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CRITICAL BEHAVIOR FOR AN ORIENTED PERCOLATION WITH LONG-RANGE INTERACTIONS IN DIMENSION $d > 2$

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Abstract. We consider a model of oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}$, $d > 2$, with long-range interactions, in which the bond occupation probability decays as the α-stable distribution with $\alpha = 1$. We use the lace expansion to get an L^1 infrared bound estimate which implies several critical exponents via the triangle condition.

1. INTRODUCTION

The Model

In this paper we introduce a certain type of oriented percolation model which may be regarded as an *infinite layer long-range* model. It is defined as follows. We consider the graph $\mathbb{Z}^d \times \mathbb{Z}$ and oriented bonds $((x, n), (y, n+1))$, $x, y \in \mathbb{Z}^d$, $n \in \mathbb{Z}$. Fix a parameter $\lambda > 0$, to each $((x, n), (y, n+1))$ we associate a random variable taking value 1 (open) with probability $p^\lambda_{x,y}$ and 0 (close) with probability $1 - p^\lambda_{x,y}$; the random variables are assumed to be totally independent. We require that $p^\lambda_{x,y} = p^\lambda_{y,x} = p^\lambda_{0,y-x}$, and define $p^\lambda_{0,x}$ to be

$$p^\lambda_{0,x} = \sum_{l=1}^{\infty} \frac{\lambda \mathbb{1}_{\{(l-1)L<\|x\|_\infty \le lL\}}}{l^2 \sum_{y\in\mathbb{Z}^d} \mathbb{1}_{\{(l-1)L<\|y\|_\infty \le lL\}}}, \tag{1.1}$$

where $\|x\|_\infty = \max_{\{j=1,2\ldots,d\}} |x_j|$, $\mathbb{1}_{\{(l-1)L<\|x\|_\infty \le lL\}}$ is the indicator function and L is a controlling factor. Note that $p^\lambda_{0,x} = O(\lambda \|x\|_\infty^{-d-1} L^{-d})$; thus it decays as the α-stable distribution with $\alpha = 1$. The factor L^{-d} is necessary to control the

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convergence of the lace expansion for the dimension $d = 3$. We believe that the results of this paper also hold without the factor L^{-d} for dimension d being large enough.

We write $(y, m) \longrightarrow (x, n)$ to denote the event that there is an oriented open connected path from (y, m) to (x, n), i.e., there is a sequence of sites $(u_m, m) = (y, m)$, $(u_{m+1}, m+1), ..., (u_n, n) = (x, n)$ such that the oriented bonds $((u_{j-1}, j-1), (u_j, j))$, $j = m+1, ..., n$ are all open. The joint probability distribution of the bond random variables is denoted P_λ, with corresponding expectation E_λ. Define

$$\psi_\lambda(x,n) = \begin{cases} P_\lambda((0,0) \longrightarrow (x,n)) & \text{if } n > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\varphi_\lambda(x,n) = \delta(x,n) + \psi_\lambda(x,n), \tag{1.2}$$

where $\delta(x,n)$ is Kronecker's delta on $\mathbb{Z}^d \times \mathbb{Z}$. For brevity we write in the sequel $\sum_{(x,n)} = \sum_{x \in \mathbb{Z}^d, n \in \mathbb{Z}}$ and $\sum_x = \sum_{x \in \mathbb{Z}^d}$ in this paper. The Fourier-Laplace transforms are

$$\begin{aligned} \widehat{\psi}_\lambda(k, s+it) &= \sum_{(x,n)} e^{ik\cdot x} e^{n(s+it)} \psi_\lambda(x,n), \\ \widehat{\varphi}_\lambda(k, s+it) &= \sum_{(x,n)} e^{ik\cdot x} e^{n(s+it)} \varphi_\lambda(x,n), \\ Z_{\lambda,n}(k) &= \sum_x e^{ik\cdot x} \varphi_\lambda(x,n), \quad n \in \mathbb{Z} \end{aligned}$$

for $(k,t) \in [-\pi,\pi]^d \times [-\pi,\pi]$ and $s \in \mathbb{R}$. Let $C(0,0) = \{(x,n) : (0,0) \longrightarrow (x,n)\}$ and denotes its cardinality by $|C(0,0)|$. We have

$$\begin{aligned} E_\lambda(|C(0,0)|) &= E_\lambda\big(\sum_{(x,n)} \mathbb{1}_{\{(x,n)\in C(0,0)\}}\big) \\ &= \sum_{(x,n)} E_\lambda\big(\mathbb{1}_{\{(x,n)\in C(0,0)\}}\big) = \widehat{\varphi}_\lambda(0,0), \end{aligned} \tag{1.3}$$

and

$$\widehat{\varphi}_\lambda(0,0) = 1 + \sum_{n=1}^{\infty} Z_{\lambda,n}(0) \tag{1.4}$$

For $(0,0) \longrightarrow (x, n+m)$, there exists a vertex (y,m) such that $(0,0) \longrightarrow (y,m)$ and $(y,m) \longrightarrow (x,n)$. Since the two events are independent, by translation invariant, we have

$$
\begin{aligned}
Z_{\lambda,n+m}(0) &= \sum_x \varphi_\lambda(x, n+m) \\
&\le \sum_x \sum_y \varphi_\lambda(y, m)\varphi_\lambda(x-y, n) \qquad (1.5)\\
&= Z_{\lambda,n}(0) Z_{\lambda,m}(0).
\end{aligned}
$$

From the subadditive limit theorem, see for Example [9, Theorem II.2], for every $\lambda > 0$, there exists m_λ such that

$$
-m_\lambda = \lim_{n\to\infty} \frac{\log Z_{\lambda,n}(0)}{n} \quad \text{and} \quad Z_{\lambda,n}(0) \ge e^{-nm_\lambda} \qquad (1.6)
$$

for all $n \in \mathbb{N}$. Clearly, e^{m_λ} is the radius of convergence of the power series $\widehat{\varphi}_\lambda(0, z)$. Since $E_\lambda(|C(0,0)|)$ is non-decreasing with respect to λ, there exists a critical point $\lambda_c = \sup\{\lambda : E_\lambda(|C(0,0)|) < \infty\}$. It is seen that

$$
\lambda_0 := \frac{6}{\pi^2} \le \lambda_c,
$$

due to $\sum_x p_{0,x}^\lambda = \frac{\pi^2\lambda}{6}$. There is another critical value traditionally defined as $\lambda_T = \inf\{\lambda : P_\lambda(|C(0,0)| = \infty) > 0\}$[1,9]. For any $0 < \|x\|_\infty \le L$,

$$
p_{0,x}^\lambda = \frac{\lambda}{\sum_y \mathbb{1}_{\{0<\|y\|_\infty \le L\}}} \ge \frac{\lambda}{(2L+1)^d},
$$

which implies $\lambda_T < (2L+1)^d$. Since our model is a kind of independent translation invariant bond percolation models, we have, by [1,Theorem 1.1], $\lambda_c = \lambda_T$.

Main Results

The paper is mainly on the infrared bond estimate; there is no general proof of infrared bound for a given percolation model. There are indications that the infrared bound is violated in less than dimension six for nearest-neighbor nonoriented percolation model [8]. In [14], it is obtained the infrared bound of the nearest-neighbor percolation model in high dimensions and spread-out model for dimension $d > 6$. We obtain in this paper the following infrared bound of our model for dimension $d > 2$.

Theorem 1.1. *For our infinite layer long-range* model on $\mathbb{Z}^d \times \mathbb{Z}$ with $d > 2$, there exists an L_0 (depending on d) such that for all $L \ge L_0$, $(k,t) \in [-\pi,\pi]^d \times [-\pi,\pi]$, $s \in (0,1]$ and $\lambda \le \lambda_c$ we have

$$
|\widehat{\varphi}_\lambda(k, m_\lambda - s + it)| \le \frac{1}{c_1|t| + c_2 s + c_3\|k\|_1},
$$

where c_j, $j = 1, 2, 3$, are constants depending on d and L.

As usual, the critical exponents γ, β, δ and Δ_{t+1} are defined as follows:

$$
\begin{aligned}
&E_\lambda(|C(0,0)|) &&\sim (\lambda_c - \lambda)^{-\gamma} &&\text{as} \quad \lambda \uparrow \lambda_c, \\
&P_\lambda(|C(0,0)| = \infty) &&\sim (\lambda - \lambda_c)^{\beta} &&\text{as} \quad \lambda \downarrow \lambda_c, \\
&\sum_{1 \le n \le \infty} P_{\lambda_c}(|C(0,0)| = n)[1 - e^{-nh}] &&\sim h^{\delta} &&\text{as} \quad h \downarrow 0, \\
&\frac{E_\lambda(|C(0,0)|^{t+1})}{E_\lambda(|C(0,0)|^{t})} &&\sim (\lambda_c - \lambda)^{-\Delta_{t+1}} &&\text{as} \quad \lambda \uparrow \lambda_c
\end{aligned}
\tag{1.7}
$$

for $t \in \mathbb{N}$, where we write $A(r) \sim B(r)$ as $r \uparrow r_0$, resp. $r \downarrow r_0$, means that there are universal constants c_1, c_2 such that $c_1 B(r) \le A(r) \le c_2 B(r)$ as the parameter $r \uparrow r_0$, resp. $r \downarrow r_0$. It was proved in [18] that for the nearest-neighbor oriented percolation model in high dimensions and spread-out oriented model in dimension $d > 4$, the critical exponents β, γ, δ and Δ_{t+1} exist and take their mean-field values. The same results were extended to the contact process [22]. In the following theorem, we use Theorem 1.1 and the triangle condition to prove that $\gamma = 1$. Then the other critical exponents δ, β and Δ_{t+1} can be obtained (see [5],[26]).

Theorem 1.2. *For our infinite layer long-range model on $\mathbb{Z}^d \times \mathbb{Z}$ with $d > 2$, there exists an L_0 (depending on d) such that for all $L \ge L_0$, the critical exponents are $\gamma = 1$, $\beta = 1$, $\delta = \frac{1}{2}$ and $\Delta_{t+1} = 2$ for $t \in \mathbb{N}$.*

Remark 1.1. Theorem 1.2 implies that there is no infinite cluster at the critical value $\lambda = \lambda_c$ for our model in dimension $d > 2$. There have been literatures[3, 4, 21] to discuss the cluster infinity and related properties at the critical values for non-oriented long-range percolation models with polynomial decays.

Remark 1.2. For each percolation model, there exists a upper critical dimension d_c such that the critical behavior is the same as the mean-field behavior when dimension $d > d_c$. If the random walk with one-step transition function $p^\lambda_{o,x} / \sum_x p^\lambda_{o,x}$ belongs to the domain of an α-stable law, then the upper critical dimension of the oriented percolation is believed to be 2α. For the case $\alpha = 2$, it is proved that, by using the *hyperscaling inequalities* , the upper critical dimension is four [23]. The upper critical dimension is two in our model; this will be the content of a coming paper.

To prove Theorem 1.1, we use the lace expansion which is introduced in a seminal paper of [7] for studying the weakly self-avoiding walk in dimension $d > 4$. The method has also been applied successfully to study the strictly self-avoiding walk

([12, 13]), percolation models ([14], [11]), oriented percolation models ([18, 19]), lattice trees and lattice animals ([15]), networks of self-avoiding walks ([20, 10]), etc. The basic idea of the present work is closely related to that in [18]; however, it should be emphasized that our infrared bound is an L^1 estimates, rather than the L^2 estimate as that appeared in [18] and other works. It is the L^1 estimate makes us to be significantly different from the L^2 arguments in [18]. An L^1 infrared bound estimate for self-avoiding random walks has been studied by Y. Cheng (a 2000 PhD thesis of Temple University).

From the lace expansion, there is a connected function $\Pi_\lambda(x,n)$ such that its Fourier-Laplace transform $\widehat{\Pi}_\lambda(k,z)$ is defined by the renewal equation (see [17])

$$\widehat{\varphi}_\lambda(k,z) = \frac{1+\widehat{\Pi}_\lambda(k,z)}{F_\lambda(k,z)} \tag{1.8}$$

for $\lambda \le \lambda_c$, $\mathrm{Re}(z) < m_\lambda$, where

$$F_\lambda(k,z) = 1 - \lambda_0^{-1}\lambda e^z \widehat{D}(k)(1+\widehat{\Pi}_\lambda(k,z)), \tag{1.9}$$

$$\widehat{D}(k) = \sum_x \varphi_{\lambda_0}(x,1)e^{ik\cdot x}. \tag{1.10}$$

To prove Theorem 1.1, we need the following continuity of two-point functions.

Proposition 1.3. *For our infinite layer long-range model on $\mathbb{Z}^d \times \mathbb{Z}$ with $d > 0$, we have*

(a) $E_\lambda(|C(0,0)|) < \infty$ *if and only if* $m_\lambda > 0$,

(b) $E_\lambda(|C(0,0)|) = \infty$ *and* $m_\lambda = 0$ *if* $\lambda = \lambda_c$,

(c) $\widehat{\varphi}_\lambda(0,r)$ *is continuous at* λ *for* $0 < \lambda < \lambda_c, r < m_\lambda$ *and* $\lim_{\lambda\uparrow\lambda_c}\widehat{\varphi}_\lambda(0,r) = \widehat{\varphi}_{\lambda_c}(0,r)$ *for* $r < 0$.

From [18], we know that Proposition 1.3 holds for finite-range models, and we show that it can also be extended to our model.

Next, we need to estimate $\widehat{D}(k)$ as follows:

Proposition 1.4. *For our infinite layer long-range model on $\mathbb{Z}^d \times \mathbb{Z}$ with $d > 2$, there exists an L_0 (depending on d) such that for $L \ge L_0$, we have*

(a) $|\widehat{D}(k)| \le 1 - \frac{0.12L}{d}\|k\|_1$ *for* $\|k\|_\infty \in [0, \frac{\pi}{4L+1}]$,

(b) $|\widehat{D}(k)| < 0.95$ *for* $\|k\|_\infty \in (\frac{\pi}{4L+1}, \frac{\pi}{L}]$,

(c) $|\widehat{D}(k)| < \frac{9}{10n}$ *for* $\|k\|_\infty \in (\frac{n\pi}{L}, \frac{(n+1)\pi}{L}]$ *with* $n = 1, 2, ..., L-1$.

Finally, we want to control $|\widehat{\Pi}_\lambda(k,z)|$. The following two propositions give us that $|\widehat{\Pi}_\lambda(k,z)|$ decays to zero as L tends to infinity for $\lambda=\lambda_0$ and satisfies a bootstrapping argument for $\lambda\le\lambda_c$, respectively.

Proposition 1.5. *For our infinite layer long-range model on $\mathbb{Z}^d\times\mathbb{Z}$ with $d>2$, there exists an L_1 (depending on d) such that for $L\ge L_1$, we have*

$$\sum_{(x,n)}|\Pi_{\lambda_0}(x,n)|\le\frac{\tau_0}{L},$$
$$\sum_{(x,n)}|n\Pi_{\lambda_0}(x,n)|\le\frac{\tau_1}{L},$$
$$\sum_{(x,n)}\|x\|_1\big|\Pi_{\lambda_0}(x,n)\big|\le\frac{\tau_2(\log L)^{\frac{1}{3}}}{L}$$

for some universal constants τ_0, τ_1 and τ_2.

Proposition 1.6. *For our infinite layer long-range model on $\mathbb{Z}^d\times\mathbb{Z}$ with $d>2$, there exists an L_0 (depending on d) such that for $L\ge L_0$, $\lambda\le\lambda_c$ and $r\le m_\lambda$, (P_4) implies (P_2), where (P_α) means that the following inequalities hold*

$$\sum_{(x,n)}\big|\Pi_\lambda(x,n)e^{rn}\big|\le\frac{\alpha\tau_0'}{L}, \tag{1.11}$$

$$\sum_{(x,n)}\big|n\Pi_\lambda(x,n)e^{rn}\big|\le\frac{\alpha\tau_1'}{L}, \tag{1.12}$$

$$\sum_{(x,n)}\|x\|_1\big|\Pi_\lambda(x,n)e^{rn}\big|\le\frac{\alpha\tau_2'(\log L)^{\frac{1}{3}}}{L} \tag{1.13}$$

for some universal constants τ_0', τ_1' and τ_2' with $\tau_j'\ge\tau_j$, τ_j as in Proposition 1.5.

We denote c to be a positive constant, whose precise value is not important to us and may vary from line to line. In **Section** 2, we prove the main theorems by assuming Propositions 1.3, 1.5 and 1.6. In **Section** 3, we define the Feynman diagrams which are the same as in [18]. Proposition 1.3 is proved in **Section** 4 and Proposition 1.4 is proved in **Section** 5. In **Section** 6, we prove Proposition 1.5 and 1.6 by Proposition 1.4 and the inequalities in **Section** 3.

2. Proof of the Main Theorems

The following inequality is used to prove Theorem 1.1.

Lemma 2.1. *For our infinite layer long-range model on $\mathbb{Z}^d \times \mathbb{Z}$ with $d > 2$, there exists an L_0 (depending on d) such that for $L \geq L_0$, $\lambda \leq \lambda_c$ and $r \leq m_\lambda$, (P_4) and Proposition 1.4 imply*

$$|\widehat{\Pi}_\lambda(0,r) - \widehat{\Pi}_\lambda(k, r-s+it)e^{-s+it}\widehat{D}(k)| \leq \frac{1}{3}|1 - e^{-s+it}\widehat{D}(k)|$$

Proof. Since

$$\begin{aligned} &|\widehat{\Pi}_\lambda(0,r) - \widehat{\Pi}_\lambda(k, r-s+it)e^{-s+it}\widehat{D}(k)| \\ &\quad\leq |\widehat{\Pi}_\lambda(0,r) - \widehat{\Pi}_\lambda(0, r-s+it)| \\ &\quad+ |\widehat{\Pi}_\lambda(0, r-s+it) - \widehat{\Pi}_\lambda(k, r-s+it)| \\ &\quad+ |\widehat{\Pi}_\lambda(k, r-s+it)||1 - e^{-s+it}\widehat{D}(k)|, \end{aligned} \tag{2.1}$$

we have, by Mean-Value theorem and (P_4),

$$\begin{aligned} |\widehat{\Pi}_\lambda(k, r-s+it) - \widehat{\Pi}_\lambda(0, r-s+it)| &\leq \Big[\sum_{(x,n)} \|x\|_1 |\Pi_\lambda(x,n)e^{r-s+it}|\Big]\|k\|_1 \\ &\leq \frac{4\tau_2'(\log L)^{\frac{1}{3}}}{L}\|k\|_1, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} |\widehat{\Pi}_\lambda(0,r) - \widehat{\Pi}_\lambda(0, r-s+it)| &\leq \Big[\sum_{(x,n)} |n\Pi_\lambda(x,n)e^{r+it}|\Big]\big(|s-it|\big) \\ &\leq \tfrac{4\tau_1'}{L}(|s-it|). \end{aligned} \tag{2.3}$$

Then by (2.1)-(2.3),

$$\begin{aligned} |\widehat{\Pi}_\lambda(0,r) - \widehat{\Pi}_\lambda(k, r-s+it)e^{-s+it}\widehat{D}(k)| &\leq \frac{4\tau_2'(\log L)^{\frac{1}{3}}}{L}\big(\|k\|_1\big) \\ + \frac{4\tau_1'}{L}(|s-it|) + \frac{4\tau_0'}{L}|1 - e^{-s+it}\widehat{D}(k)|. \end{aligned} \tag{2.4}$$

On the other hand, we have, by Proposition 1.4,

$$\begin{aligned} |1 - e^{-s+it}\widehat{D}(k)|^2 &= \big|(1 - e^{-s+it}) + e^{-s+it}[1 - \widehat{D}(k)]\big|^2 \\ &\geq |1 - e^{-s+it}|^2 + ce^{-2s}\|k\|_1^2 \\ &\geq \frac{(c'|s-it| + ce^{-s}\|k\|_1)^2}{2} \end{aligned} \tag{2.5}$$

for some universal constants $c, c' > 0$. From (2.4) and (2.5), let $L > 0$ large enough, this lemma follows. ■

Proof of Theorem 1.1. By Proposition 1.5, for $L \geq L_1$, (P_1) is stisfied at $\lambda = \lambda_0$ and $r = 0$. Then $\widehat{\varphi}_{\lambda_0}(0,0) = E_{\lambda_0}(|C(0,0)|) < \infty$, by (1.8). From Proposition 1.3 (a) and (b), we have $\lambda_c > \lambda_0$. According to (1.8)-(1.9) and Proposition 1.3 (c), the left-hand sides of (1.11)-(1.13) are continuous at λ for every $\lambda < \lambda_c$ and $r < m_\lambda$. Then from Proposition 1.6 and inductive method, (P_4) is satisfied for every $\lambda \in (0, \lambda_c)$ and $r < m_\lambda$. By the Dominated Convergence theorem, we have

$$\widehat{\Pi}_\lambda(0, m_\lambda) = \lim_{s\downarrow 0} \widehat{\Pi}_\lambda(0, m_\lambda - s)$$

which implies (P_4) is satisfied for every $\lambda \in (0, \lambda_c)$ and $r \leq m_\lambda$. From (1.4) and (1.6), we have

$$\begin{aligned} \widehat{\varphi}_\lambda(0, m_\lambda) &= \lim_{s\downarrow 0} \widehat{\varphi}_\lambda(0, m_\lambda - s) = 1 + \lim_{s\downarrow 0} \sum_{n=1}^{\infty} Z_{\lambda,n}(0) e^{(m_\lambda - s)n} \\ &\geq 1 + \lim_{s\downarrow 0} \sum_{n=1}^{\infty} e^{-sn} = \infty, \end{aligned}$$

and

$$1 + \widehat{\Pi}_\lambda(0, m_\lambda) = \lim_{s\downarrow 0} \frac{\widehat{\varphi}_\lambda(0, m_\lambda - s)}{1 + \lambda_0^{-1} \lambda e^{m_\lambda - s} \widehat{\varphi}_\lambda(0, m_\lambda - s)} = \frac{1}{\frac{1}{\widehat{\varphi}_\lambda(0, m_\lambda)} + \lambda_0^{-1} \lambda e^{m_\lambda}} < \infty.$$

Then

$$F_\lambda(0, m_\lambda) = \lim_{s\downarrow 0} \frac{1 + \widehat{\Pi}_\lambda(0, m_\lambda - s)}{\widehat{\varphi}_\lambda(0, m_\lambda - s)} = 0 \tag{2.6}$$

and $1 + \widehat{\Pi}_\lambda(0, m_\lambda) = \lambda_0 \lambda^{-1} e^{-m_\lambda}$. (2.6) implies

$$\begin{aligned} F_\lambda(k, m_\lambda - s + it) &= F_\lambda(k, m_\lambda - s + it) - F_\lambda(0, m_\lambda) \\ &= \lambda_0^{-1} \lambda e^{m_\lambda} (1 - e^{-s+it} \widehat{D}(k)) + \lambda_0^{-1} \lambda e^{m_\lambda} \big[\widehat{\Pi}_\lambda(0, m_\lambda) \\ &\quad - e^{-s+it} \widehat{D}(k) \widehat{\Pi}_\lambda(k, m_\lambda - s + it) \big]. \end{aligned}$$

Since (P_4) is satisfied for all $\lambda \in (0, \lambda_c)$ with $r \leq m_\lambda$, there exists $L_0 > 0$ such that for $L \geq L_0$ and $\lambda \in (0, \lambda_c)$, $|\widehat{\Pi}_\lambda(0, m_\lambda)| < \frac{1}{2}$, and from Lemma 2.1, we have

$$\begin{aligned} |F_\lambda(k, m_\lambda - s + it)| &\geq \tfrac{2\lambda e^{m_\lambda}}{3\lambda_0} |1 - e^{-s+it} \widehat{D}(k)| \\ &= \tfrac{2}{3[1 + \widehat{\Pi}_\lambda(0, m_\lambda)]} |1 - e^{-s+it} \widehat{D}(k)| \\ &\geq \tfrac{4}{9} |1 - e^{-s+it} \widehat{D}(k)| \end{aligned} \tag{2.7}$$

with $s \in (0,1)$. Besides, by Proposition 1.3, $\widehat{\varphi}_\lambda(k,-s+it)$ is left continuous at $\lambda=\lambda_c$ for $s\in(0,1)$. This completes the proof. ■

Proof of Theorem 1.2. Let

$$\begin{aligned}\bigtriangledown_\lambda(x,n) &= \sum_{(u_1,n_1)}\sum_{(u_2,n_2)} P_\lambda((0,0)\longrightarrow(u_1,n_1))P_\lambda((u_1,n_1)\longrightarrow(u_2,n_2)) \\ &\times P_\lambda((x,n)\longrightarrow(u_2,n_2)).\end{aligned}$$

Since the x-space is symmetric with respect to the origin, its Fourier transform is

$$\widehat{\bigtriangledown}_\lambda(k,it)=\widehat{\varphi}_\lambda(k,it)^2\widehat{\varphi}_\lambda(-k,-it)=\widehat{\varphi}_\lambda(k,it)^2\widehat{\varphi}_\lambda(k,-it).$$

Then, by Hausdorff-Young's inequality and infrared bound (we write $\int\int dkdt=\frac{1}{(2\pi)^{d+1}}\int_{t\in[-\pi,\pi]}\int_{k\in[-\pi,\pi]^d}dkdt$ and $\int dk=\frac{1}{(2\pi)^d}\int_{k\in[-\pi,\pi]^d}dk$ in this paper),

$$\begin{aligned}\left\{\sum_{(x,n)}|\bigtriangledown_\lambda(x,n)|^p\right\}^{\frac{1}{p}} &\le \left\{\int\int|\widehat{\varphi}_\lambda(k,it)^2\widehat{\varphi}_\lambda(k,-it)|^q dkdt\right\}^{\frac{1}{q}} \\ &\le \left\{\int\int\left|\frac{1}{c_1m_\lambda+c_2|t|+c_3\|k\|_1}\right|^{3q}dkdt\right\}^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $0<q\le 2$. Then for any $d>2$ and $1<q<1+\frac{1}{3}$, there exists constant c_0 (depending on d and q) such that for all $\lambda<\lambda_c$, $\sum_{(x,n)}|\bigtriangledown_\lambda(x,n)|^p\le c_0$. Since $\sum_{(x,n)}|\bigtriangledown_{\lambda_c}(x,n)|^p=\lim_{\lambda\uparrow\lambda_c}\sum_{(x,n)}|\bigtriangledown_\lambda(x,n)|^p$, which implies the triangle condition holds, that is,

$$\lim_{R\to\infty}\sup\{\bigtriangledown_{\lambda_c}(x,n):\|x\|_2+|n|\ge R\}=0.$$

Then $\gamma=1$, $\delta=\frac{1}{2},\beta=1$ and $\Delta=2$ (see [5, 18, 16] etc.). This completes the proof. ■

3. Estimates of $\Pi_\lambda(x,n)$ and Its Derivatives

As in [17], there are unique the lace parts $\Pi_\lambda^{(l)}(x,n)$ for $l=0,1,2,...$, such that

$$\widehat{\Pi}_\lambda(k,z)=\sum_{l=0}^{\infty}(-1)^l\widehat{\Pi}_\lambda^{(l)}(k,z).$$

In this section, we describe the Feynman diagrams which are adapted from [18] and use them to control the upper bound of $|\widehat{\Pi}_\lambda^{(l)}(k,z)|$ for each $l=0,1,2,....$

Given sites $(x_1, n_1), (x_2, n_2)$, (x, n) and an oriented bond b, define the triangle function:

$$T_\lambda[((x_1, n_1), (x_2, n_2)); ((x, n), b)] = P_\lambda(b: \text{ open})P_\lambda(\text{top of } b \longrightarrow (x, n)) \\ \times P_\lambda((x_2, n_2) \longrightarrow \text{bottom of } b)\psi_\lambda(x - x_1, n - n_1).$$

Let the triangle function $T_\lambda[(u, n'); ((x, n), b)] = T_\lambda[((x_1, n_1), (x_2, n_2)); ((x, n), b)]$ if $(x_1, n_1) = (x_2, n_2) = (u, n')$. We also assume

$$T_\lambda[((x_2, n_2), (x_1, n_1)); (b, (x, n))] = T_\lambda[((x_1, n_1), (x_2, n_2)); ((x, n), b)].$$

Define the bubble functions as follows

$$\begin{aligned} Q^{(\lambda,1)}_{(y,m)}(x, n) &= \varphi_\lambda(x, n)\varphi_\lambda(x - y, n - m), \\ Q^{(\lambda,2)}_{(y,m)}(x, n) &= \psi_\lambda(x, n)\Big[\sum_u \psi_\lambda(x - u, 1)\varphi_\lambda(u - y, n - m - 1)\Big], \\ Q^{(\lambda,3)}_{(y,m)}(x, n) &= \Big[\sum_u \varphi_\lambda(u, n - 1)\psi_\lambda(x - u, 1)\Big]\psi_\lambda(x - y, n - m). \end{aligned}$$

They are represented by the diagrams in Figure 1.

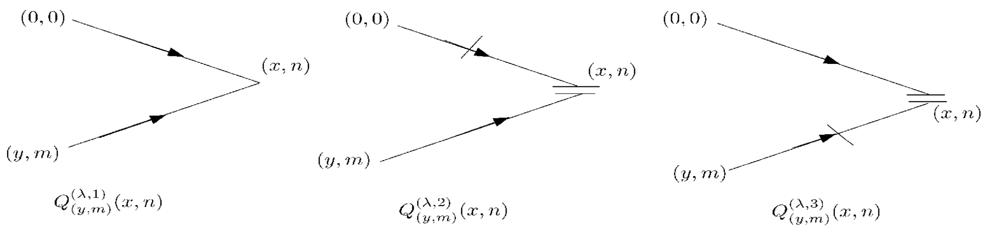

Fig. 1.

For l pairs of sites and bonds $\{((u_j, n_j), b_j), j = 1, 2, ..., l\}$, set $\sigma_j((u_j, n_j), b_j) = ((u_j, n_j), b_j)$ or $(b_j, (u_j, n_j))$, $j = 1, 2, 3, ..., l-1$ and $\sigma_l((u_l, n_l), b_l) = ((u_l, n_l), b_l)$. Let $\sigma = (\sigma_1, \sigma_2, ..., \sigma_l)$, the diagram

$$D^{(l)}_\lambda[\sigma, (0, 0), (b_j, (u_j, n_j)), (x, n); j = 1, 2, ..., l]$$

is defined by

$$T_\lambda[(0, 0); ((u_1, n_1), b_1)]\big\{\textstyle\prod_{j=2,...,l} T_\lambda[\sigma_{j-1}((u_{j-1}, n_{j-1}), b_{j-1}); \\ \sigma_j((u_j, n_j), b_j)]\big\} \times Q^{(\lambda,1)}_{b_l}(x - u_l, n - n_l). \tag{3.1}$$

The diagram $D^{(l)}_\lambda(x, n)$ is defined as the sum of $D^{(l)}_\lambda[\sigma, (0, 0), (b_j, (u_j, n_j)), (x, n); j = 1, 2, ..., l]$ over $\{(b_j, (u_j, n_j)), j = 1, 2, ..., l\}$ and σ such that $\sigma_l =$ identity and σ_i

is identity map or the permutation of sites and bonds for all $j = 1, 2, ..., l-1$ and $l \in \mathbb{N}$. Let $D_\lambda^{(0)}(x,n) = Q_{(0,0)}^{(\lambda,2)}(x,n)$. The following lemma states upper bounds of $\Pi_\lambda^{(l)}(x,n)$ and their Fourier-Laplace transforms which were proved in [18] by diagrams introduced above and Van Den Berg-Kesten's inequality (see [6]).

Lemma 3.1. *For $l \in \mathbb{N} \cup \{0\}$, we have*

$$\Pi_\lambda^{(l)}(x,n) \le D_\lambda^{(l)}(x,n) \text{ for } (x,n) \in \mathbb{Z}^d \times \mathbb{Z}, \quad \widehat{\Pi}_\lambda(0,s) \le \widehat{D}_\lambda^{(l)}(0,s) \text{ for } s \in \mathbb{R}.$$

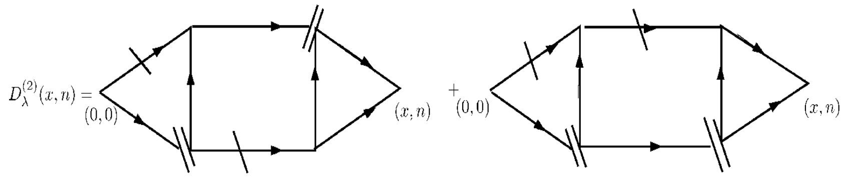

Fig. 2.

To estimate the upper bounds of the Feynman diagrams $\widehat{D}_\lambda^{(l)}(0,s)$ for all $l \in \mathbb{N}$, we have to introduce the triangle functions which are defined in [18].

$$\begin{aligned}
T_{(y,m)}^{(\lambda,1)}(x,n) &= \varphi_\lambda(x,n) \sum_{(u_1,n_1)} \varphi_\lambda(x-u_1, n-n_1)\varphi_\lambda(u_1-y, n_1-m), \\
T_{(y,m)}^{(\lambda,2)}(x,n) &= \psi_\lambda(x,n)\Big\{ \sum_{(u_1,n_1)} \sum_{u\in\mathbb{Z}^d} \varphi_\lambda(x-u_1, n-n_1)\psi_\lambda(u_1-u, 1) \\
&\quad \times\varphi_\lambda(u-y, n_1-1-m)\Big\}, \\
T_{(y,m)}^{(\lambda,3)}(x,n) &= \sum_{(u_1,n_1)} \sum_{u\in\mathbb{Z}^d} \varphi_\lambda(u, n-1)\psi_\lambda(x-u, 1)\varphi_\lambda(x-u_1, n-n_1) \\
&\quad \times\psi_\lambda(u_1-y, n_1-m).
\end{aligned}$$

They are represented by the diagrams in Figure 3. We have the following lemma which is the same as (32) in [18].

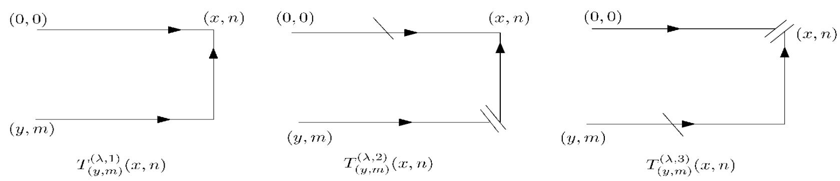

Fig. 3.

Lemma 3.2.

$$\widehat{D}_\lambda^{(l)}(0,s) \le 2^{l-1}\big[\sup_{(y,m)} \widehat{Q}^{(\lambda,1)}_{(y,m)}(0,s)\big]\big[\sup_{(y,m)} \widehat{T}_{\lambda,(y,m)}(0,s)\big]^l \quad \textit{for } l \in \mathbb{N},$$

where

$$\widehat{T}_{\lambda,(y,m)}(0,s) = \max\Big\{\widehat{T}^{(\lambda,2)}_{(y,m)}(0,s), \quad \widehat{T}^{(\lambda,3)}_{(y,m)}(0,s)\Big\}.$$

Define

$$\delta_{k_j}\widehat{f}(0,s) = \sum_{(x,n)} |x_j| f(x,n)e^{sn}, \qquad \delta_z \widehat{f}(0,s) = \sum_{(x,n)} |n| f(x,n)e^{sn},$$

where $f(x,n)$ is any function on $\mathbb{Z}^d \times \mathbb{Z}$ and $j \in \{1,2,...,d\}$. Then

$$|\delta_{k_j}\widehat{\Pi}_\lambda^{(0)}(0,s)| \le \sup_{(y,m)}\{\sum_{(x,n)} |x_j| Q^{(\lambda,2)}_{(y,m)}(x,n)e^{sn}\} = \sup_{(y,m)}[\delta_{k_j}\widehat{Q}^{(\lambda,2)}_{(y,m)}(0,s)],$$

$$|\delta_z\widehat{\Pi}_\lambda^{(0)}(0,s)| \le \sup_{(y,m)}\{\sum_{(x,n)} |n| Q^{(\lambda,2)}_{(y,m)}(x,n)e^{sn}\} = \sup_{(y,m)}[\frac{\partial}{\partial z}\widehat{Q}^{(\lambda,2)}_{(y,m)}(0,s)].$$

Clearly, the upper bound of $\delta_a \widehat{D}_\lambda^{(l)}(0,s)$ is also an upper bound of $\delta_a \widehat{\Pi}_\lambda^{(l)}(0,s)$ with $l \in \mathbb{N}\cup\{0\}$ for $a = k_1,...,k_d$ or $a = z$. To estimate $\delta_a \widehat{D}_\lambda^{(l)}(0,s)$, we need to distribute the factors such that $|x_j|$ or n is along the top of the diagram. Using the same technique as in Section 3.2 of [14], we have the following lemma.

Lemma 3.3. *For $l \in \mathbb{N}$, $a \in \{k_1,...,k_d\}$ or $a = z$, we have*

$$|\delta_a\widehat{\Pi}_\lambda^{(l)}(0,s)| \le 2^{l-1} l [\sup_{(y,m)} \widehat{T}_{\lambda,(y,m)}(0,s)]^{l-1} [\sup_{(y,m)} \widehat{T}^{(\lambda,1)}_{(y,m)}(0,s)][\sup_{(y,m)} \delta_a \widehat{Q}_{\lambda,(y,m)}(0,s)]$$
$$+2^{l-1}[\sup_{(y,m)} \delta_a \widehat{Q}^{(\lambda,1)}_{(y,m)}(0,s)][\sup_{(y,m)} \widehat{T}_{\lambda,(y,m)}(0,s)]^l,$$

where

$$\widehat{Q}_{\lambda,(y,m)}(0,s) = \max\{\widehat{Q}^{(\lambda,2)}_{(y,m)}(0,s), \widehat{Q}^{(\lambda,3)}_{(y,m)}(0,s)\}.$$

The upper bounds of the triangle functions and bubble functions in terms of related Fourier-Laplace transforms are stated in the following lemma which was proved in [17].

Lemma 3.4.

$$\sup_{(y,m)} \widehat{Q}^{(\lambda,1)}_{(y,m)}(0,s) \le \int\int |\widehat{\varphi}_\lambda(k,s+it)\widehat{\varphi}_\lambda(k,it)|dkdt,$$

$$\sup_{(y,m)} \widehat{Q}^{(\lambda,2)}_{(y,m)}(0,s) \le \int\int |\widehat{D}(k)\widehat{\psi}_\lambda(k,s+it)\widehat{\varphi}_\lambda(k,it)|dkdt,$$

$$\sup_{(y,m)} \widehat{Q}^{(\lambda,3)}_{(y,m)}(0,s) \le e^s\int\int |\widehat{D}(k)\widehat{\varphi}_\lambda(k,s+it)\widehat{\psi}_\lambda(k,it)|dkdt,$$

$$\sup_{(y,m)} \widehat{T}^{(\lambda,1)}_{(y,m)}(0,s) \le \int\int |\widehat{\varphi}_\lambda(k,s+it)\widehat{\varphi}^2_\lambda(k,it)|dkdt,$$

$$\sup_{(y,m)} \widehat{T}^{(\lambda,2)}_{(y,m)}(0,s) \le \int\int |\widehat{D}(k)\widehat{\psi}_\lambda(k,s+it)\widehat{\varphi}^2_\lambda(k,it)|dkdt,$$

$$\sup_{(y,m)} \widehat{T}^{(\lambda,3)}_{(y,m)}(0,s) \le e^s\int\int |\widehat{D}(k)\widehat{\varphi}_\lambda(k,it)\widehat{\psi}_\lambda(k,it)\widehat{\varphi}_\lambda(k,s+it)|dkdt.$$

Next, we want to estimate the derivatives of the bubble functions in terms of $\widehat{\varphi}_\lambda(k,z)$ and its derivatives. Note that $\varphi_\lambda(x,n)=0$ if $n<0$. Using Hausdorff-Young's inequality, we have

$$\begin{aligned}\sup_{(y,m)} \delta_z\widehat{Q}^{(\lambda,1)}_{(y,m)}(0,s) &= \sup_{(y,m)}\sum_{(x,n)}|n|Q^{(\lambda,1)}_{(y,m)}(x,n)e^{sn} = \sup_{(y,m)}\sum_{(x,n)} nQ^{(\lambda,1)}_{(y,m)}(x,n)e^{sn}\\ &= \sup_{(y,m)} \varphi_{\lambda,s,z}*\varphi_\lambda(y,m) \le \int\int |\widehat{\varphi}_{\lambda,s,z}(k,it)\widehat{\varphi}_\lambda(k,it)|dkdt,\end{aligned}$$

where $\varphi_{\lambda,s,z}(x,n)=\varphi_\lambda(x,n)e^{sn}n$, and

$$\widehat{\varphi}_{\lambda,s,z}(k,it)=\sum_{(x,n)}\varphi_\lambda(x,n)e^{sn}ne^{ik\cdot x}e^{itn}=\frac{\partial}{\partial z}\widehat{\varphi}_\lambda(k,s+it).$$

By this argument, we have the following lemma:

Lemma 3.5.

$$\sup_{(y,m)} \delta_z\widehat{Q}^{(\lambda,1)}_{(y,m)}(0,s) \le \int\int \Big|\widehat{\varphi}_\lambda(k,it)\frac{\partial}{\partial z}\widehat{\varphi}_\lambda(k,s+it)\Big|dkdt,$$

$$\sup_{(y,m)} \delta_z\widehat{Q}^{(\lambda,2)}_{(y,m)}(0,s) \le \int\int \Big|\widehat{D}(k)\widehat{\varphi}_\lambda(k,it)\frac{\partial}{\partial z}\widehat{\psi}_\lambda(k,s+it)\Big|dkdt,$$

$$\sup_{(y,m)} \delta_z\widehat{Q}^{(\lambda,3)}_{(y,m)}(0,s) \le e^s\int\int \Big|\widehat{\varphi}_\lambda(k,it)\frac{\partial}{\partial z}[\widehat{D}(k)\widehat{\varphi}_\lambda(k,s+it)]\Big|dkdt.$$

4. Proof of Proposition 1.3

In order to prove Proposition 1.3, we need the following lemma.

Lemma 4.1. *In our infinite layer long-range model on* $\mathbb{Z}^d \times \mathbb{Z}$ *with* $d > 0$, *we have*

(a) *for any* n *finite,* $\varphi_\lambda(x, n)$ *and* $Z_{\lambda,n}(0)$ *are continuous functions of* λ *on* $\lambda \in \big(0, (2L+1)^d\big)$,

(b) m_λ *is a continuous function of* λ *on* $\lambda \in (0, \lambda_c]$.

Proof of Proposition 1.3 (a). If $m_\lambda > 0$, by the definition of m_λ, $E_\lambda(|C(0,0)|) < \infty$. On the other hand, if $E_\lambda(|C(0,0)|) < \infty$, by (1.4), there exists $n_1 > 0$ such that $Z_{\lambda,n}(0) \le u < 1$ for $n \ge n_1$. Then by (1.5), we have

$$\lim_{n\to\infty} Z_{\lambda,n}(0) \le \lim_{n\to\infty} u^{\frac{n}{n_1}},$$

which implies $m_\lambda > 0$ by (1.6). This completes the proof. ∎

Proof of Proposition 1.3 (b). From Proposition 1.5 (a) and Lemma 4.1 (c), we have $m_{\lambda_c} \ge 0$. Suppose $m_{\lambda_c} > 0$, since

$$\widehat{\varphi}_{\lambda_c}(0,0) = 1 + \sum_{n=1}^{n_0} Z_{\lambda_c,n}(0) + \sum_{n=n_0+1}^{\infty} Z_{\lambda_c,n}(0),$$

and the second term can be made arbitrarily small since $Z_{\lambda_c,n}(0) \sim e^{-nm_{\lambda_c}}$ as n_0 large, by (1.6), and the first term is finite sum of the continuous functions, by Lemma 4.1(a), we have $\widehat{\varphi}_\lambda(0,0)$ is continuous at $\lambda = \lambda_c$. Then there exists $\lambda_1 > \lambda_c$ such that $\widehat{\varphi}_{\lambda_1}(0,0) < \infty$, which is contradictory to the definition of λ_c. Hence, $m_{\lambda_c} = 0$. This completes the proof of (b). ∎

Proof of Proposition 1.3 (c). For any $0 < \lambda_1 < \lambda_c$, from Lemma 4.1 (b), there exists $\lambda_1 < \lambda' < \lambda_c$ such that $0 < m_{\lambda'} - r < m_{\lambda_1} - r$. This implies $\widehat{\varphi}_{\lambda'}(0, r) < \infty$, by the Dominated Convergence theorem and Lemma 4.1 (a), we have

$$\lim_{\lambda\to\lambda_1} \widehat{\varphi}_\lambda(0,r) = \lim_{\lambda\to\lambda_1} \sum_{(x,n)} \varphi_\lambda(x,n)e^{rn} = \sum_{(x,n)} \lim_{\lambda\to\lambda_1} \varphi_\lambda(x,n)e^{rn} = \widehat{\varphi}_{\lambda_1}(0,r),$$

Besides, by the Monotone Convergence theorem, for $r < 0$ we have $\lim_{\lambda\uparrow\lambda_c} \widehat{\varphi}_\lambda(0,r) = \widehat{\varphi}_{\lambda_c}(0,r)$. This completes the proof of (c). ∎

For any n, let $C_{\le n}(0,0) = \{x : (x,m) \in C_m(0,0) \text{ for } m \le n\}$, and denotes its cardinality by $|C_{\le n}(0,0)|$. To prove Lemma 4.1, we use the following lemma. The proof of Lemma 4.2 is the same as the one of Lemma A.5 [1].

Lemma 4.2. *In our infinite layer long-range model, for any finite number n and m, $P_\lambda\big(|C_{\le n}(0,0)| = m\big)$ is a continuous function of λ.*

Proof of Lemma 4.1. (a). Since for $n < \infty$, $P_\lambda(|C_{\le n}(0,0)| = \infty) = 0$, we have

$$\begin{aligned}\lim_{\lambda\to\lambda_1} \varphi_\lambda(x,n) &= \lim_{\lambda\to\lambda_1} P_\lambda\big((x,n)\in C_n(0,0)\big)\\ &= \lim_{\lambda\to\lambda_1} P_\lambda\big((x,n)\in C(0,0), |C_{\le n}(0,0)| < \infty\big)\\ &= \varphi_{\lambda_1}(x,n),\end{aligned}$$

where the last equality is by Lemma 4.2. Then $\varphi_\lambda(x,n)$ is a continuous function of λ for any (x,n) with $n<\infty$. Moreover, for any $\lambda \in (0,(2L+1)^d)$ and $n<\infty$ we have

$$Z_{\lambda,n}(0) = \sum_{x:\|x\|_\infty \le m} \varphi_\lambda(x,n) + \sum_{x:\|x\|_\infty > m} \varphi_\lambda(x,n) < \infty,$$

where the second term can be made arbitrarily small uniformly, by choosing m large enough and the first term is finite sum of the continuous functions. The proof of (a) is completed. ■

Proof of Lemma 4.1. (b). For any $n < \infty$ and $\lambda_1 \in (0,\lambda_c)$, we have (i). $Z_{\lambda,n}(0) = \lim_{m\to\infty} \sum_{x:\|x\|_\infty \le mL} \varphi_\lambda(x,n)$ pointwise on $\lambda \in [\lambda_1,\lambda_c]$, (ii). $\{\sum_{x:\|x\|_\infty\le mL}\varphi_\lambda(x,n)\}_{m<\infty}$ is a sequence of continuous functions on $[\lambda_1,\lambda_c]$ and $Z_{\lambda,n}(0)$ is also a continuous function on $[\lambda_1,\lambda_c]$, (iii) $\sum_{x:\|x\|_\infty\le mL}\varphi_\lambda(x,n) \le \sum_{x:\|x\|_\infty \le (m+1)L}\varphi_\lambda(x,n)$ for all $m\in\mathbb{N}$ and $\lambda\in[\lambda_1,\lambda_c]$. Then, by (i) (ii) and (iii), for any $n<\infty$, we have

$$Z_{\lambda,n}(0) = \lim_{m\to\infty} \sum_{x:\|x\|_\infty \le mL} \varphi_\lambda(x,n)$$

uniformly on $[\lambda_1,\lambda_c]$. This implies for any $n<\infty$, there exists a $M_n > 1$ which is independent of $\lambda \in [\lambda_1,\lambda_c]$ such that

$$\sum_{x:\|x\|_\infty > M_nL} \varphi_\lambda(x,n) \le \sum_{x:\|x\|_\infty \le M_nL} \varphi_\lambda(x,n). \tag{4.1}$$

By the definition of $Z_{\lambda,n}(0)$, for any $\lambda \le \lambda_c$ and $t \ge 1$,

$$\lim_{n\to\infty} \sum_{x:\|x\|_\infty > ntL} \varphi_\lambda(x,n) \le \lim_{n\to\infty} Z_{\lambda,n}(0) = \lim_{n\to\infty} \sum_{x:\|x\|_\infty \le ntL} \varphi_\lambda(x,n). \tag{4.2}$$

By $(4.1)-(4.2)$, there exists a constant $M>1$ (depending only on λ_1) such that for any $\lambda\in[\lambda_1,\lambda_c]$ and $n\in\mathbb{Z}$,

$$\sum_{x:\|x\|_\infty>nML}\varphi_\lambda(x,n)\qquad\leq\sum_{x:\|x\|_\infty\leq nML}\varphi_\lambda(x,n).$$

Then

$$Z_{\lambda,n}(0)\leq 2\sum_{x:\|x\|_\infty\leq nML}\varphi_\lambda(x,n)\leq 2(2nML+1)^d\Big\{\sup_{x:\|x\|_\infty\leq nML}\varphi_\lambda(x,n)\Big\}, \tag{4.3}$$

and

$$\sup_{\|x\|_\infty>nML}\varphi_\lambda(x,n)\leq 2(2nML+1)^d\big\{\sup_{\|x\|_\infty\leq nML}\varphi_\lambda(x,n)\big\}$$

with $n\in\mathbb{Z},\lambda\in[\lambda_1,\lambda_c]$. By (1.5), we have

$$\begin{aligned}\sup_{x:\|x\|_\infty\leq(n+m)ML}\varphi_\lambda(x,n+m)&=\sup_{x:\|x+y\|_\infty\leq(n+m)ML}\varphi_\lambda(x+y,n+m)\\&\leq\sup_{x:\|x+y\|_\infty\leq(n+m)ML}\sum_y\varphi_\lambda(y,n)\varphi_\lambda(x,m)\\&\leq\sum_{y:\|y\|_\infty\leq nML}\varphi_\lambda(y,n)\big[\sup_x\varphi_\lambda(x,m)\big]\\&\quad+\sum_{y:\|y\|_\infty>nML}\varphi_\lambda(y,n)\big[\sup_{x:\|x\|_\infty\leq mML}\varphi_\lambda(x,m)\big]\\&\leq c(2mML)^d\sum_y\varphi_\lambda(y,n)\big[\sup_{\|x\|_\infty\leq mML}\varphi_\lambda(x,m)\big]\\&\leq c(2mML)^d(2nML)^d\big[\sup_{\|y\|_\infty\leq nML}\varphi_\lambda(y,n)\big]\\&\quad\times\big[\sup_{\|x\|_\infty\leq mML}\varphi_\lambda(x,m)\big]\end{aligned}$$

for all n,m. Thus, there is a universal constant c such that

$$\gamma_{n+m}(\lambda)\leq cd[\log(2nML)+\log(2mML)]+\gamma_n(\lambda)+\gamma_m(\lambda), \tag{4.4}$$

where

$$\gamma_n(\lambda)=\log\big\{\sup_{x:\|x\|_\infty\leq nML}\varphi_\lambda(x,n)\big\}.$$

Let $b_n(\lambda)=-\frac{\gamma_n}{n}$, then, by (4.4) and the Generalized Subadditive Limit theorem (see Appendix II in [9]), we have $\lim_{n\to\infty}b_n(\lambda)$ exists. From (1.6) and (4.3), we have

$$|m_\lambda-b_n(\lambda)|\leq\frac{cd\log(2MnL+1)}{n}, \tag{4.5}$$

for some universal constant c. Thus by (4.5), $b_n(\lambda) \to m_\lambda$ uniformly as $n \to \infty$. Also, $b_n(\lambda)$ is a continuous function of λ on $\lambda \in [\lambda_1, \lambda_c]$, by Lemma 4.1 (a) and $\{x : \|x\|_\infty \le cnL\}$ be only the finite collection. Therefore, m_λ is a continuous function of λ on $\lambda \in [\lambda_1, \lambda_c]$. This completes the proof. ■

5. Proof of Proposition 1.4

5.1 Estimates $\widehat{D}(k)$

To prove Proposition 1.4, we first need to analyze $\widehat{D}(k)$. Let $L > 0$ be fixed, and define $B_l = \{x \in \mathbb{Z}^d : (l-1)L < \|x\|_\infty \le lL\}$ and denote its cardinality by $|B_l|$. Since all B_l are symmetric with respect to the origin, the part of sine in the following sum vanish. Thus we have

$$\begin{aligned}
\widehat{D}(k) &= \sum_x \sum_{l=1}^{\infty} \frac{\lambda_0 1_{\{(l-1)L<\|x\|_\infty \le lL\}}}{l^2|B_l|} e^{ik\cdot x} \\
&= \sum_{l=1}^{\infty} \sum_x \frac{\lambda_0 1_{\{(l-1)L<\|x\|_\infty \le lL\}}}{l^2|B_l|} \big[\cos(k\cdot x) + i\sin(k\cdot x)\big] \\
&= \sum_{l=1}^{\infty} \sum_{x\in B_l} \frac{\lambda_0 \cos(k\cdot x)}{l^2|B_l|},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{x\in B_l} \cos(k_1x_1 + k_2x_2 + \cdots + k_dx_d) \\
&\quad = \sum_{x\in B_l} \Big\{ \cos(k_1x_1 + k_2x_2 + \cdots + k_{d-1}x_{d-1}) \cos k_dx_d \\
&\qquad - \sin(k_1x_1 + k_2x_2 + \cdots + k_{d-1}x_{d-1}) \sin k_dx_d \Big\} \\
&\quad = \sum_{x\in B_l} \cos(k_1x_1 + k_2x_2 + \cdots + k_{d-1}x_{d-1}) \cos k_dx_d \\
&\quad = \sum_{x\in B_l} \prod_{j=1}^{d} \cos k_jx_j,
\end{aligned}$$

so

$$\widehat{D}(k) = \sum_{l=1}^{\infty} \frac{\lambda_0}{l^2|B_l|} \Big\{ \sum_{x\in B_l} \prod_{j=1}^{d} \cos k_jx_j \Big\}.$$

For $l \in \mathbb{N}$,

$$\begin{aligned}\sum_{x\in B_l} = & \sum_{\substack{(l-1)L<x_1\le lL\\ -lL\le x_1<-(l-1)L}} \sum_{x_2=-lL}^{lL} \cdots \sum_{x_d=-lL}^{lL}\\ & + \sum_{x_1=-(l-1)L}^{(l-1)L} \sum_{\substack{(l-1)L<x_2\le lL\\ -lL\le x_2<-(l-1)L}} \sum_{x_3=-lL}^{lL} \cdots \sum_{x_d=-lL}^{lL}\\ & + \quad \cdots\\ & + \sum_{x_1=-(l-1)L}^{(l-1)L} \cdots \sum_{x_{j-1}=-(l-1)L}^{(l-1)L} \sum_{\substack{(l-1)L<x_j\le lL\\ -lL\le x_j<-(l-1)L}} \sum_{x_{j+1}=-lL}^{lL} \cdots \sum_{x_d=-lL}^{lL}\\ & + \quad \cdots\\ & + \sum_{x_1=-(l-1)L}^{(l-1)L} \cdots \sum_{x_{d-1}=-(l-1)L}^{(l-1)L} \sum_{\substack{(l-1)L<x_d\le lL\\ -lL\le x_d<-(l-1)L}}\end{aligned}$$

and for $j = 1, 2, ..., d$

$$\begin{aligned}&\sum_{x_1=-(l-1)L}^{(l-1)L} \cdots \sum_{x_{j-1}=-(l-1)L}^{(l-1)L} \sum_{\substack{(l-1)L<x_j\le lL\\ -lL\le x_j<-(l-1)L}} \sum_{x_{j+1}=-lL}^{lL} \cdots \sum_{x_d=-lL}^{lL} \prod_{j=1}^{d} \cos k_j x_j\\ &= [\sum_{x_1=-(l-1)L}^{(l-1)L} \cos k_1 x_1] \cdots [\sum_{x_{j-1}=-(l-1)L}^{(l-1)L} \cos k_{j-1}x_{j-1}][2\sum_{x_j=(l-1)L+1}^{Ll} \cos k_j x_j]\\ &\quad \times [\sum_{x_{j+1}=-lL}^{lL} \cos k_{j+1}x_{j+1}] \cdots [\sum_{x_d=-lL}^{lL} \cos k_d x_d]\\ &= [1+2\sum_{x_1=1}^{(l-1)L} \cos k_1 x_1] \cdots [1+2\sum_{x_{j-1}=1}^{(l-1)L} \cos k_{j-1}x_{j-1}][2\sum_{m=0}^{L-1} \cos(lL-m)k_j]\\ &\quad \times [1+2\sum_{x_{j+1}=1}^{lL} \cos k_{j+1}x_{j+1}] \cdots [1+2\sum_{x_d=1}^{lL} \cos k_d x_d].\end{aligned}$$

We have

$$\widehat{D}(k) = \sum_{l=1}^{\infty} \sum_{x\in B_l} \frac{\lambda_0 \prod_{j=1}^{d} \cos(k_j x_j)}{l^2 |B_l|}$$

$$= \lambda_0 \sum_{l=1}^{\infty} \frac{1}{l^2|B_l|}\Big\{\sum_{j=1}^{d}\Big[\prod_{\mu=1}^{j-1}(1+2\sum_{x_\mu=1}^{(l-1)L}\cos k_\mu x_\mu)\Big]\Big[2\sum_{m=0}^{L-1}\cos(lL-m)k_j\Big]$$

$$\times\Big[\prod_{\nu=j+1}^{d}(1+2\sum_{x_\nu=1}^{lL}\cos k_\nu x_\nu)\Big]\Big\}$$

$$(5.1)\qquad = \lambda_0 \sum_{l=1}^{\infty}\frac{2^dL}{|B_l|}\sum_{j=1}^{d}\Big\{\Big[\sum_{m=0}^{L-1}\frac{\cos(lL-m)k_j}{l^2L}\Big]\prod_{\mu=1}^{j-1}\Big[\frac{1}{2}+\sum_{x_\mu=1}^{(l-1)L}\cos k_\mu x_\mu\Big]$$

$$\times\prod_{\nu=j+1}^{d}\Big[\frac{1}{2}+\sum_{x_\nu=1}^{lL}\cos k_\nu x_\nu\Big]\Big\}$$

$$= \lambda_0\sum_{l=1}^{\infty}\sum_{j=1}^{d}\Big[\sum_{m=0}^{L-1}\frac{\cos(Ll-m)k_j}{Ll^2}\Big]J_l^j(k),$$

where

$$(5.2)\qquad J_l^j(k) = \frac{\prod_{\mu=1}^{j-1}\Big[\frac{1}{2}+\sum_{x_\mu=1}^{L(l-1)}\cos k_\mu x_\mu\Big]\prod_{\nu=j+1}^{d}\Big[\frac{1}{2}+\sum_{x_\nu=1}^{Ll}\cos k_\nu x_\nu\Big]}{A_l},$$

and

$$(5.3)\qquad A_l = \frac{|B_l|}{2^dL} = \frac{\left(\frac{1}{2}+lL\right)^d}{L}\left[1-\left(\frac{\frac{1}{2}+(l-1)L}{\frac{1}{2}+lL}\right)^d\right].$$

Let $g_l(r) = \frac{6}{\pi^2}\sum_{m=0}^{L-1}\frac{\cos(Ll-m)r}{Ll^2}$ for $l \in \mathbb{N}$ and $g_l(r) = 0$ for $l \le 0$. By (5.1) and recall $\lambda_0 = \frac{6}{\pi^2}$, we have

$$(5.4)\qquad \widehat{D}(k) = \frac{\pi^2\lambda_0}{6}\sum_{j=1}^{d}\sum_{l=1}^{\infty}[g_l(k_j)J_l^j(k)] = \sum_{j=1}^{d}\sum_{l=1}^{\infty}[g_l(k_j)J_l^j(k)].$$

Suppose k with $\|k\|_\infty$ tends to 0, it is easy to see that $\sum_{j=1}^{d}J_l^j(k)$ tends to 1. Then we define $G(r) = \sum_{l=1}^{\infty}g_l(r)$ for $r \in [-\pi,\pi]$ and use it to control $\widehat{D}(k)$. From trigonometric series [24], we have

$$(5.5)\qquad G(r) = \sum_{l=1}^{\infty}g_l(r) = f_1(r)f_2(r)+f_3(r)f_4(r)$$

with

$$f_1(r) = \frac{6}{\pi^2}\sum_{l=1}^{\infty}\frac{\cos(Llr)}{l^2} = 1 - \frac{3}{\pi}L|r| + \frac{3}{2\pi^2}L^2r^2, \tag{5.6}$$

$$f_3(r) = \frac{6}{\pi^2}\sum_{l=1}^{\infty}\frac{\sin(Llr)}{l^2} = \frac{6}{\pi^2}\{-(\log 2)Lr - \int_0^{Lr}\log|\sin\frac{t}{2}|dt\}, \tag{5.7}$$

$$f_2(r) = \frac{1}{L}\sum_{m=0}^{L-1}\cos(mr) = \frac{1}{L}\frac{\sin(\frac{2L-1}{2}r)+\sin(\frac{r}{2})}{2\sin(\frac{r}{2})}, \tag{5.8}$$

$$f_4(r) = \frac{1}{L}\sum_{m=0}^{L-1}\sin(mr) = \frac{1}{L}\frac{\cos(\frac{r}{2})-\cos(\frac{2L-1}{2}r)}{2\sin(\frac{r}{2})}. \tag{5.9}$$

The behavior of $f_1(r), f_2(r), f_3(r)$ and $f_4(r)$ is stated in the following three lemmas.

Lemma 5.1. *(a). $f_1(r)$ is an even function and strictly decreasing for $|r| \le \frac{\pi}{L}$ with $f_1(\frac{3-\sqrt{3}}{3L}) = 0$, $f_1(\frac{\pi}{L}) = \frac{-1}{2}$,*
(b). $f_3(r)$ is a odd function, strictly increasing for $r \in [0, \frac{\pi}{3L}]$ with $f_3(\frac{\pi}{3L}) \le 0.64$ and strictly decreasing for $r \in [\frac{\pi}{3L}, \frac{\pi}{L}]$ with $f_3(\frac{\pi}{L}) \ge 0$,
(c). For $|r| \le \frac{\pi}{4L+1}$,

$$1 - \frac{3}{\pi}L|r| \le f_1(r) \le 1 - \frac{21L|r|}{8\pi},$$

and

$$\frac{6Lr}{\pi^2}\{1-\log(Lr)\} \le |f_3(r)| \le \frac{6L|r|}{\pi^2}\{1.12-\log(L|r|)\}.$$

Proof. (a) and (b) are obvious by (5.6) and (5.7). Since $0.89u \le u - \frac{u^3}{6} \le \sin u \le u$ for $u \in [0, \frac{\pi}{4}]$, we have, by (5.7),

$$\begin{aligned}|f_3(r)| &\le \frac{6}{\pi^2}\{-(\log 2)Lr - \int_0^{Lr}\log(0.89\frac{t}{2})dt\}\\ &= \frac{6Lr}{\pi^2}\{-\log 0.89 + 1 - \log(Lr)\} \le \frac{6Lr}{\pi^2}\{-\log(L|r|)+1.12\},\end{aligned}$$

and $|f_3(r)| \ge \frac{6Lr}{\pi^2}\{1-\log(Lr)\}$ for $r \in [0, \frac{\pi}{4L+1}]$. By (5.6), we have $\frac{21}{8\pi}L|r| \le 1 - f_1(r) \le \frac{3L|r|}{\pi}$ for $|r| \le \frac{\pi}{4L+1}$. This completes the proof. ∎

Lemma 5.2. *For $|r| \le \frac{\pi}{4L+1}$, we have $1 - \frac{(L-1)(2L-1)r^2}{12} \le f_2(r) \le 1 - \frac{0.94(L-1)(2L-1)r^2}{12}$ and $\frac{0.89r(L-1)}{2} \le f_4(r) \le \frac{r(L-1)}{2}$.*

Proof. By Taylor's formula, $1-\frac{u^2}{2}\le \cos u\le 1-\frac{u^2}{2}+\frac{u^4}{24}$. For $|u|<\frac{\pi}{4}$, we have $1-\frac{|u|^2}{2}\le\cos u\le 1-0.94\frac{|u|^2}{2}$. Then

$$1-\frac{(L-1)(2L-1)r^2}{12}\le\frac{1}{L}\sum_{m=0}^{L-1}\cos(mr)\le 1-\frac{0.94(L-1)(2L-1)r^2}{12}.$$

Similarly, by (5.9), $\frac{0.89r(L-1)}{2}\le f_4(r)\le\frac{r(L-1)}{2}$ for $|r|\le\frac{\pi}{4L+1}$. This completes the proof. ∎

Lemma 5.3.

(a) $0\le f_2(r)\le\frac{\sin Lr}{Lr(1-\frac{1}{L^2})}+\frac{1}{2L}$ *for* $r\in[\frac{\pi}{4L+1},\frac{\pi}{L}]$,

(b) $|f_4(r)|\le\frac{1-\cos Lr}{Lr(1-\frac{1}{L^2})}$ *for* $r\in[\frac{\pi}{4L+1},\frac{\pi}{L}]$,

(c) $|f_2(r)|+|f_4(r)|\le\frac{2}{n\pi}+\frac{1}{2L}$ *for* $r\in[\frac{n\pi}{L},\frac{(n+1)\pi}{L}]$ *and* $n=1,2,...,L-1$.

The proof of Lemma 5.3 is similar to the one of Lemma 5.2 and is omitted.

5.2 Proposition 1.4

Let $K_l(r)=\frac{1}{2}+\sum_{m=1}^{Ll}\cos mr$ be the l-th Dirichlet kernel for $l\in\mathbb{N}$. The following lemma is the key lemma to show Proposition 1.4.

Lemma 5.4. *There is a large constant L_1 such that for $L\ge L_1$, we have*

(a) *for* $\|k\|_\infty\in[0,\frac{\pi}{4L+1}]$, $\quad|\widehat{D}(k)|\le|G(\|k\|_\infty)|+0.48L\|k\|_\infty$,

(b) *for* $\|k\|_\infty\in(\frac{\pi}{4L+1},\frac{\pi}{L}]$, $|\widehat{D}(k)|\le\big|G(\|k\|_\infty)\big|+\frac{6}{\pi^3}+\frac{3}{L\pi^2}$, *for* $\|k\|_\infty\in(\frac{n\pi}{L},\frac{(n+1)\pi}{L}]$, *with* $n=1,2,...,L-1$, $|\widehat{D}(k)|\le|G(\|k\|_\infty)|+\frac{6}{n\pi^3}+\frac{3}{L\pi^2}$,

(c) $|\frac{\partial}{\partial k_\nu}\widehat{D}(k)|\le c|\frac{d}{dk_\nu}G(k_\nu)|$ *with* $k\in[-\pi,\pi]^d,\nu\in\{1,2,...,d\}$ *and* $c>0$.

Proof. Clearly, $\widehat{D}(0)=G(0)=1$. For any $k\in[-\pi,\pi]^d$ with $\|k\|_\infty=|k_\mu|$, there exists k^∞ such that $k^\infty=\|k\|_\infty e_\mu$. Clearly, $|\widehat{D}(k)|\le|\widehat{D}(k^\infty)|$. To estimate the upper bound of $|\widehat{D}(k)|$, it is sufficient to estimate $|\widehat{D}(k^\infty)|$.

Let $|k_\mu|=\|k\|_\infty$ for some $\mu\in\{1,2,...,d\}$ and $k=\|k\|_\infty e_\mu$. Clearly, $g_l(k_j)=\frac{6}{\pi^2l^2}$ for $j\ne\mu$. Then, by (5.2),

$$
\begin{aligned}
\widehat{D}(k) &= \sum_{l=1}^{\infty} g_l(k_\mu) J_l^\mu(k) + \sum_{l=1}^{\infty} \sum_{j=1, j\neq\mu}^{d} g_l(k_j) J_l^j(k) \\
&= \sum_{l=1}^{\infty} g_l(k_\mu) \frac{(\frac{1}{2}+lL)^{d-\mu}\big(\frac{1}{2}+(l-1)L\big)^{\mu-1}}{A_l} \\
&\quad + \sum_{l=1}^{\infty} \frac{6}{l^2\pi^2} [\sum_{j=1, j\neq\mu}^{d} J_l^j(k)] \\
&= \sum_{l=1}^{\infty} g_l(k_\mu) \frac{(\frac{1}{2}+Ll)^{d-1}(r_l)^{\mu-1}}{A_l} + \sum_{l=1}^{\infty} \frac{6}{l^2\pi^2} [\sum_{j=1, j\neq\mu}^{d} J_l^j(k)],
\end{aligned} \tag{5.10}
$$

where $r_l = \frac{\frac{1}{2}+(l-1)L}{\frac{1}{2}+lL}$. By (5.2)-(5.3),

$$
J_l^j(k) = \frac{(\frac{1}{2}+(l-1)L)^{j-1} K_l(k_\mu)(\frac{1}{2}+lL)^{d-j-1}}{A_l} \quad \text{for } j<\mu,
$$

and

$$
J_l^j(k) = \frac{(\frac{1}{2}+(l-1)L)^{j-2} K_{l-1}(k_\mu)(\frac{1}{2}+lL)^{d-j}}{A_l} \quad \text{for } j>\mu.
$$

With $\frac{6}{l^2\pi^2}[K_l(k_\mu) - K_{l-1}(k_\mu)] = L g_l(k_\mu)$ and $1 - r_l = \frac{L}{\frac{1}{2}+lL}$, we have

$$
\begin{aligned}
&\sum_{l=1}^{\infty} \frac{6}{l^2\pi^2} [\sum_{j=1, j\neq\mu}^{d} J_l^j(k)] \\
&= \sum_{l=1}^{\infty} \frac{6}{l^2\pi^2} \Big\{ [\sum_{j=1}^{\mu-1} (\frac{1}{2}+(l-1)L)^{j-1} K_l(k_\mu)(\frac{1}{2}+lL)^{d-j-1}] \\
&\quad + [\sum_{j=\mu+1}^{d} (\frac{1}{2}+(l-1)L)^{j-2} K_{l-1}(k_\mu)(\frac{1}{2}+lL)^{d-j}] \Big\} (A_l)^{-1} \\
&= \sum_{l=1}^{\infty} \frac{6}{l^2\pi^2} \Big\{ [\sum_{j=1}^{\mu-1} (\frac{1}{2}+(l-1)L)^{j-1} \big(K_l(k_\mu) - K_{l-1}(k_\mu)\big)(\frac{1}{2}+lL)^{d-j-1}] \\
&\quad + [\sum_{j=1}^{d-1} (\frac{1}{2}+(l-1)L)^{j-1} K_{l-1}(k_\mu)(\frac{1}{2}+lL)^{d-j-1}] \Big\} (A_l)^{-1} \\
&= \sum_{l=1}^{\infty} g_l(k_\mu) \frac{(\frac{1}{2}+lL)^{d-1}(1-r_l^{\mu-1})}{A_l} \\
&\quad + \tfrac{6}{l^2\pi^2 L} \tfrac{(\frac{1}{2}+lL)^{d-1}(1-r_l^{d-1})}{A_l} K_{l-1}(k_\mu).
\end{aligned} \tag{5.11}
$$

Since $A_l = \frac{(\frac{1}{2}+lL)^d(1-r_l^d)}{L}$, by (5.10)-(5.11), we have

(5.12)
$$\begin{aligned}\widehat{D}(k) &= \sum_{l=1}^{\infty}\Big\{\frac{(\frac{1}{2}+Ll)^{d-1}}{A_l}g_l(k_\mu)+\frac{6}{l^2\pi^2 L}\frac{(\frac{1}{2}+lL)^{d-1}(1-r_l^{d-1})}{A_l}K_{l-1}(k_\mu)\Big\}\\ &= \sum_{l=1}^{\infty}\Big\{\frac{g_l(k_\mu)L}{(\frac{1}{2}+lL)(1-r_l^d)}+\frac{6}{l^2\pi^2}\frac{(1-r_l^{d-1})}{(\frac{1}{2}+lL)(1-r_l^d)}K_{l-1}(k_\mu)\Big\}.\end{aligned}$$

Due to $g_l(k_\mu) = \dfrac{6}{l^2\pi^2 L}[K_l(k_\mu) - K_{l-1}(k_\mu)]$, by (5.12),

(5.13)
$$\begin{aligned}\widehat{D}(k) &= G(k_\mu)-\sum_{l=1}^{\infty}\Big\{\big(1-\frac{1-r_l}{1-r_l^d}\big)g_l(k_\mu)-\frac{6}{l^2\pi^2}\frac{(1-r_l^{d-1})}{(\frac{1}{2}+lL)(1-r_l^d)}K_{l-1}(k_\mu)\Big\}\\ &= G(k_\mu)-\sum_{l=1}^{\infty}\Big\{\frac{r_l(1-r_l^{d-1})}{1-r_l^d}g_l(k_\mu)-\frac{6}{l^2\pi^2}\frac{(1-r_l)(1-r_l^{d-1})}{L(1-r_l^d)}K_{l-1}(k_\mu)\Big\}\\ &= G(k_\mu)-\sum_{l=1}^{\infty}\Big\{\frac{6(1-r_l^{d-1})}{l^2\pi^2 L(1-r_l^d)}\big[r_lK_l(k_\mu)-K_{l-1}(k_\mu)\big]\Big\}.\end{aligned}$$

Then

(5.14)
$$\widehat{D}(k) - G(k_\mu) = \frac{6}{l^2\pi^2 L}S_1(k_\mu) - S_2(k_\mu),$$

where

$$S_1(k_\mu) = \sum_{l=1}^{\infty}(1-\frac{1-r_l^{d-1}}{1-r_l^d})\big[r_lK_l(k_\mu)-K_{l-1}(k_\mu)\big],$$

and

$$S_2(k_\mu) = \sum_{l=1}^{\infty}\frac{6}{\pi^2 l^2 L}\big[r_lK_l(k_\mu)-K_{l-1}(k_\mu)\big].$$

From $K_l(k_\mu) = \frac{\sin(lL+\frac{1}{2})k_\mu}{2\sin\frac{1}{2}k_\mu}$,

(5.15)
$$\begin{aligned}&r_lK_l(k_\mu)-K_{l-1}(k_\mu)\\ &= \frac{1}{2}\Big\{\cot(\frac{k_\mu}{2})\big[\sin(lLk_\mu)(r_l-\cos Lk_\mu)+\cos(lLk_\mu)\sin(Lk_\mu)\big]\\ &-\sin(lLk_\mu)\sin(Lk_\mu)+\cos(lLk_\mu)(r_l-\cos Lk_\mu)\Big\}.\end{aligned}$$

For $|k_\mu| \le \frac{\pi}{4L+1}$, we have $r_l - \cos(Lk_\mu) = \frac{-L}{\frac{1}{2}+lL} + 1 - \cos(Lk_\mu)$. Then by

(5.15), (5.6) and (5.7),

$$
\begin{aligned}
S_2(k_\mu) &= \sum_{l=1}^{\infty} \frac{3}{l^2\pi^2 L}\Big\{\cot(\frac{k_\mu}{2})\big[(\frac{-L}{\frac{1}{2}+lL}+1-\cos Lk_\mu)\sin lLk_\mu \\
&\quad +\sin Lk_\mu \cos lLk_\mu\big] - \sin Lk_\mu \sin lLk_\mu \\
&\quad +(\frac{-L}{\frac{1}{2}+lL}+1-\cos Lk_\mu)\cos lLk_\mu\Big\} \\
&= \sum_{l=1}^{\infty} \frac{3}{l^2\pi^2 L}\Big\{\frac{-L\sin(lL+\frac{1}{2})k_\mu}{(\frac{1}{2}+lL)(\sin\frac{k_\mu}{2})} + \cot\frac{k_\mu}{2}\big[(1-\cos Lk_\mu)\sin lLk_\mu \\
&\quad +\sin Lk_\mu \cos lLk_\mu\big] - \sin Lk_\mu \sin lLk_\mu + (1-\cos Lk_\mu)\cos lLk_\mu\Big\} \\
&= \frac{-\int_0^{k_\mu}[\sum_{l=1}^{\infty}\frac{3}{l^2\pi^2}\cos(lL+\frac{1}{2})t]dt}{\sin\frac{k_\mu}{2}} + \frac{1}{2L}\Big\{\cot\frac{k_\mu}{2}\big[(1-\cos Lk_\mu)f_3(k_\mu) \\
&\quad +\sin Lk_\mu f_1(k_\mu)\big] - \sin Lk_\mu f_3(k_\mu) + (1-\cos Lk_\mu)f_1(k_\mu)\Big\} \qquad (5.16)\\
&= \frac{-\int_0^{k_\mu}\big[f_1(t)\cos\frac{t}{2} - f_3(t)\sin\frac{t}{2}\big]dt + \frac{\cos\frac{k_\mu}{2}}{L}\sin Lk_\mu f_1(k_\mu)}{2\sin\frac{k_\mu}{2}} \\
&\quad + \frac{(1-\cos Lk_\mu)}{2L}\big[\cot\frac{k_\mu}{2}f_3(k_\mu) - \frac{\sin Lk_\mu f_3(k_\mu)}{(1-\cos Lk_\mu)} + f_1(k_\mu)\big] \\
&\geq \frac{-\int_0^{k_\mu} f_1(t)\cos\frac{t}{2}dt + \frac{\cos\frac{k_\mu}{2}}{L}\sin Lk_\mu f_1(k_\mu)}{2\sin\frac{k_\mu}{2}} \\
&\geq \frac{-[k_\mu - \frac{3Lk_\mu^2}{2\pi} + \frac{L^2k_\mu^3}{6\pi^2}] + \frac{1}{L}[Lk_\mu - \frac{(Lk_\mu)^3}{6}][1-\frac{3Lk_\mu}{\pi}+\frac{3L^2k_\mu^2}{2\pi^2}]}{k_\mu} \\
&\geq -\frac{3Lk_\mu}{2\pi},
\end{aligned}
$$

for $|k_\mu| \leq \frac{\pi}{4L+1}$. By the definition of the l-th Dirichlet's kernel $K_l(r)$ and r_l, it is easy to see that

$$r_1K_1(k_\mu) - K_0(k_\mu) = \big(\frac{\frac{1}{2}}{\frac{1}{2}+L}\big)\frac{\sin(\frac{1}{2}+L)k_\mu}{2\sin\frac{k_\mu}{2}} - \frac{1}{2} \leq 0.$$

For $l > 1$, since $\frac{1}{r_l} = 1 + \frac{L}{\frac{1}{2}+(l-1)L}$, let $u = Lk_\mu \in (0, \frac{\pi}{4}]$, we have

$$1-\frac{1-r_l^{d-1}}{1-r_l^d}=\frac{r_l^{d-1}}{1+r_l+r_l^2+\cdots+r_l^{d-1}}=\frac{L}{[\frac{1}{2}+(l-1)L]\big[(1+\frac{L}{\frac{1}{2}+(l-1)L})^d-1\big]}$$
$$=\frac{1}{d+c_l},$$

with $c_l\geq 0$. For $|Lk_\mu|=|u|\leq\frac{\pi}{4}$, we have, by (5.15),

$$\begin{aligned}S_1(k_\mu)&\leq\sum_{l=2}^{\infty}\frac{L}{u(d+c_l)}\Big\{\sin lu(\frac{-1}{l}+\frac{u^2}{2})+\cos lu\big[u+\frac{u}{L}(\frac{-1}{l}+\frac{u^2}{2})\big]\Big\}\\&\leq\sum_{l=2}^{[\frac{\pi}{u}]}\frac{L}{ud}\Big\{\sin lu(\frac{-1}{l}+\frac{u^2}{2})+\cos lu\big[u+\frac{u}{L}(\frac{-1}{l}+\frac{u^2}{2})\big]\Big\}\\&\quad+\sum_{n=1}^{\infty}(-1)^n\Big\{\sum_{l=1}^{[\frac{\pi}{u}]}\frac{L}{u(d+c_{n[\frac{\pi}{u}]+l})}\times\big[\sin lu(\frac{-1}{n[\frac{\pi}{u}]+l}+\frac{u^2}{2})\\&\qquad+\cos lu\big(u+\frac{u}{L}(\frac{-1}{n[\frac{\pi}{u}]+l}+\frac{u^2}{2})\big)\big]\Big\}\\&=\sum_{l=2}^{[\frac{\pi}{u}]}\frac{L}{ud}\Big\{\sin lu(\frac{-1}{l}+\frac{u^2}{2})+\cos lu\big[u+\frac{u}{L}(\frac{-1}{l}+\frac{u^2}{2})\big]\Big\}\\&\qquad+\sum_{n=1}^{\infty}(-1)^nR_n(u).\end{aligned}\tag{5.17}$$

For $lu\leq\pi$ with $l>1$, we have

$$\begin{aligned}&\sin lu(\frac{-1}{l}+\frac{u^2}{2})+\cos lu\big[u+\frac{u}{L}(\frac{-1}{l}+\frac{u^2}{2})\big]\\&\leq(lu-\frac{l^3u^3}{6})(\frac{-1}{l}+\frac{u^2}{2})+u(1-\frac{u^2l^2}{2}+\frac{u^4l^4}{24})<0.\end{aligned}$$

Similarly, we have $\sum_{n=1}^{\infty}(-1)^nR_n(u)<0$ since $R_n(u)$ is positive and strictly decreasing of n. This implies $S_1(k_\mu)<0$, by (5.17). Therefore, for $0<k_\mu\leq\frac{\pi}{4L+1}$ and large L, we have, by (5.14) and (5.16)-(5.17),

$$|\widehat{D}(k)-G(k_\mu)|\leq\frac{3Lk_\mu}{2\pi}\leq 0.48Lk_\mu.\tag{5.18}$$

This completes the proof of (a).

To show (b), since

$$\begin{aligned}&\frac{r_l}{l^2}(\frac{1-r_l^{d-1}}{1-r_l^d})-\frac{1}{(l+1)^2}(\frac{1-r_{l+1}^{d-1}}{1-r_{l+1}^d})\\&=\frac{1}{l^2}[1-\frac{1}{1+r_l+\cdots+r_l^{d-1}}]-\frac{1}{(l+1)^2}[1-\frac{r_{l+1}^d}{1+r_{l+1}+\cdots+r_{l+1}^{d-1}}],\end{aligned}$$

we get

$$\begin{aligned}\frac{1}{l^2}-\frac{1}{(l+1)^2}&\ge\frac{r_l}{l^2}(\frac{1-r_l^{d-1}}{1-r_l^d})-\frac{1}{(l+1)^2}(\frac{1-r_{l+1}^{d-1}}{1-r_{l+1}^d})\\&\ge\frac{1}{l^2}-\frac{1}{(l+1)^2}-\frac{1}{l^2[1+r_l+\cdots+r_l^{d-1}]}.\end{aligned}$$

This implies $\frac{r_l}{l^2}(\frac{1-r_l^{d-1}}{1-r_l^d})-\frac{1}{(l+1)^2}(\frac{1-r_{l+1}^{d-1}}{1-r_{l+1}^d})$ is non-negative and monotone decreasing sequence. For $k=\|k\|_\infty e_\mu$ and $\|k\|_\infty\in(\frac{\pi}{4L+1},\frac{\pi}{L})$, we have, by (5.13),

$$\begin{aligned}|\widehat{D}(k)-G(k_\mu)|&=\left|-\sum_{l=1}^{\infty}\frac{6(1-r_l^{d-1})}{\pi^2(1-r_l^d)l^2L}[r_lK_l(k_\mu)-K_{l-1}(k_\mu)]\right|\\&=\left|\frac{3}{L\pi^2}\frac{1-r_1^{d-1}}{1-r_1^d}-\frac{6}{L\pi^2}\sum_{l=1}^{\infty}\big[\frac{r_l}{l^2}(\frac{1-r_l^{d-1}}{1-r_l^d})\right.\\&\qquad\left.-\frac{1}{(l+1)^2}(\frac{1-r_{l+1}^{d-1}}{1-r_{l+1}^d})\big]\times\frac{\sin(lL+\frac{1}{2})k_\mu}{2\sin\frac{1}{2}k_\mu}\right|\\&\le\frac{3}{L\pi^2}+\frac{6}{L\pi^2}\sum_{l=2}^{\infty}\big[\frac{1}{l^2}-\frac{1}{(l+1)^2}\big]\frac{1}{k_\mu}\\&\le\frac{3}{L\pi^2}+\frac{6}{\pi^3}.\end{aligned}\tag{5.19}$$

By the same way, we have $|\widehat{D}(k)-G(k_\mu)|\le\frac{3}{L\pi^2}+\frac{6}{n\pi^3}$ for $k=\|k\|_\infty e_\mu$ and $\|k\|_\infty\in(\frac{n\pi}{L},\frac{(n+1)\pi}{L}]$ with $n=1,2,...,L-1$. This completes the proof of (b).

To prove (c), for $\nu\in\{1,2,...,d\}$, clearly, $|\frac{\partial}{\partial\nu}\widehat{D}(k)|\le|\frac{\partial}{\partial\nu}\widehat{D}(k^\nu)|$, where $k^\nu=k_\nu e_\nu$. Since $\frac{6}{\pi^2}K_{l-1}(r)=L[\frac{6}{2\pi^2}+\sum_{m=l}^{l-1}g_m(r)m^2]$, by (5.12) and Fubini's theorem, we have

$$|\frac{\partial}{\partial\nu}\widehat{D}(k^\nu)|=\left|\frac{d}{d_\nu}\{\sum_{l=1}^{\infty}\frac{g_l(k_\nu)L}{(\frac{1}{2}+lL)(1-r_l^d)}+\frac{6}{l^2\pi^2}\frac{(1-r_l^{d-1})}{(\frac{1}{2}+lL)(1-r_l^d)}K_{l-1}(k_\nu)\}\right|$$

$$=\left|\frac{d}{d_\nu}\{\sum_{l=1}^{\infty}\frac{g_l(k_\nu)L}{(\frac{1}{2}+lL)(1-r_l^d)}+\frac{L(1-r_l^{d-1})}{l^2(\frac{1}{2}+lL)(1-r_l^d)}[\frac{6}{2\pi^2}+\sum_{m=1}^{l-1}g_m(k_\nu)m^2]\}\right|$$

$$\le\left|\frac{d}{d_\nu}\{\sum_{l=1}^{\infty}\frac{g_l(k_\nu)L}{(\frac{1}{2}+lL)(1-r_l^d)}+\sum_{m=1}^{\infty}[cg_m(k_\nu)+c']\}\right|\le c_1|\frac{d}{d_\nu}G(k_\nu)|$$

with some positive constants c, c' and c_1. This completes the proof of (c). ∎

Proof of Proposition 1.4. By Lemma $5.1-5.2$, we have

$$\begin{aligned}
&|G(k_j)| \\
&\le \left[1-\frac{21}{8\pi}L|k_j|\right]\left[1-0.156(L-1)^2k_j^2\right]+\frac{3|k_j|^2L(L-1)}{\pi^2}\left[1.2-\log|Lk_j|\right] \\
&\le 1-0.6L|k_j|
\end{aligned}$$

for $k\in\{k:\|k\|_\infty\le\frac{\pi}{4L+1}\}$. By Lemma 5.4 (a), there exists $L_1>0$, for any $L\ge L_1$,

$$|\widehat{D}(k)|\le 1-0.6L\|k_j\|_\infty+0.48L\|k_j\|_\infty=1-0.12L\|k\|_\infty\le 1-\frac{0.12L}{d}\|k\|_1.$$

Similarly, by Lemma 5.4 and Lemma 5.3,

$$\begin{aligned}
|\widehat{D}(k)| &\le \sup_{j\in\{1,2,\dots,d\}}|G(k_j)|+\frac{3}{L\pi^2}+\frac{6}{\pi^3} \\
&\le (\frac{1}{2})[\frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}(1-\frac{1}{L^2})}+\frac{1}{2L}]+0.64(\frac{1-\frac{\sqrt{2}}{2}}{\frac{\pi}{4}(1-\frac{1}{L^2})})+\frac{3}{L\pi^2}+\frac{6}{\pi^3}<0.95
\end{aligned}$$

with $k\in\{k:\frac{\pi}{4L+1}<\|k\|_\infty<\frac{\pi}{L}\}$, and

$$\begin{aligned}
|\widehat{D}(k)| &\le |G(\|k\|_\infty)|+\frac{6}{n\pi^3}\le|f_2(\|k\|_\infty)|+|f_4(\|k\|_\infty)|+\frac{6}{n\pi^3} \\
&\frac{2}{n\pi}+\frac{1}{2L}+\frac{6}{n\pi^3}\le\frac{9}{10n}
\end{aligned}$$

with $\|k\|_\infty\in(\frac{n\pi}{L},\frac{(n+1)\pi}{L}]$, $n=1,...,L-1$. This completes the proof. ∎

6. Estimates for $\widehat{\Pi}_\lambda(k,z)$

Proof of Proposition 1.5. We use the the following propositions to prove Proposition 1.5.

Proposition 6.1. *For any dimension $d>2$, there exist constants L_0, c_1, c_2 and c_3 such that for $L\ge L_0$ and $n=1,2$, we have*

$$\int\frac{|\widehat{D}(k)|}{(1-|\widehat{D}(k)|)^n}dk\le\frac{c_1\log L}{L}, \tag{6.1}$$

(6.2) $$\int \frac{|\widehat{D}(k)|^2}{(1-|\widehat{D}(k)|)^n} dk \leq \frac{c_2}{L}.$$

Proof. Let $R_1 = [-\frac{\pi}{4L+1}, \frac{\pi}{4L+1}]^d$, $R_2 = [-\frac{\pi}{L}, \frac{\pi}{L}]^d$, by Proposition 1.6, for $d > 2$ there exists $\sigma \in (0, 1)$ such that

$$\begin{aligned}\int \frac{|\widehat{D}(k)|}{1-|\widehat{D}(k)|} dk &= (\frac{1}{2\pi})^d \Big\{ \int_{k\in R_2} \frac{|\widehat{D}(k)|}{1-|\widehat{D}(k)|} dk + \int_{k\in[-\pi,\pi]^d\setminus R_2} \frac{|\widehat{D}(k)|}{1-|\widehat{D}(k)|} dk \Big\} \\ &\leq (\frac{1}{2\pi})^d \Big\{ \int_{k\in R_1} \frac{1}{\frac{0.12L}{d}\|k\|_1} dk + \frac{1}{1-0.95} \int_{k\in R_2\setminus R_1} |\widehat{D}(k)| dk \\ &\quad + \sum_{l=1}^{L-1} 2 \int_{\frac{l\pi}{L}}^{\frac{(l+1)\pi}{L}} (\frac{9(2\pi)^{d-1}}{10l(1-\frac{9}{10l})}) dk_\mu \Big\} \\ &\leq c\frac{\log L}{L},\end{aligned}$$

and

$$\int \frac{|\widehat{D}(k)|}{(1-|\widehat{D}(k)|)^2} dk \qquad \leq \frac{c}{L^d} + \sum_{l=1}^{L-1} \frac{9}{10l(1-\frac{9}{10l})^2 L} \leq c\frac{\log L}{L}.$$

By above argument, we obtain the inequalities (6.2) for $d > 2$. This completes the proof. ∎

Proposition 6.2. *For any dimension $d > 2$, there exists $L_1 > 0$ and universal constant c such that for $L \geq L_1$, $r > 1$ $n = 1, 2$ and $\nu \in \{1, 2, ..., d\}$, we have*

$$\int \frac{|\frac{\partial}{\partial k_\nu}\widehat{D}(k)|^r}{(1-|\widehat{D}(k)|)^n} dk \leq \frac{c}{L}, \quad \int \frac{|\frac{\partial}{\partial k_\nu}\widehat{D}(k)|}{(1-|\widehat{D}(k)|)^n} dk \leq \frac{c\log L}{L}$$

Proof. By (5.6)-(5.9), for $|r| \in [\frac{n\pi}{L}, \frac{(n+1)\pi}{L}]$,$n \in \{0, 1, ..., L-1\}$, we have $f_j(r) \leq 1$ with $j = 2, 4$

$$\begin{aligned}&|f_j(r)| \leq \min\{\frac{c}{Lr}, 1\}, \quad |f_j'(r)| \leq \frac{c}{|Lr|^2}, \\ &|f_1'(r)| \leq cL, \quad |f_3'(r)| \leq cL + c'L|\log L(r - \frac{n\pi}{L})|.\end{aligned}$$

Therefore, for $|r| \leq \frac{\pi}{L}$, we have $|\frac{d}{dr}G(r)| \leq c_1 + c_2|\log Lr|$ with some constants c_1, c_2. For $\frac{n\pi}{L} \leq r \leq \frac{(n+1)\pi}{L}$, we have $|\frac{d}{dr}G(r)| \leq \frac{c_1'}{n} + c_2'\frac{|\log L(r-\frac{n\pi}{L})|}{n}$, $n \in$

$\{1,...,L-1\}$. Then by Lemma 5.4 (c) and Proposition 1.6, for $d>2$, there exists $\sigma_1>0$, such that

$$\int \frac{|\frac{\partial}{\partial k_\nu}\widehat{D}(k)|^r}{1-|\widehat{D}(k)|}|dk \le (\frac{1}{\pi})^d \int_{k\in[0,\frac{\pi}{L}]^d} \frac{(c_1+c_2|\log Lk_\nu|)^r}{\frac{\sigma_1 L\|k\|_1}{d}}dk$$
$$+c\sum_{l=1}^{L-1}\int_{\frac{l\pi}{L}}^{\frac{(l+1)\pi}{L}} \frac{[c_1'+c_2'|\log L(k_\nu-\frac{l\pi}{L})|]^r}{l^r(1-\frac{9}{10l})}dk_\nu$$
$$\le \frac{c}{L}\Big\{\int_0^\pi t^{d-2}[(c_1)+(c_2)|\log t|]dt+\sum_{l=1}^{L-1}\int_0^\pi \frac{(c_1'+c_2'|\log t|)^r}{l^r(1-\frac{9}{10l})}dt\Big\}$$
$$\le c\sum_{l=1}^{L-1}\frac{1}{l^r}(L)^{-1}.$$

By above argument, this lemma follows. ■

Let $S(x,n)$ denote the two-point function of the random walk on $\mathbb{Z}^d$ with 1-step transition function $D(x)$ for $n\in\mathbb{N}$, $S(x,n)=0$ for all $x\in\mathbb{Z}^d$, $n\le 0$ and $S_0(x,n)=S(x,n)+\delta(x,n)$. For $\lambda=\lambda_0$, we have, by Hölder's inequality,

$$\begin{aligned}\sup_{(y,m)}\delta_{k_\mu}\widehat{Q}^{(\lambda,1)}_{(y,m)}(0,0) &= \sup_{(y,m)}\sum_{(x,n)}|x_\mu|\varphi_\lambda(x-y,n-m)\varphi_\lambda(x,n)\\ &\le \|\varphi_\lambda(x,n)\|_{\frac{3}{2}}\|x_\mu\varphi_\lambda(x,n)\|_3\\ &\le \|S_0(x,n)\|_{\frac{3}{2}}\|x_\mu S_0(x,n)\|_3.\end{aligned} \tag{6.3}$$

Since $\sum_x S(x,n)=1$ for all n

$$\sum_{(x,n)}S(x,n)^{\frac{3}{2}}=\sum_{n=1}^{\infty}[\sum_x S(x,n)^{\frac{3}{2}}]\le\sum_{n=1}^{\infty}\{\sup_x S(x,n)^{\frac{1}{2}}\}=\sum_{n=1}^{\infty}\{\sup_x S(x,n)\}^{\frac{1}{2}},$$

by Hausdorff-Young's inequality, let $R_1=[-\frac{\pi}{4L+1},\frac{\pi}{4L+1}]^d$ and $R_2=[-\frac{\pi}{L},\frac{\pi}{L}]^d$, we have, for $d>2$,

$$\begin{aligned}\sum_{(x,n)}S(x,n)^{\frac{3}{2}} &\le \sum_{n=1}^{\infty}\{\int|\widehat{D}(k)|^n dk\}^{\frac{1}{2}}\\ &=\sum_{n=1}^{\infty}(\frac{1}{2\pi})^d\Big\{\int_{k_\mu\in R_1}|\widehat{D}(k)|^n dk+\int_{k_\mu\in R_2\setminus R_1}|\widehat{D}(k)|^n dk\\ &\quad+\int_{k_\mu\in[-\pi,\pi]^d\setminus R_2}|\widehat{D}(k)|^n dk\Big\}^{\frac{1}{2}}\end{aligned} \tag{6.4}$$

$$\le \sum_{n=1}^{\infty}\left\{\frac{c_0}{L^d(n+1)\cdots(n+d)} + \frac{c_1(0.95)^n}{L^d} + \frac{c_2}{L}\sum_{l=1}^{L}(\frac{9}{10l})^n\right\}^{\frac{1}{2}}$$
$$\le \sum_{n=1}^{\infty}(\frac{c}{n^d L^d})^{\frac{1}{2}} + (\frac{c' \log L}{L})^{\frac{1}{2}} \le c(\frac{\log L}{L})^{\frac{1}{2}}$$

with universal constants c. From (6.3), (6.4) and Hausdorff-Young's inequality, we have

$$\begin{aligned} \sup_{(y,m)} \delta_{k_\mu}\widehat{Q}^{(\lambda,1)}_{(y,m)}(0,0) &\le \left\{1+\tau_{\frac{3}{2}}\left(\frac{\log L}{L}\right)^{\frac{1}{3}}\right\}\|x_\mu S_0(x,n)\|_3 \\ &\le \left\{1+\tau_{\frac{3}{2}}\left(\frac{\log L}{L}\right)^{\frac{1}{3}}\right\}\left\{\iint |\frac{\partial}{\partial k_\mu}\widehat{S}_0(k,it)|^{\frac{3}{2}} dkdt\right\}^{\frac{2}{3}} \end{aligned} \tag{6.5}$$

for some universal constants $\tau_{\frac{3}{2}}$. By the same argument, we also have

$$\sup_{(y,m)} \delta_{k_\mu}\widehat{Q}^{(\lambda,j)}_{(y,m)}(0,0) \le \tau_{\frac{3}{2}}\left(\frac{\log L}{L}\right)^{\frac{1}{3}}\left\{\iint |\frac{\partial}{\partial k_\mu}\widehat{S}_0(k,it)|^{\frac{3}{2}} dkdt\right\}^{\frac{2}{3}}. \tag{6.6}$$

with $j = 2, 3$.

Remark 6.1. In (6.5), we obtain the upper bound of $\delta_{k_\mu}\widehat{Q}^{(\lambda,1)}_{(y,m)}(0,0)$ which is different from the upper bound of $\delta_z\widehat{Q}^{(\lambda,1)}_{(y,m)}(0,0)$ in Lemma 3.5. If we follows this method, we have

$$\sup_{(y,m)} \delta_{k_\mu}\widehat{Q}^{(\lambda,j)}_{(y,m)}(0,0) = \sup_{(y,m)} \varphi^\mu_\lambda * \varphi_\lambda(y,m) \le \iint |\widehat{\varphi}^\mu_\lambda(k,it)\widehat{\varphi}_\lambda(k,it)|dkdt,$$

where $\varphi^\mu_\lambda(x,n) = |x_\mu|\varphi_\lambda(x,n)$. We can not control $\widehat{\varphi}^\mu_\lambda(k,it)$ since $\widehat{\varphi}^\mu_\lambda(k,it)$ is not equal to $\frac{\partial}{\partial k_\mu}\widehat{\varphi}_\lambda(k,it)$ for any $\mu \in \{1,2,...,d\}$. If we use Hausdorff-Young inequality

$$\sup_{(y,m)} \delta_{k_\mu}\widehat{Q}^{(\lambda,j)}_{(y,m)}(0,0) \le \{\iint |\widehat{\varphi}_\lambda(k,it)|^2 dkdt\}^{\frac{1}{2}}\{\iint |\frac{\partial}{\partial k_\mu}\widehat{\varphi}_\lambda(k,it)|^2 dkdt\}^{\frac{1}{2}},$$

this right hand side is divergence for the dimension $d = 3$.

Proof of Proposition 1.5 For $\lambda = \lambda_0$, by Proposition 6.2, (6.5)-(6.6) and Lemma $3.4 - 3.5$, for any $d > 2$ there exists an L_1 (depending on d) and universal constant

c such that

$$
\begin{aligned}
\sup_{(y,m)} \widehat{Q}^{(\lambda,1)}_{(y,m)}(0,0) &\le c, \quad \sup_{(y,m)} \widehat{T}^{(\lambda,1)}_{(y,m)}(0,0) \le c,\\
\sup_{(y,m)} \delta_z \widehat{Q}^{(\lambda,1)}_{(y,m)}(0,0) &\le c\frac{\log L}{L}, \quad \sup_{(y,m)} \delta_{k_\mu} \widehat{Q}^{(\lambda,1)}_{(y,m)}(0,0) \le \frac{c}{L^{\frac{2}{3}}},\\
\sup_{(y,m)} \widehat{Q}^{(\lambda,j)}_{(y,m)}(0,0) &\le c\frac{\log L}{L}, \quad \sup_{(y,m)} \widehat{T}^{(\lambda,j)}_{(y,m)}(0,0) \le \frac{c}{L},\\
\sup_{(y,m)} \delta_z \widehat{Q}^{(\lambda,j)}_{(y,m)}(0,0) &\le \frac{c}{L}, \quad \sup_{(y,m)} \delta_{k_\mu} \widehat{Q}^{(\lambda,j)}_{(y,m)}(0,0) \le c\frac{(\log L)^{\frac{1}{3}}}{L},
\end{aligned}
$$

and

$$
\sup_{(y,m)} \widehat{T}^{(\lambda,j)}_{(y,m)}(0,0) \le c\int \frac{\widehat{D}(k)^2}{[1-\widehat{D}(k)]^2} dk \le \frac{c}{L_1} < \frac{1}{2},
$$

for $j \in \{2,3\}$. By Lemma $3.1 - 3.3$, we obtain Proposition 1.5. This completes the proof. ■

Proof of Proposition 1.6

Since (P_4) is satisfied, from (1.5), (2.7) and (6.3), $|\widehat{\varphi}_\lambda(k, m_\lambda - s + it)| \le c|\widehat{S}_0(k, -s+it)|$ and $|\widehat{\psi}_\lambda(k, m_\lambda - s + it)| \le c|\widehat{S}(k, -s+it)|$, moreover, from (1.17), we have

$$
\begin{aligned}
|\tfrac{\partial}{\partial k_\mu}\widehat{\varphi}_\lambda(k, m_\lambda - s + it)| &= |\frac{\partial}{\partial k_\mu}[\frac{1+\widehat{\Pi}_\lambda(k, m_\lambda - s + it)}{F(k, m_\lambda - s + it)}]|\\
&\le \frac{c}{|1-\widehat{D}(k)e^{-s+it}|^2}\\
&\le c|\frac{\partial}{\partial k_\mu}\widehat{S}(k, -s+it)|
\end{aligned}
\tag{6.7}
$$

with universal constant c for any $k \in [-\pi,\pi]^d$ and $s \in (0,1)$. By Hölder's inequality,

$$
\begin{aligned}
&\sup_{(y,m)} \delta_{k_\mu} \widehat{Q}^{(\lambda,1)}_{(y,m)}(0, m_\lambda - s)\\
&= \sup_{(y,m)} \sum_{(x,n)} |x_\mu| \varphi_\lambda(x-y, n-m)\varphi_\lambda(x,n)e^{(m_\lambda - s)n}\\
&\le \|\varphi_\lambda(x,n)\|_{\frac{3}{2}} \|x_\mu \varphi_\lambda(x,n)e^{(m_\lambda - s)n}\|_3.
\end{aligned}
\tag{6.8}
$$

Since $m_\lambda > 0$ for $\lambda \in (0, \lambda_c)$, from (6.4),

$$
(6.9)\qquad \begin{aligned}
\sum_{(x,n)} \psi_\lambda(x,n)^{\frac{3}{2}} &= \lim_{s\uparrow m_\lambda} \sum_{(x,n)} \{\psi_\lambda(x,n)e^{(m_\lambda - s)n}\}^{\frac{3}{2}} \\
&\le c \lim_{s\uparrow m_\lambda} \sum_{(x,n)} \{S(x,n)e^{-sn}\}^{\frac{3}{2}} \\
&= c \sum_{(x,n)} \{S(x,n)e^{-m_\lambda n}\}^{\frac{3}{2}} \\
&\le c \sum_{(x,n)} \{S(x,n)\}^{\frac{3}{2}} \le c(\frac{\log L}{L})^{\frac{1}{2}},
\end{aligned}
$$

By (6.7)-(6.9), we have

$$
(6.10)\qquad \begin{aligned}
&\sup_{(y,m)} \delta_{k_\mu} \widehat{Q}^{(\lambda,1)}_{(y,m)}(0, m_\lambda - s) \le \Big\{1 + \tau_{\frac{3}{2}}\Big\} \left(\frac{\log L}{L}\right)^{\frac{1}{3}} \left\{\iint |\frac{\partial}{\partial k_\mu} \widehat{S}_0(k, it)|^{\frac{3}{2}} dk dt\right\}^{\frac{2}{3}} \\
&\sup_{(y,m)} \delta_{k_\mu} \widehat{Q}^{(\lambda,j)}_{(y,m)}(0, m_\lambda - s) \le \tau_{\frac{3}{2}} \left(\frac{\log L}{L}\right)^{\frac{1}{3}} \left\{\iint |\frac{\partial}{\partial k_\mu} \widehat{S}_0(k, it)|^{\frac{3}{2}} dk dt\right\}^{\frac{2}{3}}
\end{aligned}
$$

with $j = 2, 3$. By Lemma $3.4 - 3.5$, (6.7)-(6.9) and Proposition 6.2, for any $d > 2$, we have

$$
\begin{aligned}
\sup\nolimits_{(y,m)} \widehat{Q}^{(\lambda,1)}_{(y,m)}(0,0) &\le c, \quad \sup_{(y,m)} \widehat{T}^{(\lambda,1)}_{(y,m)}(0,0) \le c, \\
\sup\nolimits_{(y,m)} \delta_z \widehat{Q}^{(\lambda,1)}_{(y,m)}(0,0) &\le c\frac{\log L}{L}, \quad \sup_{(y,m)} \delta_{k_\mu} \widehat{Q}^{(\lambda,1)}_{(y,m)}(0,0) \le \frac{c}{L^{\frac{2}{3}}}, \\
\sup\nolimits_{(y,m)} \widehat{Q}^{(\lambda,j)}_{(y,m)}(0,0) &\le \frac{c}{L}, \quad \sup_{(y,m)} \widehat{T}^{(\lambda,j)}_{(y,m)}(0,0) \le \frac{c}{L}, \\
\sup\nolimits_{(y,m)} \delta_z \widehat{Q}^{(\lambda,j)}_{(y,m)}(0,0) &\le \frac{c}{L}, \quad \sup_{(y,m)} \delta_{k_\mu} \widehat{Q}^{(\lambda,j)}_{(y,m)}(0,0) \le c\frac{(\log L)^{\frac{1}{3}}}{L},
\end{aligned}
$$

with $j \in \{2,3\}$ and $\mu = 1, 2, ..., d$. Let $L_0 \ge L_1$ sufficiently large such that

$$
\sup_{(y,m)} \widehat{T}^{(\lambda,j)}_{(y,m)}(0,r) \le \frac{c}{L} < \frac{1}{2},
$$

for any $L \ge L_0$ and $j = 2, 3$. From Lemma $3.1 - 3.3$, we have

$$
\begin{aligned}
&\sum_{(x,n)} |\Pi_\lambda(x,n)e^{rn}| \le \frac{c_0}{L}, \qquad \sum_{(x,n)} |n\Pi_\lambda(x,n)e^{rn}| \le \frac{c_1}{L}, \\
&\sum_{(x,n)} |x_\mu \Pi_\lambda(x,n)e^{rn}| \le \frac{c_2(\log L)^{\frac{1}{3}}}{L},
\end{aligned}
$$

where c_0, c_1 and c_2 are constants which are independent of τ_0', τ_1', τ_2' for any $r < m_\lambda$ and $\lambda \in (0, \lambda_c)$. Let

$$\tau_0' = \max\{\tau_0, \frac{c_0}{2}\}, \tau_1' = \max\{\tau_1, \frac{c_1}{2}\}, \text{ and } \tau_2' = \{\tau_2, \frac{c_2}{2}\},$$

where c_i as in the Proposition 1.6. Therefore (P_4) implies (P_2). This completes the proof. ■

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References

1. M. Aizenman and D. J. Barsky, Sharpness of The Phase Transition in Percolation Models, *Commum. Math. Phys.*, **108** (1987), 489-526.
2. M. Aizenman and C. M. Newman, Tree Graph Inequalities and Critical Behavior in Percolation Models, *J. Stat. Phys.*, **36** (1984), 107-143.
3. M. Aizenman and C. M. Newman, Discontinuity of The Percolation Density in One Dimension $1/|x-y|^2$ Percolation Models, *Commum. Math. Phys.*, **107** (1986), 611-647.
4. D. J. Berger, Transience, Recurrence and Critical Behavior for Long-Range Percolation, *Comm. Math. Phys.*, **226** (2001), 531-558.
5. D. J. Barsky and M. Aizenman, Percolation Critical Exponents under The Triangle Condition, *Ann. Prob.* (4), **19** (1991), 1520-1536.
6. J. Van Den Berg and H. Kesten, Inequalities With Applications to Percolation and Reliability, *J. Appl. Prob.*, **22** (1985), 556-569.
7. D. Brydges and T. Spencer, Self-Avoiding Walk in 5 or More Dimensions, *Commum. Math. Phys.*, **97** (1985), 125-148.
8. J. W. Essam, percolation Theory, *Z. Rep. Prog. Phys*, **43** (1980), 833-912.
9. G. Grimmett, *Percolation*, Springer-Verlag, New York Inc., 1989.
10. M. Holmes, A. A. Jarai, A. Sakai and G. Slade, High-dimensional graphical networks of self-avoiding walks, *Canad. J. Math.*, **56** (2004), 77-114.
11. T. Hara, Remco van der Hofstad and G. Slade, Critical Two-Point Functions and The Lace Expansion for Spread-Out High-Dimensional Percolation and Related Models, *The Ann. Probab.*, **31** (2003), 349-408.

12. T. Hara and G. Slade, Self-Avoiding Walk in Five or More dimensions. I, *Commun. Math. Phys.*, **147** (1992), 101-136.
13. T. Hara and G. Slade, The Lace Expansion for Self-Avoiding Walk in Five or More Dimensions, *Rev. Math. Phys.*, **4** (1992), 235-327.
14. T. Hara and G. Slade, Mean-Field Critical Behavior for Percolation in High Dimension, *Commun. Math. Phys.*, **128** (1990), 233-391.
15. T. Hara and G. Slade, On The Upper Critical Dimension of Lattice Trees and Lattice Animals, *J. Stat. Phys.*, **59** (1990), 1469-1510.
16. B. G. Nguyen, Gap Exponents for Percolation Processes with Triangle Condition, *J. Stat. Phys.*, **49** (1987), 235-243.
17. B. G. Nguyen and W. S. Yang, Triangle Condition for Oriented Percolation in High Dimensions, *Ann. Probab.* (4), **21** (1993), 1809-1844.
18. B. G. Nguyen and W. S. Yang, Gaussian Limit for Critical Oriented Percolation in High Dimensions, *J. Stat. Phys.*, **21** (1995), 841-876.
19. Remco van der Hofstad and G. Slade, Convergence of Critical Oriented Percolation to Super-Brownian Motion above $4+1$ Dimensions, *Ann. Inst. H. Poincaré Probab. Statist.*, **39** (2003), 413-485.
20. Remco van der Hofstad and G. Slade, The lace expansion on a tree with application to networks of self-avoiding walks, *Adv. Appl. Math.*, **30** (2003), 471-528.
21. L. S. Schulman, Long-Range Percolation in One Dimension, *J. Phys. A*, **16(17)** (1983), L639-L641.
22. A. Sakai, Mean-Field Critical Behavior for the Contact Process, *J. stat. Phys.*, **104** (2001), 111-143.
23. A. Sakai, Hyperscaling Inequalities for the Contact Process and Oriented Percolation, *J. Stat. Phys.*, **106** (2002), 201-211.
24. A. Zygmund, *Trigonometric Series*, (2nd ed.), Cambridge Univ. Press. Cambridge, 1968.

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