexity of the space B is such that

$$\delta_B(\epsilon) \geq D\epsilon^2 \tag{3.1}$$

for some constant $D > 0$, and the modulus of smoothness $\rho_B(\tau)$ is arbitrary. This assumption holds, for example, if B are Lebegue spaces l^p, L^p and Sobolev spaces W^p_m for $p \in$ (1,2].

We need the following lemma.

Lemma 3.1. *Let* $\{\mu_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $n = 0, 1, ...,$ *be nonnegative real numbers satisfying the recurrent inequality*

$$\mu_{n+1} \cdot \quad \mu_n - \alpha_n\beta_n + \gamma_n. \tag{3.2}$$

Assume that $\alpha_n > 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and*

$$\lim_{n\to\infty} \frac{\gamma_n}{\alpha_n} = 0. \tag{3.3}$$

Then

$$\omega_m = \frac{\sum_{n=0}^{m} \alpha_n\beta_n}{\sum_{n=0}^{m} \alpha_n} \to 0, \quad \text{as} \quad m \to \infty.$$

Proof. By iterations, we have from (3.2)

$$\frac{\sum_{n=0}^{m} \alpha_n\beta_n}{\sum_{n=0}^{m} \alpha_n} \cdot \quad \frac{\mu_1 + \sum_{n=0}^{m} \gamma_n}{\sum_{n=0}^{m} \alpha_n}. \tag{3.4}$$

We claim that

$$\frac{\sum_{n=0}^{m} \gamma_n}{\sum_{n=0}^{m} \alpha_n} \to 0, \quad \text{as} \quad m \to \infty. \tag{3.5}$$

Indeed, given an arbitrary $\epsilon > 0$, by virtue of (3.3), there exists $\bar{N} > 0$ such that for all $n \geq \bar{N}$,

$$\frac{\gamma_n}{\alpha_n} \cdot \ \epsilon,$$

and for any $m \geq \bar{N}$,

$$\sum_{n=\bar{N}}^{m} \gamma_n \cdot \ \epsilon \sum_{n=\bar{N}}^{m} \alpha_n.$$

It follows from the last inequality that

$$\begin{aligned} \frac{\sum_{n=0}^{m} \gamma_n}{\sum_{n=0}^{m} \alpha_n} &= \frac{\sum_{n=0}^{\bar{N}} \gamma_n + \sum_{n=\bar{N}+1}^{m} \gamma_n}{\sum_{n=0}^{\bar{N}} \alpha_n + \sum_{n=\bar{N}+1}^{m} \alpha_n} \\ &\cdot \ \frac{\sum_{n=0}^{\bar{N}} \gamma_n + \sum_{n=\bar{N}+1}^{m} \gamma_n}{\sum_{n=\bar{N}+1}^{m} \alpha_n} \\ &\cdot \ \frac{\sum_{n=0}^{\bar{N}} \gamma_n + \epsilon \sum_{n=\bar{N}+1}^{m} \alpha_n}{\sum_{n=\bar{N}+1}^{m} \alpha_n} \\ &= \frac{\sum_{n=0}^{\bar{N}} \gamma_n}{\sum_{n=\bar{N}+1}^{m} \alpha_n} + \epsilon. \end{aligned}$$

It is clear that there exists sufficiently large $N > 0$ such that for all $m > N$,

$$\frac{\sum_{n=0}^{\bar{N}} \gamma_n}{\sum_{n=\bar{N}+1}^{m} \alpha_n} \cdot \ \epsilon,$$

and then

$$\frac{\sum_{n=0}^{m} \gamma_n}{\sum_{n=0}^{m} \alpha_n} \cdot \ 2\epsilon.$$

This proves the claim. The whole statement is obtained from (3.4). ■

We apply this lemma in order to prove the following statement.

Theorem 3.2. *Let $f : B \to \mathbb{R}$ be a convex functional, be a convex closed set. Suppose that the conditions* (i)-(iii) *hold, and* (iv) *the intersection of with any level set of f bounded. Let $\{x^n\}$ be any sequence of iterates generated by*

(1.3), (1.4) *and let* $\{v^m\}$ *be the sequence of its Cesàro averages. It follows that* $\lim_{m\to\infty} f(v^m) = f^*$ *and all weak accumulation points of* $\{v^m\}$ *belong to the solution set* M. *In particular, if* M *is a singleton, i.e.,* $M = \{x^*\}$, *then* $\{v^m\}$ *converges weakly to* x^*.

Proof. First of all, let us observe that the boundedness of the sequence $\{x^n\}$ follows from [5] by reason of (iii), (iv) and (3.1). Then it is clear that the Cesàro averages are also bounded, i.e.,

$$\|v^m\| \cdot \;\; C_1 = \text{const.} \tag{3.6}$$

Denote $\varphi^n = Jx^n - \alpha_n u^n/\|u^n\|_*$. With this definition, we have $x^{n+1} = \pi \;\; [\varphi^n]$ and

$$\|\varphi^n - Jx^n\|_* = \alpha_n.$$

Take any $x^* \in M$. By the properties of the functional $V(\varphi, x^*)$ (see (2.3)), one gets

$$V(\varphi^n, x^*) \cdot \;\; V(Jx^n, x^*) + 2\langle \varphi^n - Jx^n, J^*\varphi^n - x^*\rangle.$$

Further from the property j) of the generalized projections π , we conclude that

$$V(Jx^{n+1}, x^*) \cdot \;\; V(\varphi^n, x^*).$$

Hence

$$V(Jx^{n+1}, x^*) \cdot \;\; V(Jx^n, x^*) + 2\langle \varphi^n - Jx^n, J^*\varphi^n - x^*\rangle. \tag{3.7}$$

We now use the convexity condition of $f(x)$ and Theorem 2.1:

$$\begin{aligned}
&\langle \varphi^n - Jx^n, J^*\varphi^n - x^*\rangle \\
&= \langle \varphi^n - Jx^n, x^n - x^*\rangle + \langle \varphi^n - Jx^n, J^*\varphi^n - x^n\rangle \\
&= -\frac{\alpha_n}{\|u^n\|_*}\langle u^n, x^n - x^*\rangle + \langle \varphi^n - Jx^n, J^*\varphi^n - J^*Jx^n\rangle \\
&\cdot \;\; -\frac{\alpha_n}{\|u^n\|_*}(f(x^n) - f^*) + C^2(\|J^*\varphi^n\|, \|J^*Jx^n\|)\rho_{B^*}(4\|\varphi^n - Jx^n\|_*/C) \\
&= -\frac{\alpha_n}{\|u^n\|_*}(f(x^n) - f^*) + C^2\rho_{B^*}(4\alpha_n/C),
\end{aligned}$$

where

$$C = C(\|\varphi^n\|_*, \|x^n\|) = \sqrt{(\|\varphi^n\|_*^2 + \|x^n\|^2)/2}.$$

Thus, (3.7) yields

$$V(Jx^{n+1}, x^*) \cdot \;\; V(Jx^n, x^*) - 2\frac{\alpha_n}{\|u^n\|_*}(f(x^n) - f^*) + 2C^2\rho_{B^*}(4\alpha_n/C). \tag{3.8}$$

Since $\delta_B(\epsilon) \geq D\epsilon^2, D > 0$, then $\rho_{B^*}(\tau) \cdot \ \ D_1\tau^2,\ D_1 > 0$ [13]. Combining the last relation with (3.8), we obtain

$$V(Jx^{n+1}, x^*) \cdot \ \ V(Jx^n, x^*) - 2\frac{\alpha_n}{\|u^n\|_*}(f(x^n) - f^*) + 32D_1\alpha_n^2.$$

Obviously, from the property (ii) and boundedness of the sequence $\{x^n\}$, it follows $\|u^n\| \cdot \ \ C_2$, and then

$$V(Jx^{n+1}, x^*) \cdot \ \ V(Jx^n, x^*) - 2C_2^{-1}\alpha_n(f(x^n) - f^*) + 32D_1\alpha_n^2. \tag{3.9}$$

It is not difficult to see that (3.9) is the inequality of type (3.2), where $\mu_n = V(Jx^n, x^*)$, and

$$\lim_{n\to\infty} \frac{\gamma_n}{\alpha_n} = \lim_{n\to\infty} 16C_2D_1\alpha_n = 0.$$

In this case, Lemma 3.11 gives

$$\omega_m = \frac{\sum_{n=0}^m \alpha_n(f(x^n) - f^*)}{\sum_{n=0}^m \alpha_n} \to 0, \quad \text{as} \quad m \to \infty.$$

Provided that $f(x)$ is a convex functional, we estimate easily

$$f\left(\frac{\sum_{n=0}^m \alpha_n x^n}{\sum_{n=0}^m \alpha_n}\right) \cdot \ \ \frac{\sum_{n=0}^m \alpha_n f(x^n)}{\sum_{n=0}^m \alpha_n}.$$

We have now for the Cesàro averages v^m:

$$f(v^m) - f^* = f\left(\frac{\sum_{n=0}^m \alpha_n x^n}{\sum_{n=0}^m \alpha_n}\right) - f^* \cdot \ \ \frac{\sum_{n=0}^m \alpha_n f(x^n)}{\sum_{n=0}^m \alpha_n} - f^*$$

$$= \frac{\sum_{n=0}^m \alpha_n(f(x^n) - f^*)}{\sum_{n=0}^m \alpha_n} = \omega_m \to 0, \quad \text{as} \quad m \to \infty.$$

It follows that $\lim_{m\to\infty} f(v^m) = f^*$. Let v be any weak accumulation point of $\{v^m\}$ (it exists in view of (3.6), and $\{v^{m_k}\}$ be any subsequence which weakly converges to v. Finally, the weak lower semicontinuity of convex functionals gives

$$f(v) \cdot \ \ \liminf_{k\to\infty} f(v^{m_k}) = \lim_{m\to\infty} f(v^m) = f^*.$$

Thus, the set of weak accumulation points of $\{v^m\}$ is contained in M. It is obviously that if M is a singleton then the whole sequence $\{v^m\}$ converges weakly to x^*. The theorem is accomplished. ■

It is well-known that a Hilbert space H is uniformly convex and uniformly smooth, and that [4]

$$\frac{\epsilon^2}{8} \cdot \ \ \delta_H(\varepsilon) \cdot \ \ \frac{\epsilon^2}{4},$$

i.e., (3.1) is satisfied. Therefore, the following corollary is valid.

Corollary 3.3. *Let H be a Hilbert space, $f : B \to \mathbb{R}$ be a convex functional, be a convex closed set. Suppose that the conditions* (ii)-(iv) *hold* (*see Theorem* 3.2). *Let $\{x^n\}$ be any sequence of iterates generated by* (1.5), (1.4) *and let $\{v^m\}$ be the sequence of its Cesàro averages. It follows that $\lim_{m\to\infty} f(v^m) = f^*$ and all weak accumulation points of $\{v^m\}$ belong to the solution set M. In particular, if M is a singleton, i.e., $M = \{x^*\}$, then $\{v^m\}$ converges weakly to x^*.*

Remark 3.4. If in addition to the conditions of this corollary $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$, then $\{v^m\}$ converges weakly to some $x^* \in M$.

Further we omit the condition $\delta_B(\epsilon) \geq D\epsilon^2$, $D > 0$, i.e., assume that B is an arbitrary uniformly convex and uniformly smooth Banach space. At the same time, we raise the stronger claims to the set .

Theorem 3.5. *Let $f : B \to \mathbb{R}$ be a convex functional, be a convex closed bounded set. Suppose that the conditions* (i)-(ii) *hold. Let $\{x^i\}$ be any sequence of iterates generated by* (1.3), (1.4) *and let $\{v^m\}$ be the sequence of its Cesàro averages. Then all the conclusions of Theorem* 3.2 *are valid.*

Proof. It follows from the assumptions of this theorem that $M \neq \emptyset$. Since $x^n \in$ for all $n \geq 1$, x^n are bounded, say, by R_1. Without lost of generality, we can consider $\alpha_n \cdot \ \bar{\alpha}$. In this case we obtain

$$\|\varphi^n\|_* \cdot \ \|Jx^n\|_* + \alpha_n \cdot \ R_1 + \bar{\alpha} \cdot \ \|x^n\| + \alpha_n \cdot \ R_1 + \bar{\alpha} = R$$

and

$$C = C(\|\varphi^n\|_*, \|x^n\|) \cdot \ \max\{R_1, R\} = R.$$

From the property (ii), we get again that $\|u^n\|_* \cdot \ C_2$. Finally, using the estimate (2.1) one can write analogous to (3.9):

$$V(Jx^{n+1}, x^*) \cdot \ V(Jx^n, x^*) - 2C_2^{-1}\alpha_n(f(x^n) - f^*) + 2LR^2\rho_{B^*}(4\alpha_n/R). \tag{3.10}$$

Recall that the space B is uniformly smooth if and only if

$$\frac{\rho_B(\tau)}{\tau} \to 0 \quad \text{as} \quad \tau \to 0.$$

This allows us to apply Lemma 3.1 because (up to constants):

$$\frac{\gamma_n}{\alpha_n} = \frac{\rho_{B^*}(\alpha_n)}{\alpha_n} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.11}$$

The rest of the proof follows the pattern of Theorem 3.2. ∎

We recall that $\{x^n\}$ will be a bounded sequence of the iterates above if

1) is a bounded set (consequently, M is not empty);
2) $\sum_{n=0}^{\infty} \rho_{B^*}(\alpha_n) < \infty$ and M is not empty;
3) $\delta_B(\epsilon) \geq D\epsilon^2$, $D =$ const. > 0 and the intersection of with any level set of f is bounded.

Therefore in the sequel, for more generality, we prefer to suppose that $\{x^n\}$ is *apriori* a bounded sequence. The following theorem holds for the method (1.8).

Theorem 3.6. *Let B be a uniformly smooth and uniformly convex Banach space $f : B \to \mathbb{R}$ be a convex functional, and be a convex closed set. Let $\{x^n\}$ be any bounded sequence of the iterates generated by* (1.8), (1.4) *and let $\{v^m\}$ be the sequence of $\{\bar{x}^n\}$ Cesàro averages. It follows that $\lim_{m\to\infty} f(v^m) = f^*$ and all weak accumulation points of $\{v^m\}$ belong to the solution set M. In particular, if $M = \{x^*\}$ then $\{v^m\}$ converges weakly to x^*.*

We omit the proof of this theorem, and note only that the inequality of type (3.8) is obtained from the relations (see [7]):

$$
\begin{aligned}
V(Jx^{n+1}, x^*) \cdot \; & V(Jx^n, x^*) + \langle Jx^{n+1} - Jx^n, x^{n+1} - x^* \rangle \\
&= V(Jx^n, x^*) + \langle Jx^{n+1} - Jx^n, x^n - \bar{x}^n \rangle \\
&+ \langle Jx^{n+1} - Jx^n, \bar{x}^n - x^* \rangle + \langle Jx^{n+1} - Jx^n, x^{n+1} - x^n \rangle.
\end{aligned}
$$

To investigate the stability of the iterations (1.3), (1.4), let us denote by $\partial_\epsilon f(x)$ an ϵ-subdifferential of $f(x)$ at $x \in B$, that is,

$$\partial_\epsilon f(x) = \{w \in B^* : f(y) - f(x) \geq \langle w, y - x \rangle - \epsilon, \text{ for all } y \in B\}.$$

Consider the perturbed iterative method

$$z^{n+1} = \pi \quad Jz^n - \alpha_n \frac{w^n}{\|w^n\|_*} \Big], \; n = 1, 2, ..., \tag{3.12}$$

where $w^n \in \partial_{\epsilon_n} f(z^n)$ is an arbitrary ϵ_n-subgradient of $f(x)$ at $z^n \in B$.

Theorem 3.7. *Let B be a uniformly convex and uniformly smooth Banach space, $f : B \to \mathbb{R}$ be a convex functional, and be a convex closed set. Let $\{z^n\}$ be any bounded sequence of the iterates z^n generated by* (3.12), (1.4) *and let $\{v^m\}$ be the sequence of its Cesàro averages. If $\epsilon_n \to 0$, it follows that $\lim_{m\to\infty} f(v^m) = f^*$ and all weak accumulation points of $\{v^m\}$ belong to the solution set M. In particular, if M is a singleton then $\{v^m\}$ converges weakly to x^*.*

Proof. Instead of the inequality (3.10), we get

$$V(Jz^{n+1}, x^*) \cdot \quad V(Jz^n, x^*) - 2C_2^{-1}\alpha_n(f(z^n) - f^*) + 2C_2^{-1}\alpha_n\epsilon_n$$
$$+ 2LR^2\rho_{B^*}(4\alpha_n/R).$$

Now in Lemma 3.1,

$$\gamma_n = 2C_2^{-1}\alpha_n\epsilon_n + 2LR^2\rho_{B^*}(4\alpha_n/R).$$

It is clear that

$$\frac{\gamma_n}{\alpha_n} \to 0, \quad \text{as} \quad \epsilon_n \to 0 \quad \text{and} \quad \alpha_n \to 0.$$

Together with (3.11) this is enough for the conclusion of the theorem. ■

The condition $\epsilon_n \to 0$ in Theorem 3.7 is the weakest requirement among those that guarantee stability of the weak convergence in optimization problems. Note for a comparison that in the corresponding theorems of [6, 7], for weak convergence of x^n to x^*, we have needed one additional condition, namely, $\epsilon_n \cdot \quad \mu\alpha_n$ for some $\mu > 0$. We call readers' attention to the fact that ϵ_n in (3.12) describe perturbations of the functional $f(x)$ in the points x^n. Below in Section 4, we investigate another variant – perturbations of subgradients in the points x^n, and more general perturbations of the monotone operators in variational inequalities.

Remark 3.8. Similarly to Lemma 3.1, one can obtain the assertion that if

$$\lim_{n\to\infty} \frac{\gamma_n}{\alpha_n} = \omega,$$

then

$$\lim_{n\to\infty} \omega_m = \lim_{n\to\infty} \frac{\sum_{n=0}^m \alpha_n\beta_n}{\sum_{n=0}^m \alpha_n} \cdot \quad \omega.$$

Let $\lim_{n\to\infty} \epsilon_n = \epsilon$. Then Theorem 3.7 gives in this case the following limit-relation:

$$\lim_{m\to\infty} f(v^m) \cdot \quad f^* + \epsilon.$$

4. Convergence and Stability Analysis for Variational Inequalities

Up to now, we dealt just with potential operators which were the subgradients of convex functionals. We next consider the more general problem of solving the variational inequalities

$$\langle Ax, x - x^*\rangle \geq 0, \quad x^* \in \quad, \ \forall x \in \quad, \tag{4.1}$$

with nonsmooth and monotone (not necessary potential) operators A. Variational inequalities allow us to investigate, in the framework of a single scheme, such problems as: constrained and unconstrained minimization problems, operator equations, convex-concave minimax problems, evolution equations, etc.

Definition 4.1. *The element* $x^* \in \quad$ *such that for all* $x \in \quad$ *and for all* $z \in Ax$,

$$\langle z, x - x^* \rangle \geq 0$$

will be called the solution of the variational inequality (4.1).

We study the iterative process

$$x^{n+1} = \pi \quad Jx^n - \alpha_n \frac{u^n}{\|u^n\|_*} \Big], \quad u^n \in Ax^n, \;\; n = 1, 2, ..., \tag{4.2}$$

where

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \;\; \alpha_n > 0, \;\; \alpha_n \to 0. \tag{4.3}$$

Theorem 4.2. *Suppose that* B *is a uniformly convex and uniformly smooth Banach space*, $A : D(A) \subset B \to B^*$ *is a monotone and bounded operator* (*i.e.*, *it carries bounded sets from* $D(A)$ *into bounded sets of* B^*), $\quad$ *is a convex closed set*, $\quad \subset \int D(A)$ [8]. *Let* $\{x^n\}$ *be any bounded sequence of iterates generated by* (4.2), (4.3) *and let* $\{v^m\}$ *be the sequence of its Cesàro averages. It follows that*
1) *a solution* x^* *of the variational inequality* (4.1) *exists*, *possible nonunique*;
2) *all weak accumulation points of* $\{v^m\}$ *belong to the solution set* M. *If* M *is a singleton, then* $\{v^m\}$ *converges weakly to* x^*.

Proof. First of all, let us observe that the sequence $\{x^n\}$ is bounded, for example, if $\quad$ is bounded or $\sum_{n=0}^{\infty} \rho_{B^*}(\alpha_n) < \infty$ and M is nonempty. Denote $\varphi^n = Jx^n - \alpha_n u^n / \|u^n\|_*$. Then similarly to Theorem 3.5, we have for all $y \in \quad, u^n \in Ax^n$:

$$V(Jx^{n+1}, y) \cdot \;\; V(Jx^n, y) - 2C^{-1}\alpha_n \langle u^n, x^n - y \rangle + 2LR^2 \rho_{B^*}(4\alpha_n/R),$$

where $\|x^n\| \cdot \;\; R$ and $\|u^n\| \cdot \;\; C$. And now we can write

$$\alpha_n \langle u^n, y - x^n \rangle \geq 2^{-1} C V(Jx^{n+1}, y) - V(Jx^n, y)) - CLR^2 \rho_{B^*}(4\alpha_n/R). \tag{4.4}$$

Let $z \in Ay$. Then by the monotonicity of the operator A, (4.4) gives

$$\alpha_n \langle z, y - x^n \rangle \geq 2^{-1} C (V(Jx^{n+1}, y) - V(Jx^n, y)) - CLR^2 \rho_{B^*}(4\alpha_n/R).$$

Since $V(Jx^n, y) \geq 0$ for all $y \in$, one can obtain the following estimate for the Cesàro averages:

$$\langle z, y - v^m \rangle \geq -2^{-1} C \frac{V(Jx^1, y))}{\sum_1^m \alpha_n} - CLR^2 \frac{\sum_1^m \rho_{B^*}(4\alpha_n / R)}{\sum_1^m \alpha_n}.$$

If $\{v^m\}$ has a weak limit, then (3.5), (3.11) and (4.3) imply

$$\lim_{m \to \infty} \langle z, y - v^m \rangle = \langle z, y - \lim_{m \to \infty} v^m \rangle \geq 0.$$

The sequence $\{v^m\}$ is bounded because $\{x^n\}$ is bounded. Therefore, there exists a subsequence $\{v^{m_j}\}$ which converges weakly to $\tilde{x}$. By virtue of the convexity and closeness of , all v^{m_j} belong to and the limit element $\tilde{x}$ also belongs to . In this case,

$$\langle z, y - \lim_{m_j \to \infty} v^{m_j} \rangle = \langle z, y - \tilde{x} \rangle \geq 0, \ \ \forall y \in \ .$$

This means that $\tilde{x}$ is the solution of (4.1), i.e., $\tilde{x} = x^*$. If x^* is unique, then the whole sequence $\{v^m\}$ converges weakly to x^*. The theorem is proved. ■

Remark 4.3. We showed that if the iterative sequence (4.2) is bounded then the variational inequality (4.1) has a solution. The contrary assertion is unknown in general. However, the boundedness of each iterative sequence in [3, 5, 6, 7] was obtained provided that a solution of the corresponding problem exists (see also Section 3).

Remark 4.4. In Theorem 4.2, in reality, it is sufficient to demand for the operator A to be bounded only on the bounded sequence $\{x^n\}$.

Corollary 4.5. *Under the conditions of Theorems* 4.2, *if* $M = \emptyset$ *then the sequence* $\{x_n\}$ *is unbounded.*

Consider now the stability of the iterations

$$z^{n+1} = \pi \ \ \Big[Jz^n - \alpha_n \frac{w^n}{\|w^n\|_*} \Big], \ \ w^n \in A^{h_n} z^n, \ \ n = 1, 2, ..., \tag{4.5}$$

for the perturbed variational inequality (4.1), where $A^{h_n} z^n$ are perturbed values of the operator A in the points z^n, and h_n is perturbation parameter. Suppose for simplicity that A and A^{h_n} are maximal monotone (possibly multivalued) operators [9, 16], and $D(A^{h_n}) = D(A)$ for all $h_n \geq 0$. Besides, we assume that the following proximity estimate is given:

$$\mathcal{H}_{B^*}(S^n, S^{h_n}) \cdot \ \zeta(\|z^n\|) h_n, \quad 0 \cdot \ h_n \cdot \ \bar{h}, \tag{4.6}$$

where $\mathcal{H}_{B^*}(Q_1, Q_2)$ is the Hausdorff distance between the sets Q_1 and Q_2 in the space B^* (see, for example, [8]), S^n and S^{h_n} are the ranges of the operators A and A^{h_n} in z^n, respectively. Let, finally, the function $\zeta(t)$ in (4.6) be continuous and nondecreasing for all $t \geq 0$, and bounded on bounded sets. The following result is proved according to the scheme of the previous Theorems 3.7 and 4.2.

Theorem 4.6. *Suppose that all conditions of Theorem* 4.2 *for the iterative process* (4.5), (4.3) *and all assumptions above for the operators* A *and* A^{h_n} *are satisfied. If* $h_n \to 0$, *then the assertions* 1), 2) *of this theorem are valid.*

Remark 4.7. In the theorems above, the iterations (4.2) can be replaced by

$$x^{n+1} = \pi \quad [Jx^n - \alpha_n u^n], \quad u^n \in Ax^n, \ \ n = 1, 2, ...,$$

provided that the sequence $\{x^n\}$ is bounded.

So, we proved the weak convergence and stability of the average iterations v^m for finding solutions of the variational inequality (4.1) with the monotone (possibly) non-potential operator A. Let us emphasize for comparison that, in general, the original iterations x^n, generated by (4.2), (4.3) (or (4.2), (4.3) and (1.7)) do not converge for this problem even weakly.

Remark 4.8. The method (1.8) for the variational inequality (4.1) is considered by the same way.

Remark 4.9. All results of this Section are carried out for variational inequalities with an arbitrary "right-hand part" $f \in B^*$

$$\langle Ax - f, x - x^* \rangle \geq 0, \ \ x^* \in \quad , \ \ \forall x \in \quad .$$

The proof method of Theorem 4.2 can be applied to the problem of finding fixed points of the nonexpansive operator $A : \quad \to \quad$ in Hilbert and Banach spaces [14]. The iterative process is given by the formula:

$$x^{n+1} = (1 - \alpha_n)x^n + \alpha_n Ax^n, \ \ n = 1, 2, ..., \tag{4.7}$$

where α_n obey the rule (4.3) and $0 < \alpha_n \cdot \ \ 1$. Obviously, (4.7) is equivalent to

$$x^{n+1} = x^n - \alpha_n Tx^n, \ \ n = 1, 2, ..., \tag{4.8}$$

with $T = I - A$. Recall that the fixed point x^* is a solution of the equation $Tx^* = 0$. Since A is nonexpansive, the operator $T : \quad \to H$ is accretive [9], and it satisfies the Lipschitz condition. It is easy to see that if $x^1 \in \quad$ then all $x^n \in \quad$ and we do not need any projection operator in (4.8).

The parameters α_n in Theorems 4.2 tend to 0 because this theorem has been intended for general nonsmooth case. Therefore, the weak convergence of v^m to a fixed point x^* is also obtained when $\alpha_n \to 0$. However, the smooth (Lipschitz) operators allow to use the constant $\alpha_n = \alpha$ in the iterative method (4.8) in order to get average weak convergence or even weak convergence of the original sequence $\{x^n\}$, but in the framework of a different approach. Such stronger results were obtained in [11, 17, 18, 19].

In conclusion, let us make several general remarks.

Remark 4.10. We showed above that every time when M is a singleton, $\{v^m\}$ converges weakly to x^*. Otherwise, one asserts the following: if J is a sequentially weakly continuous operator (on some bounded set containing $\{v^m\}$) and $V(Jv^m, x^*)$ has a limit as $m \to \infty$ for any $x^* \in M$, then $\{v^m\}$ is weakly convergent to a point in M.

Remark 4.11. In finite-dimensional Hilbert and Banach spaces, the theorems above assert a strong convergeence of the Cesàro averaged approximations to solutions of the corresponding variational problems.

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Ya. I. Alber
Department of Mathematics
The Technion-Israel Institute of Technology
32000 Haifa, Israel