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SINGULARITIES AND SOME INVARIANTS OF SINGULARITIES IN CONTACT 3-MANIFOLDS

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Abstract. In this paper, we study the singularities in contact 3-Manifolds. We give simple presentations for singularities, make an initial classification of them and show that the space of all singularities has continuous moduli. We also study the stabilities of singularities and obtain that a nondegenerate singular point is isolated. Finally, by means of the transversality theorem of Thom, we show that generically every immersion contains no degenerate singular points. Thus the singular set of a closed surface is finite.

1. INTRODUCTION

A contact 3-manifold (M, ξ) is a 3-manifold equipped with a contact structure ξ which is defined by a completely nonintegrable tangent plane field. The complete nonintegrability of ξ can be expressed by the inequality $\theta \wedge d\theta \neq 0$ for a 1-form θ which defines the plane field ξ, that is, $\xi = \ker \theta$.

Let Σ be a surface and (M, ξ) be a contact 3-manifold. A differentiable map $F : \Sigma \to M$ is an immersion if the rank of F at each point p of Σ is 2. An immersion F is an embedding if F is a homeomorphism of Σ into M. A point $p \in \Sigma$ is said to be a **singular point** of an immersion $F : \Sigma \to M$ if $F_{*p}(T\Sigma_p) = \xi_{F(p)}$ (the contact plane at the point $F(p)$), that is, the surface is tangent to the contact plane at $F(p)$. Denote $S_F \subset \Sigma$ as the set of all singular points of F. In this paper, we always do not distinguish Σ from $F(\Sigma)$ if $F : \Sigma \to M$ is an embedding. Outside the singular set S_F the contact structure ξ intersects $T\Sigma$ along a line field which integrates to a 1-dimensional foliation on Σ with singularities at points S_F. This singular foliation on Σ or on $F(\Sigma)$ is called the **characteristic foliation** of Σ. The reader can refer to [3] for contact manifolds and the related subjects.

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It is well known from Darboux's theorem that all contact 3-manifolds are locally contact diffeomorphic. Therefore, for each $p \in \Sigma$, we can always regard F as an immersion $F : U \to W$ from a neighborhood $U \subset \mathbb{R}^2$ of p into a neighborhood $W \subset H^1$ of $F(p)$. Here H^1 is the Heisenberg group with the standard contact form $\hat{\theta} = dz + xdy - ydx$ and the contact bundle $\hat{\xi} = Ker\hat{\theta}$. Also, suppose $\mathbb{R}^2$ is equipped with a coordinates (u, v), we can present F to be

$$F(u, v) = \big(x(u, v), y(u, v), z(u, v)\big),$$

for three smooth functions $x(u, v), y(u, v), z(u, v)$ on U. The immersion F is associated with a vector field X_F on U defined by

$$X_F = -\hat{\theta}(F_v)\frac{\partial}{\partial u} + \hat{\theta}(F_u)\frac{\partial}{\partial v} \tag{1.1}$$

It is easy to see that $p \in U$ is a singular point of F if and only if $X_F(p) = 0$, or, equivalently, p is a root of the following equation system

$$\begin{aligned} \hat{\theta}(F_u) &= z_u + xy_u - yx_u = 0 \\ \hat{\theta}(F_v) &= z_v + xy_v - yx_v = 0. \end{aligned} \tag{1.2}$$

In this paper, inspired by [2], we define the concept of **positions** in contact 3-manifolds (see Section 2). Intuitively a position is a description of how to display a surface into a contact 3-manifold. It is defined by an equivalence class of immersion-germs (see Definitions 2.1 and 2.2). A position which is defined by an immersion-germ of F at some singular point is called a **singularity**. We wonder how many kinds of singularities can be found. One of our main theorems in this paper is that we classify singularities by a kind of value called the **discriminant** (see Example 2.11 in Section 2 and Example 3.2 in Section 3) and finally show that there are continuous moduli in the space of all singularities. Actually, suppose p is a singular point of F, we can associate a value I to the immersion-germ $[F]_p$ by

$$\begin{aligned} I\left([F]_p\right) &= \frac{\det\left(X_F\right)_*(p)}{\left(\det\left(\pi\circ F\right)_*(p)\right)^2} \\ &= \frac{\det\left(Hess_p(z) + q_1\cdot Hess_p(y) - q_2\cdot Hess_p(x)\right)}{\left(\det\left(\pi\circ F\right)_*(p)\right)^2} + 1. \end{aligned} \tag{1.3}$$

Here $F(p) = q = (q_1, q_2, q_3)$, $\pi(x, y, z) = (x, y)$ is the projection from H^1 onto $\mathbb{R}^2$ and

$$Hess_p(f) = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix}(p)$$

is the Hessian of $f(u,v)$ at p. Note that we regard X_F as a map from U into $\mathbb{R}^2$ such that the differential $(X_F)_*$ makes sense. We obtain our first main theorem:

Theorem 1.1. *We have that I is an invariant of singularities. That is, if $[F_1]_p \sim [F_2]_q$ then $I\left([F_1]_p\right) = I\left([F_2]_q\right)$.*

Moreover, if $I > 0$ then the singularity is elliptic; if $I < 0$ then the singularity is hyperbolic. Therefore, $I = 0$ means that the singularity is degenerate.

Since we can always find a singularity $\Im$ such that $I(\Im) = r$ for each real value $r \in \mathbb{R}$. We immediately have the following corollary:

Corollary 1.2. *The space of all nondegenerate singularities has continuous moduli.*

This result says that surfaces at some singular points may define different singularities even though, topologically, they have the same characteristic foliations.

Remark 1.3. In general, if Σ is closed or with collared Legendrian boundary, then the characteristic foliation (up to a diffeomorphism fixed at Σ) uniquely defines the germ of a contact structure along Σ. The reader can refer to [4, 5].

On the other hand, it is also a really interesting problem if there still exists continuous moduli in regular positions, that is, a position defined by an immersion at a nonsingular point. We guess that there should be just one regular position although we are not able to give a proof here.

Remark 1.4. The study of positions or singularities is itself an interesting problem as well as we guess that it would help us deal with the global L^p estimates for some subelliptic operators, because we may locally find a suitable normal form for a boundary of a suitable domain in a pseudohermitian 3-manifold.

We also study the stabilities of singularities and obtain some interesting results (see Section 4). The second main theorem is about the problem of the stabilities of singularities. We can tell from Theorem 1.1 that a C^2-perturbation will change singularity from one into another one. That is, the type of singularity is not stable under a C^2-perturbation. However, we can weaken the concept of the stability and finally obtain a weakly stable theorem in some sense. We show that a nondegenerate singular point is stable under a C^2-perturbation. That is, after a small perturbation, the singular point does not vanish, but simply shifts slightly (see Theorem 1.5). Such a phenomenon is not available for a degenerate singular point. In the end of Section 4 we give some examples to say that degenerate singular points behave completely differently under perturbation.

Let *Map*(Σ, M) be the space of all C^∞ maps from Σ to M and *Map*$^s(\Sigma, M)$ be the space *Map*(Σ, M) equipped with the C^s-topology. Let *Imm*$(\Sigma, M) \subset$*Map*(Σ, M) be the set of all immersions. Clearly if Σ is compact, then *Imm*(Σ, M) is an open subset of *Map*$^s(\Sigma, M),\ s \geq 1$. Also, we denote *Imm*$^s(\Sigma, M)$ the space *Imm*(Σ, M) equipped with the C^s-topology.

Theorem 1.5. *Suppose $p \in \Sigma$ is a nondegenerate singular point of $F \in$Imm^2 (Σ, M). Then for any neighborhood W of p, there is a neighborhood $\Omega \subset$Imm^2 (Σ, M) of the immersion F such that for any immersion $\widetilde{F} \in \Omega$ there is a unique singular point $\tilde{p} \in W$ of $\widetilde{F}$ which is also nondegenerate. A C^2-perturbation preserves elliptic, or hyperbolic, singularities. That is they are **stable** in C^2 topology in this sense.*

Corollary 1.6. *A nondegenerate singular point is isolated.*

We say an immersion $F : \Sigma \to M$ is **good** if its singular set S_F involves no degenerate singular point. We immediately have the following corollary:

Corollary 1.7. *Let $F : \Sigma \to M$ be a good immersion. Then the singular set S_F is discrete. Moreover if Σ is compact then S_F is finite.*

The last main theorem is Theorem 1.8 which shows that generically every immersion is good, that is, it contains no degenerate singular point. From Section 4, we see that degenerate singular points have more complicated behavior and hence are hard to control. Fortunately, Theorem 1.8 says that generic immersions are **good**. Therefore, intuitively, we can say that an immersion at a degenerate singular point is not in a general position. This result together with Corollary 1.7 show that generically the singular set S_F of an immersion F is finite. In addition, from Theorem 1.5, we see that both the numbers of elliptic singular points and hyperbolic singular points of a good map are invariant under a small C^2-perturbation. We prove Theorem 1.8 by means of the transversality theorem of Thom.

Theorem 1.8. *Suppose Σ is a closed surface. Then the set of all **good** immersions $F : \Sigma \to M$ is an open and dense subset of $Imm^s(\Sigma, M),\ s > 2$.*

2. Positions and Singularities

The Heisenberg group is the Lie group H^1 whose underlying manifold is $\mathbb{R}^3$ with coordinates (x, y, z) and whose group law is given by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + y_1x_2 - x_1y_2).$$

As a contact manifold, the Heisenberg group is equipped with the standard contact form $\hat{\theta} = dz + xdy - ydx$ and the contact bundle $\hat{\xi} = Ker\hat{\theta}$. An immersion-germ $\mathbb{R}^2 \to H^1$ at a point p of $\mathbb{R}^2$ is an equivalent class of immersions $F : U \to H^1$ (each of which is defined on some neighborhood U of p, not necessarily the same for each); here two immersions are regarded as equivalence if they coincide on some neighborhood of p. Two immersions of the same class are said to have the same germ at the point p. We denote by $[F]_p$ the immersion-germ of F at p.

Definition 2.1. Two immersion-germs $[F_1]_p$, $[F_2]_q$ are said to be equivalent if there exist neighborhoods U_1 and U_2 of p and q, respectively such that the following diagram commutes:

$$(2.1) \qquad \begin{array}{ccc} p \in U_1 & \xrightarrow{F_1} & (W_1, \hat{\theta}) \\ \Big\downarrow h & & \Big\downarrow f \\ q \in U_2 & \xrightarrow{F_2} & (W_2, \hat{\theta}) \end{array}$$

for some diffeomorphism $h : U_1 \to U_2,\ h(p) = q$ and some contact diffeomorphism $f : W_1 \to W_2$ from a neighborhood W_1 of $F_1(U_1)$ onto a neighborhood W_2 of $F_2(U_2)$. We denote the equivalence relation by $\sim$.

Note that every left translation on H^1 is a contact diffeomorphism.

Definition 2.2. A **position** is an equivalence class of an immersion-germ at a point. If F is an immersion defined on a neighborhood of p, then we say the germ of F at p defines a position (or simply say F defines a position at p). A position defined by an immersion-germ at a singular point is said to be a **singularity**.

Suppose that $F : U \to H^1$ is an immersion. Here U is an open subset of $\mathbb{R}^2$ with coordinates (u, v). F is associated with a vector field X_F on U defined by

$$(2.2) \qquad X_F = -\hat{\theta}(F_v)\frac{\partial}{\partial u} + \hat{\theta}(F_u)\frac{\partial}{\partial v}$$

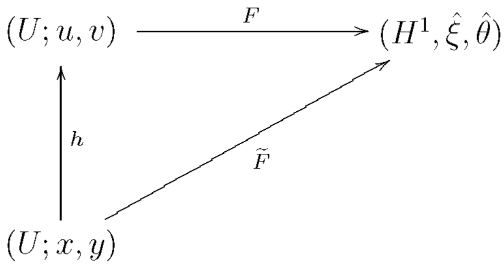
It is easy to see that $p \in U$ is a singular point of F if and only if $X_F(p) = 0$.

Definition 2.3. A singular point $p \in U$ of an immersion F is said to be **nondegenerate** if the differential $(X_F)_*(p)$ of X_F at p is invertible (here we regard X_F as a map from $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$). In addition, an nondegenerate singular point p is said to be **elliptic** if $\det(X_F)_*(p) > 0$ and **hyperbolic** if $\det(X_F)_*(p) < 0$.

Remark 2.4. Let $F : \Sigma \to (M, \theta)$ be an immersion from a surface into a contact 3-manifold M with a contact form θ. Locally, for each $p \in \Sigma$, we can choose a coordinates neighborhood $(U; u, v)$ of p and a coordinates neighborhood $(W; x, y, z)$ of $F(p)$ such that $F(u, v) = (x(u, v), y(u, v), z(u, v))$ is an immersion from an open subset $U \subset \mathbb{R}^2$ into $W \subset H^1$ with the standard contact form $\hat{\theta} = dz + xdy - ydx$. Thus we can define nondegenerate, elliptic and hyperbolic singular point like Definition 2.3. Proposition 2.8 shows that the definition is independent of the choice of local coordinates.

In order to have Proposition 2.8, we need the following three Lemmas.

Lemma 2.5. *Let $F : U \longrightarrow H^1$ be an embedding and $h : U \longrightarrow U$ a diffeomorphism. Suppose $\widetilde{F} = F \circ h$ as shown in the following diagram. Then we have that $h_* X_{\widetilde{F}} = (\det h_*) X_F$.*



Proof. Since that

$$h_* \frac{\partial}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial}{\partial v}$$
$$h_* \frac{\partial}{\partial y} = \frac{\partial u}{\partial y} \frac{\partial}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial}{\partial v},$$

and

$$\widetilde{F}_x = F_u \frac{\partial u}{\partial x} + F_v \frac{\partial v}{\partial x}$$
$$\widetilde{F}_y = F_u \frac{\partial u}{\partial y} + F_v \frac{\partial v}{\partial y},$$

we have

$$\begin{aligned} h_* X_{\widetilde{F}} &= -\hat{\theta}(\widetilde{F}_y) h_* \frac{\partial}{\partial x} + \hat{\theta}(\widetilde{F}_x) h_* \frac{\partial}{\partial y} \\ &= -\hat{\theta}(\widetilde{F}_y) \left(\frac{\partial u}{\partial x} \frac{\partial}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial}{\partial v} \right) + \hat{\theta}(\widetilde{F}_x) \left(\frac{\partial u}{\partial y} \frac{\partial}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial}{\partial v} \right) \\ &= \left(\frac{\partial v}{\partial y} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right) \left(-\hat{\theta}(F_v) \frac{\partial}{\partial u} + \hat{\theta}(F_u) \frac{\partial}{\partial v} \right) \\ &= \det h_* \cdot X_F. \end{aligned}$$

■

Lemma 2.6. *Let* $F: U \longrightarrow W_1 \subseteq H^1$ *be an embedding and* $f: W_1 \longrightarrow W_2$ *a contact transformation such that* $f^*\hat{\theta} = \varphi\hat{\theta}$. *Suppose* $\widetilde{F} = f \circ F$ *as shown in the following diagram. Then we have that* $X_{\widetilde{F}} = \varphi X_F$.

$$\begin{array}{ccc} (U; u, v) & \xrightarrow{\quad F \quad} & (W_1, \hat{\xi}, \hat{\theta}) \\ & \searrow^{\widetilde{F}} & \downarrow f \\ & & (W_2, \hat{\xi}, \hat{\theta}) \end{array}$$

Proof. We have

$$\begin{aligned} X_{\widetilde{F}} &= -\hat{\theta}(\widetilde{F}_v)\frac{\partial}{\partial u} + \hat{\theta}(\widetilde{F}_u)\frac{\partial}{\partial v} \\ &= -\hat{\theta}(f_*F_v)\frac{\partial}{\partial u} + \hat{\theta}(f_*F_u)\frac{\partial}{\partial v} \\ &= -f^*\hat{\theta}(F_v)\frac{\partial}{\partial u} + f^*\hat{\theta}(F_u)\frac{\partial}{\partial v} \\ &= -\varphi\hat{\theta}(F_v)\frac{\partial}{\partial u} + \varphi\hat{\theta}(F_u)\frac{\partial}{\partial v} \\ &= \varphi \cdot X_F. \end{aligned}$$

■

Lemma 2.7. *Let* X *be a vector field on* $U \subset \mathbb{R}^2$, $h: U \longrightarrow U$ *a diffeomorphism, and* φ *a function on* U. *Put* $Y = h_*X$ *and* $Z = \varphi X$. *If* p *is a singular point of* X, *i.e.,* $X(p) = 0$, *then we have*

$$\begin{aligned} Y_*(q) &= h_*(p) \cdot X_*(p) \cdot h_*(p)^{-1}, \\ Z_*(p) &= \varphi \cdot X_*(p), \end{aligned}$$

where $q = h(p)$.

Proof. Since $Y = h_*X$ means that $Y(h(p)) = h_*(p)X(p)$ for each $p \in U$. Therefore, by elementary calculus, we have $Y_*(h(p))h_*(p) = h_*(p)X_*(p)$ at any singular point p of X. Finally $Z_*(p) = \varphi \cdot X_*(p)$ is obvious. ■

Proposition 2.8. *Suppose that* F_i, $i = 1, 2$ *are two immersions defined on neighborhoods of* p *and* q, *respectively. If* $[F_1]_p$ *and* $[F_2]_q$ *define the same position, then, without lose of generality, we can assume that both* F_i, $i = 1, 2$ *are embeddings and satisfy the commutative diagram* (2.1). *We have*

$$X_{F_2} = \frac{\varphi}{\det h_*} \cdot h_* X_{F_1}. \tag{2.3}$$

Moreover if they define a singularity, we have

$$(X_{F_2})_*(q) = \frac{\varphi(p)}{\det h_*(p)} \cdot h_*(p) \cdot (X_{F_1})_*(p) \cdot h_*(p)^{-1}, \tag{2.4}$$

where $q = h(p)$ *and* $f^*\hat{\theta} = \varphi\hat{\theta}$.

Proof. This Proposition just follows from Lemmas 2.5, 2.6 and 2.7. Let $F = f \circ F_1 = F_2 \circ h$. Then, by Lemma 2.5 and 2.6,

$$\begin{aligned} X_F &= \varphi X_{F_1} \\ h_* X_F &= \det h_* \cdot X_{F_2}. \end{aligned}$$

This implies (2.3). If p is a singular point of F_1, then (2.4) follows from (2.3) by Lemma 2.7. ∎

Denote by $M_2(\mathbb{R})$ the space of all 2×2 matrices over $\mathbb{R}$. Let $P : M_2(\mathbb{R}) \longrightarrow \mathbb{R}$ be an invariant, homogeneous function of degree m, that is, it satisfies (i) $P(QAQ^{-1}) = P(A)$ for all $Q \in GL(2)$, and (ii) $P(cA) = c^m P(A)$ for some $m \in \mathbb{Z}$. Then we have

$$P\left((X_{F_2})_*(q)\right) = c^m P\left((X_{F_1})_*(p)\right),$$

where $c = \frac{\varphi(p)}{\det h_*(p)}$. Therefore we immediately have

Proposition 2.9. *If m is even and F defines a singularity at p, then the sign of $P\Big((X_F)_*(p)\Big)$ is an invariant of singularity.*

Example 2.10. The following are invariant homogeneous polynomials of degrees as indicated:

(1) $\mathrm{Tr(A)}$, the trace of A, is of degree 1.

(2) $\det(A)$, the determinant of A, is of degree 2.

(3) $P_c(A) = (\mathrm{Tr}^2 + \mathrm{c} \cdot \det)(\mathrm{A})$ is of degree 2 for any number $c \in \mathbb{R}$.

Example 2.11. Let $F(x, y) = (x, y, z(x, y))$ be an immersion into H^1, defined on a neighborhood of $(0, 0)$ with $z(x, y) = Ax^2 + 2Bxy + Cy^2$ for some constant A, B and C. We have

$$\begin{aligned} X_F &= -\hat{\theta}(F_y)\frac{\partial}{\partial x} + \hat{\theta}(F_x)\frac{\partial}{\partial y} \\ &= (\ (-2B-1)x - 2Cy,\ 2Ax + (2B-1)y\ ). \end{aligned}$$

Clearly $X_F(0,0) = 0$. So $(0,0)$ is a singular point of F. What is the type of the singularity defined by F at $(0,0)$? It is up to the value of $D = B^2 - AC$ which is called the **discriminant** of F. In fact, after a simple calculation, we have (putting $S = (X_F)_*(0,0)$)

$$S = \begin{pmatrix} -2B-1 & -2C \\ 2A & 2B-1 \end{pmatrix}$$
$$\det(S) = 1 - 4(B^2 - AC).$$

Clearly $D = \frac{1}{4}$ means that F at $(0,0)$ defines a degenerate singularity. So suppose $D \neq \frac{1}{4}$ we have

Lemma 2.12. *Let $P_c : M_2(\mathbb{R}) \to \mathbb{R}$ be the invariant homogeneous polynomial of degree 2 defined by $P_c = \mathrm{Tr}^2 + \mathrm{c} \cdot \det$ for some $c \in \mathbb{R}$. Then $c = \frac{4}{4D-1}$ is the only value such that $P_c(S) = 0$.*

Proof. We have

$$\begin{aligned} 0 &= P_c(S) \\ &= (\mathrm{Tr}^2 + \mathrm{c}\det)(\mathrm{S}) \\ &= 4 + c(1 - 4D). \end{aligned}$$

So $c = \frac{4}{4D-1}$. ■

For $i = 1, 2$, let $F_i(x,y) = (x, y, z_i(x,y))$ be two immersions into H^1, defined on a neighborhood of $(0,0)$ with $z_i(x,y) = A_i x^2 + 2B_i xy + C_i y^2$ for some constants A_i, B_i and C_i. Suppose the respective **discriminants** are $D_i = B_i^2 - A_i C_i$. The following Proposition shows that different **discriminants** define different singularities.

Proposition 2.13. *If $D_1 \neq D_2$, then F_1 and F_2 define the different singularities at $(0,0)$.*

Proof. Putting $S_i = (X_{F_i})_*(0,0)$ and $c_i = \frac{4}{4D_i-1}$. $D_1 \neq D_2$ means $c_1 \neq c_2$. We have by Lemma 2.12 that $P_{c_1}(S_1) = 0$ and $P_{c_1}(S_2) \neq 0$. This shows that they have different signs, and hence different singularities by Proposition 2.9. ■

3. Presentation of Singularities

Let $F(x,y) = (x, y, z(x,y))$ be an immersion into H^1 with $z(x,y) = a \cdot x^2 + 2b \cdot xy + c \cdot y^2$. Here $a(x,y), b(x,y)$ and $c(x,y)$ are three C^∞ functions (each is defined on a neighborhood of $(0,0)$). It is easy to see that $(0,0)$ is a singular point of F, so F at $(0,0)$ defines a singularity. Conversely it is also true that every singularity $\Im$ has such a presentation.

Proposition 3.1. *Suppose $\Im$ is a singularity. Then there exist C^∞ functions $a(x,y), b(x,y)$ and $c(x,y)$ defined on a neighborhood of $(0,0)$ such that $\Im$ can be defined by the immersion $F(x,y) = (x,y,z(x,y))$ at $(0,0)$. Here $z(x,y) = a \cdot x^2 + 2b \cdot xy + c \cdot y^2$.*

Proof. We choose a arbitrary presentation of $\Im$ and denote it by $\varphi : U \to H^1$. After a translation and a left translation on $\mathbb{R}^2$ and H_1, respectively, we can assume that $(0,0) \in U,\ \varphi(0,0) = (0,0,0)$ and $\Im$ is defined by φ at $(0,0)$.

Suppose $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$, then $x(0,0) = y(0,0) = z(0,0) = 0$. We have

$$X_\varphi = -\hat{\theta}(\varphi_v)\frac{\partial}{\partial u} + \hat{\theta}(\varphi_u)\frac{\partial}{\partial v} \tag{3.1}$$
$$= (\ -(z_v + xy_v - yx_v),\ z_u + xy_u - yx_u\ ).$$

Since $\Im$ is a singularity, so $(0,0)$ is a singular point of φ, that is, $X_\varphi(0,0) = 0$. From (3.1) we can see that this is equivalent to that $(0,0)$ is a critical point of $z(u,v)$, i.e., $dz(0,0) = 0$.

On the other hand, let $\pi : H_1 \to \mathbb{R}^2$ be the projection from H_1 to $\mathbb{R}^2$ along z-axis. Putting $h = \pi \circ \varphi$, that is, $h(u,v) = (x(u,v), y(u,v))$. We will show that (V,h) defines a coordinates chart for some neighborhood $V \subset U$ of $(0,0)$ and by the inverse function theorem this just follows from that $h_*(0,0)$ is invertible. In fact, since φ is an immersion, we see that $\varphi_u = (x_u, y_u, z_u)$ and $\varphi_v = (x_v, y_v, z_v)$ are independent at each point (u,v). However

$$\varphi_u(0,0) = (x_u, y_u, z_u)(0,0) = (x_u(0,0), y_u(0,0), 0)$$
$$\varphi_v(0,0) = (x_v, y_v, z_v)(0,0) = (x_v(0,0), y_v(0,0), 0).$$

Therefore

$$h_*(0,0) = \begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix}(0,0)$$

is invertible.

Let $F = \varphi \circ h^{-1}$. Then F can be expressed by a graph, that is,

$$F(x,y) = (x,y,z(x,y)),$$

and $z(0,0) = 0,\ dz(0,0) = 0$. By definition F at $(0,0)$ defines the same singularity $\Im$.

Finally that $z(0,0) = 0,\ dz(0,0) = 0$ imply that $z(x,y) = a \cdot x^2 + 2b \cdot xy + c \cdot y^2$ for some suitable smooth functions $a(x,y), b(x,y)$ and $c(x,y)$ defined on a neighborhood of $(0,0)$. ∎

Let $z(x,y) = a \cdot x^2 + 2b \cdot xy + c \cdot y^2$. What is the type of the singularity defined by $F(x,y) = (x, y, z(x,y))$ at $(0,0)$? The answer to this question is given by the following Example:

Example 3.2. Suppose $F(x,y) = (x, y, z(x,y) = a \cdot x^2 + 2b \cdot xy + c \cdot y^2)$. From (3.1), we have

$$X_F = (\ -(z_y + x),\ z_x - y\ ),$$

and hence

$$(X_F)_*(0,0) = \begin{pmatrix} -2B-1 & -2C \\ 2A & 2B-1 \end{pmatrix},$$

where $A = a(0,0), B = b(0,0), C = c(0,0)$. We also call the value $D = B^2 - AC$ the **discriminant** of F. Therefore, Putting $\tilde{z}(x,y) = Ax^2 + 2Bxy + Cy^2$, we see that $F(x,y) = (x, y, z(x,y))$ at $(0,0)$ defined the same type of singularity as the one defined by $\tilde{F}(x,y) = (x, y, \tilde{z}(x,y))$ at $(0,0)$ as we can tell from Example 2.11. Here the same type means that they have the same value of **discriminant**. Again singularities belonging to different types are different. It is easy to see that

$$\det\Big(Hess_{(0,0)}(z)\Big) = -4D, \tag{3.2}$$

here $Hess_{(0,0)}(z)$ is the Hessian of $z(x,y)$ at $(0,0)$, and hence the type of the singularity defined by F at $(0,0)$ is up to the determinant of the Hessian of $z(x,y)$ at $(0,0)$.

As a consequence, we have

Proposition 3.3. *The space of all singularities has continuous moduli.*

Proposition 3.4. *We have that $D < \frac{1}{4}$ ($> \frac{1}{4}$) if and only if the singularity is elliptic (hyperbolic). Therefore, $D = \frac{1}{4}$ means that the singularity is degenerate.*

Proof of Theorem 1.1. Let $F(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$ be an immersion from $U \subset \mathbb{R}^2$ into $W \subset H^1$ such that $F(p) = q = (q_1, q_2, q_3)$. Suppose $p = (p_1, p_2)$ is a singular point of F. After a translation by $-(p_1, p_2)$ and a left translation by$-(q_1, q_2, q_3)$ on $\mathbb{R}^2$ and on H^1, respectively, we obtain a new immersion $\widetilde{F}(\widetilde{u}, \widetilde{v}) = \big(\widetilde{x}(\widetilde{u}, \widetilde{v}), \widetilde{y}(\widetilde{u}, \widetilde{v}), \widetilde{z}(\widetilde{u}, \widetilde{v})\big)$ such that $\widetilde{F}(0,0) = (0,0,0)$, that is,

$$
\begin{aligned}
\widetilde{u} &= u - p_1,\\
\widetilde{v} &= v - p_2;\\
\widetilde{x}(\widetilde{u},\widetilde{v}) &= x(u,v) - q_1,\\
\widetilde{y}(\widetilde{u},\widetilde{v}) &= y(u,v) - q_2,\\
\widetilde{z}(\widetilde{u},\widetilde{v}) &= z(u,v) + q_1 y(u,v) - q_2 x(u,v) - q_3.
\end{aligned}
\tag{3.3}
$$

We note that F at p and $\widetilde{F}$ at $(0,0)$ define the same singularity. We compute, at p,

$$
\begin{aligned}
\det (X_F)_* &= \det \begin{pmatrix} z_{uu}+xy_{uu}-yx_{uu} & z_{uv}+xy_{uv}-yx_{uv}+x_v y_u - x_u y_v \\ z_{vu}+xy_{vu}-yx_{vu}+x_u y_v - x_v y_u & z_{vv}+xy_{vv}-yx_{vv} \end{pmatrix}\\
&= \det \Big(Hess_p(z) + q_1 \cdot Hess_p(y) - q_2 \cdot Hess_p(x) \Big) + \Big(\det (\pi \circ F)_* (p) \Big)^2\\
&= \det Hess_{(0,0)}(\widetilde{z}) + \Big(\det \big(\pi \circ \widetilde{F}\big)_* (0,0) \Big)^2\\
&= \det \begin{pmatrix} \widetilde{z}_{\widetilde{u}\widetilde{u}} & \widetilde{z}_{\widetilde{u}\widetilde{v}} \\ \widetilde{z}_{\widetilde{v}\widetilde{u}} & \widetilde{z}_{\widetilde{v}\widetilde{v}} \end{pmatrix}(0,0) + \left(\det \begin{pmatrix} \widetilde{x}_{\widetilde{u}} & \widetilde{x}_{\widetilde{v}} \\ \widetilde{y}_{\widetilde{u}} & \widetilde{y}_{\widetilde{v}} \end{pmatrix}(0,0) \right)^2 .
\end{aligned}
\tag{3.4}
$$

Where $\pi : H^1 \to \mathbb{R}^2$ is the projection onto the first two variables.

On the other hand, from the argument of the proof of Proposition 3.1, let $\widetilde{h}(\widetilde{u},\widetilde{v}) = \big(\widetilde{x}(\widetilde{u},\widetilde{v}), \widetilde{y}(\widetilde{u},\widetilde{v})\big)$, that is, $\widetilde{h} = \pi \circ \widetilde{F}$. Then $F = \widetilde{F} \circ \widetilde{h}^{-1}$ can be expressed by a graph $F(\widetilde{x},\widetilde{y}) = (\widetilde{x},\widetilde{y},\widetilde{z}(\widetilde{x},\widetilde{y}) = a\widetilde{x}^2 + 2b\widetilde{x}\widetilde{y} + c\widetilde{y}^2)$. By a direct computation and note that $d\widetilde{z}(0,0) = 0$, we get that

$$
\begin{aligned}
&\begin{pmatrix} \widetilde{z}_{\widetilde{x}\widetilde{x}} & \widetilde{z}_{\widetilde{x}\widetilde{y}} \\ \widetilde{z}_{\widetilde{y}\widetilde{x}} & \widetilde{z}_{\widetilde{y}\widetilde{y}} \end{pmatrix}(0,0)\\
&= \begin{pmatrix} \widetilde{u}_{\widetilde{x}} & \widetilde{v}_{\widetilde{x}} \\ \widetilde{u}_{\widetilde{y}} & \widetilde{v}_{\widetilde{y}} \end{pmatrix} \begin{pmatrix} \widetilde{z}_{\widetilde{u}\widetilde{u}} & \widetilde{z}_{\widetilde{u}\widetilde{v}} \\ \widetilde{z}_{\widetilde{v}\widetilde{u}} & \widetilde{z}_{\widetilde{v}\widetilde{v}} \end{pmatrix} \begin{pmatrix} \widetilde{u}_{\widetilde{x}} & \widetilde{u}_{\widetilde{y}} \\ \widetilde{v}_{\widetilde{x}} & \widetilde{v}_{\widetilde{y}} \end{pmatrix}(0,0).
\end{aligned}
\tag{3.5}
$$

Combining (3.2), (3.4) and (3.5), we get,

$$
\begin{aligned}
\det (X_F)_*(p) &= \Bigg[\det \begin{pmatrix} \widetilde{z}_{\widetilde{x}\widetilde{x}} & \widetilde{z}_{\widetilde{x}\widetilde{y}} \\ \widetilde{z}_{\widetilde{y}\widetilde{x}} & \widetilde{z}_{\widetilde{y}\widetilde{y}} \end{pmatrix} \det \begin{pmatrix} \widetilde{x}_{\widetilde{u}} & \widetilde{x}_{\widetilde{v}} \\ \widetilde{y}_{\widetilde{u}} & \widetilde{y}_{\widetilde{v}} \end{pmatrix}^2\\
&\quad + \det \begin{pmatrix} \widetilde{x}_{\widetilde{u}} & \widetilde{x}_{\widetilde{v}} \\ \widetilde{y}_{\widetilde{u}} & \widetilde{y}_{\widetilde{v}} \end{pmatrix}^2 \Bigg] (0,0)\\
&= (-4D+1) \left(\det \begin{pmatrix} \widetilde{x}_{\widetilde{u}} & \widetilde{x}_{\widetilde{v}} \\ \widetilde{y}_{\widetilde{u}} & \widetilde{y}_{\widetilde{v}} \end{pmatrix}(0,0) \right)^2 ,
\end{aligned}
\tag{3.6}
$$

where $D = \det \begin{pmatrix} a(0,0) & b(0,0) \\ b(0,0) & c(0,0) \end{pmatrix}$. Thus $I = -4D + 1$. This completes the proof of Theorem 1.1 because Proposition 3.4.

4. Stable Singularities

Let $U \subset \mathbb{R}^2$ be an open subset and Let $\aleph^1(U)$ be the space of all smooth vector fields on U equipped with the C^1-topology. We have the following perturbation Proposition

Proposition 4.1. *Let $X : U \to \mathbb{R}^2$ be a smooth vector field and $p \in U$ is a singular point of X (i.e. $X(p) = 0$) such that $X_*(p)$ is invertible. Then for any neighborhood $W \subset U$ of p, there is a neighborhood $\Omega \subset \aleph^1(U)$ of X such that for any vector field $Y \in \Omega$ there is a unique singular point $\tilde{p} \in W$ of Y which is also nondegenerate, (i.e. $Y_*(\tilde{p}) is invertible$).*

Proof. This is a standard result. The reader can refer to Chapter 16 in [6]. ∎

As an application, let (M, θ) be a contact 3-manifold, we have

Proposition 4.2. *Suppose $p \in \Sigma$ is a nondegenerate singular point of $F \in$ Imm$^2(\Sigma, M)$. Then for any neighborhood W of p, there is a neighborhood $\Omega \subset$Imm^2 (Σ, M) of the immersion F such that for any immersion $\widetilde{F} \in \Omega$ there is a unique singular point $\tilde{p} \in W$ of $\widetilde{F}$ which is also nondegenerate.*

Proof. Let $(U; u, v)$ be a local coordinates of p. Let $\aleph^1(U)$ be the set of all smooth vector fields on U with C^1-topology. Then the mapping from $Imm^2(\Sigma, M)$ to $\aleph^1(U)$ defined by $F \longmapsto X_F = -\theta(F_v)\frac{\partial}{\partial u} + \theta(F_u)\frac{\partial}{\partial v}$ is continuous. Therefore, by Proposition 4.1, we have the Proposition. ∎

We immediately have the following Corollary

Corollary 4.3. *A nondegenerate singular point is isolated.*

We can sharpen Proposition 4.2 as following:

Proposition 4.4. *A C^2-perturbation preserves elliptic, or hyperbolic, singularities. That is they are **stable** in C^2 topology in this sense.*

Proof. This is because that the determinant of $(X_F)_*$ at some singular point just involves derivatives of F with order up to 2. ∎

The preservation of singularities is in the sense of Proposition 4.2. For example, W, Ω can be chosen so that if p is elliptic then the unique singular point $\tilde{p}$ is also elliptic.

Theorem 1.5 is just Proposition 4.2 plus Proposition 4.4.

we now give some examples to say that degenerate singular points behave completely differently under perturbation.

Example 4.5. Let $W \in \mathbb{R}^2$ be an open subset, we consider the embedding $F : W \hookrightarrow H^1$ (with the standard contact form $\hat{\theta} = dz + xdy - ydx$) which be expressed by a graph:

$$F : (x, y) \longmapsto (x, y, z(x, y)).$$

F induces a vector field $X_F = -\hat{\theta}(F_y)\frac{\partial}{\partial x} + \hat{\theta}(F_x)\frac{\partial}{\partial y}$. The set S_F is just the set of all zeros of the following system of equations:

$$\begin{aligned} \hat{\theta}(F_x) &= z_x - y = 0 \\ \hat{\theta}(F_y) &= z_y + x = 0. \end{aligned} \tag{4.1}$$

(1) Suppose $(0, 0) \in W$ and $F(x, y) = (x, y, z = x^2 + y^2)$. Then $(0, 0)$ is a nondegenerate singular point of F. Now let us consider a small perturbation $F^\epsilon(x, y) = (x, y, z^\epsilon(x, y) = x^2 + y^2 + \epsilon x)$. It is easy to show that $S_{F^\epsilon} = \left\{(\frac{-2\epsilon}{5}, \frac{\epsilon}{5})\right\}$ and also $(\frac{-2\epsilon}{5}, \frac{\epsilon}{5})$ is nondegenerate. This means that, under the perturbation F^ϵ, the singular point $(0, 0)$ of F does not vanish, but simply shifts slightly to $(\frac{-2\epsilon}{5}, \frac{\epsilon}{5})$.
The degenerate singular point of the embedding $F(x, y) = (x, y, z(x, y) = x^2y - xy)$ behaves completely differently under some perturbation.

(2) Let $z^\epsilon(x, y) = x^2y - xy + \epsilon y$. The system (4.1) is

$$\begin{aligned} (2x - 1)y &= y, \\ x^2 + \epsilon &= 0. \end{aligned}$$

We see that
  (i) If $\epsilon = 0$, then $(0, 0)$ is the only singular point and it is degenerate.
  (ii) If $\epsilon > 0$, there is no singular point.
  (iii) If $\epsilon < 0$ (and $|\epsilon|$ is small enough), then $S_{F^\epsilon} = \left\{(\pm\sqrt{|\epsilon|}, 0)\right\}$, and each singular point is nondegenerate. In addition, $(\sqrt{|\epsilon|}, 0)$ is elliptic and $(-\sqrt{|\epsilon|}, 0)$ is hyperbolic.

We see that under these perturbations the degenerate singular point either vanishes (for $\epsilon > 0$) or decomposes into two nondegenerate singular points at a distance of order $\sqrt{|\epsilon|}$ from it (for $\epsilon < 0$ small enough). Thus the singular point of $F^0(x, y) = (x, y, x^2y - xy)$ is unstable.

(3) Finally suppose $W \subset \mathbb{R}^2$ is a disc with radius small enough, consider the embeddings $F^\epsilon(x, y) = (x, y, z^\epsilon = \sin xy + \epsilon x)$. Then if $\epsilon = 0$, $S_{F^0} = \{(0, y) | y \in \mathbb{R}\} \cap W$ which is a line segment, but if $\epsilon \neq 0$, then there is no any singular point on W. So these singular points are not stable. Note that all the singular points of F^0 are degenerate.

5. Generic Immersions

In this Section, let Σ be a surface and (M,θ) be a contact 3-manifold. We will use the transversality theorem of Thom to show that almost all immersions do not have any degenerate singular point. First we give a simple review on the theorem of Thom. The reader can refer to [1, 2, 7].

Definition 5.1. Two linear subspaces of a finite-dimensional linear space are said to be transversal if their sum is the whole space.

Definition 5.2. Let $f: A \to B$ be a C^∞ map of a manifold A to a manifold B, containing a submanifold C. The map f is said to be transversal to C at the point a of A if either $f(a)$ does not belong to C or the image of the tangent space to A at a under the derivative $f_*(a)$ is transversal to the tangent space to C:

$$f_*(a)T_aA + T_{f(a)}C = T_{f(a)}B.$$

The map f is said to be transversal to C if it is transversal to C at every point of A.

Let $J^k(A,B)$ be the space of k-jets of maps from A to B. It is a differentiable manifold. Locally the manifold $J^k(A,B)$ may be represented as the space of Taylor polynomials of degree k.

Theorem 5.3. [the transversality theorem of Thom]. *Let Σ be a closed surface and C a closed submanifold of the space of 2-jets $J^2(\Sigma, M)$. Then the set of maps $F:\Sigma\to M$, whose 2-jet extensions are transversal to C, is an open and dense subset of $Map^s(\Sigma, M),\ s>2$.*

Definition 5.4. A **good** immersion is an immersion which does not have any degenerate singular point.

The following Corollary says that generically an immersion is good.

Theorem 5.5. *Suppose Σ is a closed surface. Then the set of all **good** immersions $F:\Sigma\to M$ is an open and dense subset of $Imm^s(\Sigma, M),\ s>2$.*

Proof. Consider the jet space $J^2(\Sigma, M)$. We first define a closed submanifold C of $J^2(\Sigma,M)$ as following: Choose a local coordinates (u,v) on Σ and a local coordinates (x,y,z) on M in the neighborhoods of points p and $q=F(p)$, respectively. In addition, we assume that the coordinates (x,y,z) is chosen such that $\theta = dz + xdy - ydx$. The map F is given locally by

$$F(u,v) = (x(u,v), y(u,v), z(u,v)).$$

The 2-jet is determined by the following numbers:

$$\{u,v\}\,;\{x,y,z\}\,;\{x_u,x_v,y_u,y_v,z_u,z_v\}\,;$$
$$\{x_{uu},x_{uv}=x_{vu},x_{vv},y_{uu},y_{uv}=y_{vu},y_{vv},z_{uu},z_{uv}=z_{vu},z_{vv}\}\,.$$

These numbers give a local coordinates in the space $J^2(\Sigma,M)$. Let C be the closed submanifold locally defined by the following equations

$$\tilde{f}=z_u+xy_u-yx_u=0 \qquad (5.1)$$
$$\tilde{g}=z_v+xy_v-yx_v=0.$$

Here $\tilde{f},\tilde{g}$ are two smooth functions defined on open subset of $J^2(\Sigma,M)$ and $d\tilde{f}\wedge d\tilde{g}\neq 0$. This definition is independent of the choice of local coordinates $(u,v);(x,y,z)$, so it defines a closed submanifold of $J^2(\Sigma,M)$.

Next the 2-jet extension j^2F of a map F is a map from Σ to $J^2(\Sigma,M)$, associating to each point p of Σ the 2-jet of F at that point. That is, j^2F is the map locally is defined by

$$(u,v)\to(u,v,F(u,v),F_u(u,v),F_v(u,v),F_{uu}(u,v),F_{uv}(u,v)=F_{vu}(u,v),F_{vv}(u,v)),$$

where $F(u,v)=(x(u,v),y(u,v),z(u,v))$. By definition j^2F is transversal to C if, at each point (u,v), either

(i) $(j^2F)(u,v)$ does not belong to C, or

(ii) $\{(j^2F)_u(u,v),(j^2F)_v(u,v),T_{(j^2F)(u,v)}C\}$ generates the tangent space $T_{(j^2F)(u,v)}J^2(\Sigma,M)$ at $(j^2F)(u,v)$. By the following Lemma, this just means that F is good, provided F is an immersion. Therefore this Theorem follows from Theorem 5.3 and the openness of *Imm*(Σ,M) in *Map*$^s(\Sigma,M)$. ■

Lemma 5.6. *Suppose that $F:\Sigma\to M$ is an immersion. Then j^2F is transversal to C if and only if F is good.*

Proof. We will show that

(i) $j^2F(u,v)$ does not belong to C if and only if (u,v) is a nonsingular point, and

(ii) $\{(j^2F)_u(u,v),(j^2F)_v(u,v),T_{(j^2F)(u,v)}C\}$ generates the tangent space $T_{(j^2F)(u,v)}J^2(\Sigma,M)$ if and only if (u,v) is a nondegenerate singular point. By the definition of C (see (5.1)), (i) is obvious. For (ii), since $d\tilde{f}\wedge d\tilde{g}\neq 0$ and $d\tilde{f}\wedge d\tilde{g}=0$ on C, we have that $\{(j^2F)_u(u,v),(j^2F)_v(u,v),T_{(j^2F)(u,v)}C\}$ generates the tangent space $T_{(j^2F)(u,v)}J^2(\Sigma,M)$ if and only if $\langle d\tilde{f}\wedge d\tilde{g}\,,\,(j^2F)_u(u,v)\wedge(j^2F)_v(u,v)\rangle\neq 0$. Here $<,>$ is a pairing.

On the other hand, let ∇ be the standard gradient on the coordinates neighborhood in $J^2(\Sigma, M)$. Then we have

$$\nabla\tilde{f} = (0,0;y_u,-x_u,0;-y,0,x,0,1,0;0,\cdots,0)$$
$$\nabla\tilde{g} = (0,0;y_v,-x_v,0;0,-y,0,x,0,1;0,\cdots,0).$$

So

$$(j^2F)_u\cdot\nabla\tilde{f} = -yx_{uu}+xy_{uu}+z_{uu}$$
$$(j^2F)_u\cdot\nabla\tilde{g} = x_uy_v-x_vy_u-yx_{vu}+xy_{vu}+z_{vu}$$
$$(j^2F)_v\cdot\nabla\tilde{f} = x_vy_u-y_vx_u-yx_{uv}+xy_{uv}+z_{uv}$$
$$(j^2F)_v\cdot\nabla\tilde{g} = -yx_{vv}+xy_{vv}+z_{vv}.$$

Therefore,

$$\begin{aligned}
0 &\neq \left\langle d\tilde{f}\wedge d\tilde{g}\ ,\ (j^2F)_u(u,v)\wedge(j^2F)_v(u,v)\right\rangle\\
&= \begin{vmatrix} d\tilde{f}((j^2F)_u) & d\tilde{g}((j^2F)_u) \\ d\tilde{f}((j^2F)_v) & d\tilde{g}((j^2F)_v)\end{vmatrix}(u,v)\\
&= \begin{vmatrix} (j^2F)_u\cdot\nabla\tilde{f} & (j^2F)_u\cdot\nabla\tilde{g} \\ (j^2F)_v\cdot\nabla\tilde{f} & (j^2F)_v\cdot\nabla\tilde{g}\end{vmatrix}(u,v)\\
&= (z_{uu}+xy_{uu}-yx_{uu})(z_{vv}+xy_{vv}-yx_{vv})-(z_{uv}+xy_{uv}-yx_{uv})^2+(x_uy_v-x_vy_u)^2\\
&= \begin{vmatrix} f_u & f_v \\ g_u & g_v\end{vmatrix}(u,v),
\end{aligned}$$

where $f=\theta(F_u)=z_u+xy_u-yx_u,\ g=\theta(F_v)=z_v+xy_v-yx_v$. This means that (u,v) is nondegenerate. ■

The following Theorem says that generically the singular set of an immersion from a compact surface is finite.

Theorem 5.7. *Let $F:\Sigma\to M$ be a good immersion. Then the singular set S_F is discrete. Moreover if Σ is compact then S_F is finite.*

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