

## ON INTEGER DOMINATION IN GRAPHS AND VIZING-LIKE PROBLEMS

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**Abstract.** We continue the study of  $\{k\}$ -dominating functions in graphs (or integer domination as we shall also say) started by Domke, Hedetniemi, Laskar, and Fricke [5]. For  $k \geq 1$  an integer, a function  $f: V(G) \rightarrow \{0, 1, \dots, k\}$  defined on the vertices of a graph  $G$  is called a  $\{k\}$ -dominating function if the sum of its function values over any closed neighborhood is at least  $k$ . The weight of a  $\{k\}$ -dominating function is the sum of its function values over all vertices. The  $\{k\}$ -domination number of  $G$  is the minimum weight of a  $\{k\}$ -dominating function of  $G$ . We study the  $\{k\}$ -domination number on the Cartesian product of graphs, mostly on problems related to the famous Vizing's conjecture. A connection between the  $\{k\}$ -domination number and other domination type parameters is also studied.

### 1. INTRODUCTION

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [8, 9]. For a graph  $G = (V, E)$  with vertex set  $V$  and edge set  $E$ , the *open neighborhood* of a vertex  $v \in V$  is  $N(v) = \{u \in V \mid uv \in E\}$  and the *closed neighborhood* is  $N[v] = N(v) \cup \{v\}$ . A set  $S \subseteq V$  is a *dominating set* if each vertex in  $V - S$  is adjacent to at least one vertex of  $S$ . The *domination number*  $\gamma(G)$  is the minimum cardinality of a dominating set.

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In this paper we continue the study of  $\{k\}$ -dominating functions in graphs started by Domke, Hedetniemi, Laskar, and Fricke [5] (see also page 90 in [8]). This concept can be motivated with several examples; consider for instance the problem of optimally placing fire stations. Then we can model this problem by graphs, vertices representing locations, two vertices being adjacent if the corresponding locations are (relatively) close. If we demand that each location either owns a fire station or is adjacent to a location with a fire station, we are searching for (the classical) minimum dominating set. However, in some places there can be several fire stations at one location and, moreover, whenever a fire occurs, usually more than one station participates at a fire extinguishing. In this case a more realistic requirement is that any location is safeguarded by a certain (fixed) number of fire stations.

We now formally define the concept of  $\{k\}$ -domination as follows. Let  $G = (V, E)$  be a graph. For a real-valued function  $f: V \rightarrow \mathbb{R}$  the *weight* of  $f$  is  $w(f) = \sum_{v \in V} f(v)$ , and for  $S \subseteq V$  we define  $f(S) = \sum_{v \in S} f(v)$ , so  $w(f) = f(V)$ . For a vertex  $v$  in  $V$ , we denote  $f(N[v])$  by  $f[v]$  for notational convenience. For  $k \geq 1$  an integer, a function  $f: V \rightarrow \{0, 1, \dots, k\}$  is called a  $\{k\}$ -*dominating function* if for every  $v \in V$ ,  $f[v] \geq k$ . The  $\{k\}$ -*domination number*, denoted  $\gamma_{\{k\}}(G)$ , of  $G$  is the minimum weight of a  $\{k\}$ -dominating function. Note that the characteristic function of a dominating set of  $G$  is a  $\{1\}$ -dominating function, and so  $\gamma_{\{1\}}(G) = \gamma(G)$ . Since in this type of domination we are assigning to each vertex a nonnegative integer we will also speak about *integer domination* in graphs.

In the next section we introduce two relevant, known concepts, the fractional and the  $k$ -tuple domination and recall known results on the integer domination. In particular, there is a close relation between the integer and the fractional domination. The Cartesian and the strong product of graphs are also introduced. Then we follow with a section in which we study the integer domination on the Cartesian product of graphs, mostly on problems related to the famous Vizing's conjecture. We prove two results that both generalize the result of Clark and Suen from [4]. In the last section we consider the problem when  $k\gamma(G)$  equals  $\gamma_{\{k\}}(G)$  and give a connection between the integer domination and the  $k$ -tuple domination. The connection is in terms of the strong product and this product is also further studied.

## 2. RELATED CONCEPTS AND KNOWN RESULTS

For further reference we first recall the following result from [5].

**Theorem 1.** ([5]) *If  $G$  is a graph and  $k \geq 1$  an integer, then  $\gamma_{\{k\}}(G) \leq k\gamma(G)$ .*

Let  $G = (V, E)$  be a graph. A function  $f: V \rightarrow [0, 1]$  is a *fractional-dominating function*, if  $f[v] \geq 1$  holds for any vertex  $v \in V$ . The *fractional domination number*,

$\gamma_f(G)$ , is the minimum weight over all fractional-dominating functions of  $G$ . The following result is also from [5].

**Theorem 1.** ([5]) For any graph  $G$ ,  $\gamma_f(G) = \min_{k \in \mathbb{N}} \left\{ \frac{\gamma_{\{k\}}(G)}{k} \right\}$ .

In [15, Theorem 7.4.1] it is proved that for an  $r$ -regular graph on  $n$  vertices we have  $\gamma_f(G) = n/(r+1)$ . Combining this result with Theorem 2 and considering an  $\{r+1\}$ -dominating function  $f$  defined with  $f(u) = 1$ , for any  $u \in V(G)$ , we get:

**Corollary 3.** Let  $G$  be an  $r$ -regular graph on  $n$  vertices. Then  $\inf_k \left\{ \frac{\gamma_{\{k\}}(G)}{k} \right\}$  is attained for  $k = r+1$ .

In [7] Harary and Haynes defined a generalization of domination as follows: a subset  $S$  of  $V$  is a  $k$ -tuple dominating set of  $G$  if for every vertex  $v \in V$ ,  $|N[v] \cap S| \geq k$ , that is,  $v$  is in  $S$  and has at least  $k-1$  neighbors in  $S$  or  $v$  is in  $V-S$  and has at least  $k$  neighbors in  $S$ . The  $k$ -tuple domination number  $\gamma_{\times k}(G)$  is the minimum cardinality of a  $k$ -tuple dominating set of  $G$ . Clearly,  $\gamma(G) = \gamma_{\times 1}(G) \leq \gamma_{\times k}(G)$ , while  $\gamma_t(G) \leq \gamma_{\times 2}(G)$  where  $\gamma_t(G)$  denotes the total domination number of  $G$  (see [8, 9]). Every graph  $G$  with minimum degree at least  $k-1$  has a  $k$ -tuple dominating set (since  $V(G)$  is such a set). For recent results on  $k$ -tuple domination see [13, 14].

For graphs  $G$  and  $H$ , the Cartesian product  $G \square H$  is the graph with vertex set  $V(G) \times V(H)$  where two vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  are adjacent if and only if either  $u_1 = u_2$  and  $v_1v_2 \in E(H)$  or  $v_1 = v_2$  and  $u_1u_2 \in E(G)$ . The strong product  $G \boxtimes H$  is also defined on the vertex set  $V(G) \times V(H)$  where two vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  are adjacent if and only if  $u_1 = u_2$  and  $v_1v_2 \in E(H)$ , or  $v_1 = v_2$  and  $u_1u_2 \in E(G)$ , or  $v_1v_2 \in E(H)$  and  $u_1u_2 \in E(G)$ . Both products are commutative and associative in the natural way and the one vertex graph is the unit for both multiplications. For more information on them see [11].

Finally, the 2-packing number  $P_2(G)$  of a graph  $G$  is defined as the maximum cardinality of a set  $S \subset V(G)$  such that any two vertices in  $S$  are at distance at least three and that a graph  $G$  is domination-critical if  $\gamma(G-v) < \gamma(G)$  for every vertex  $v \in V(G)$ .

### 3. INTEGER DOMINATION AND CARTESIAN PRODUCTS OF GRAPHS

In 1968 Vizing [17] made the following conjecture which he first posed as a question in 1963.

**Vizing's Conjecture** For any graphs  $G$  and  $H$ ,  $\gamma(G)\gamma(H) \leq \gamma(G \square H)$ .

The conjecture became one of the central problems in the theory of graph domination, cf. [10, 11] and references therein. It also motivated many computations of exact domination numbers of Cartesian product graphs, see, for instance, [2]. The conjecture has in particular been verified when one factor is a tree [12], for products of  $d$ -regular graphs (with few possible exceptions) [3], and when the domination number of one of the factors is three [1, 16]. The best general upper bound to date on the product of the domination numbers of two graphs in terms of their Cartesian product is due to Clark and Suen [4].

**Theorem 4.** (Clark, Suen [4]) For any graphs  $G$  and  $H$ ,  $\gamma(G)\gamma(H) \leq 2\gamma(G \square H)$ .

As an immediate consequence of Theorems 1 and 4, we have the following result.

**Corollary 5.** For any graphs  $G$  and  $H$ ,  $\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq 2k^2\gamma(G \square H)$ .

For the sake of completeness we present a self-contained and relatively short proof of this corollary. Similar (yet more involved) techniques will be used in other proofs of this section.

First we introduce some notation that will be used in all proofs of this section. We will consider the Cartesian product  $G \square H$  of graphs  $G$  and  $H$ , and partition its vertex set in the following way. Consider a minimum dominating set  $\{u_1, \dots, u_{\gamma(G)}\}$  of  $G$ . Let  $\{\pi_1, \dots, \pi_{\gamma(G)}\}$  be a partition of  $V(G)$  such that  $\{u_i\} \subseteq \pi_i \subseteq N[u_i]$ . For each  $w \in V(H)$ , let  $V_w = V(G) \times \{w\}$ . For  $i = 1, \dots, \gamma(G)$ , let  $H_i = \pi_i \times V(H)$ . We call each  $\pi_i \times \{w\}$  a *cell* and denote it by  $C_w^i$ .

*Proof of Corollary 5.* Let  $D$  be a minimum dominating set of the product  $G \square H$ . We say that a cell  $C_w^i = \pi_i \times \{w\}$  is **vertically undominated** if

$$C_w^i \cap N[D \cap H_i] = \emptyset,$$

and **vertically dominated** otherwise. Let  $\ell_i$  denote the number of vertically undominated cells in  $H_i$ , then  $\gamma(H) \leq |D \cap H_i| + \ell_i$ , by projecting  $D \cap H_i$  onto  $H$ , so that, by Theorem 1,  $\gamma_{\{k\}}(H) \leq k(|D \cap H_i| + \ell_i)$ . Thus, summing over all  $i$ ,

$$(1) \quad \gamma(G)\gamma_{\{k\}}(H) \leq k|D| + k \sum_{i=1}^{\gamma(G)} \ell_i.$$

If a cell  $C_w^i$  is vertically undominated, then, since  $D$  is a dominating set of the product  $G \square H$ ,  $C_w^i \subseteq N[D \cap V_w]$ . Hence each vertex in a vertically undominated cell  $C_w^i$  is dominated by  $D \cap V_w$ . On the other hand, each vertex in a vertically dominated cell  $C_w^i$  is dominated by  $(u_i, w)$ . Hence if  $m_w$  denotes the number of vertically undominated cells in  $V_w$ , then  $\gamma(G) \leq |D \cap V_w| + (\gamma(G) - m_w)$ , or equivalently,  $m_w \leq |D \cap V_w|$ . Hence,

$$(2) \quad \sum_{i=1}^{\gamma(G)} \ell_i = \sum_{w \in V(H)} m_w \leq \sum_{w \in V(H)} |D \cap V_w| = |D|.$$

Thus, by Equations (1) and (2),  $\gamma(G)\gamma_{\{k\}}(H) \leq 2k|D| = 2k\gamma(G \square H)$ . The desired result now follows from Theorem 1. ■

Our aim in this section is to shed some light on a version of Vizing’s conjecture for the  $\{k\}$ -domination number  $\gamma_{\{k\}}$ . We will generalize Theorem 4 in two different ways.

**Theorem 6.** *For any graphs  $G$  and  $H$ ,*

$$\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq k(k + 1)\gamma_{\{k\}}(G \square H).$$

*Proof.* Let  $V = V(G \square H)$  and let  $f: V \rightarrow \{0, 1, \dots, k\}$  be a minimum  $\{k\}$ -dominating function of  $G \square H$ , and so  $w(f) = f(V) = \gamma_{\{k\}}(G \square H)$ . Let  $D = \{v \in V \mid f(v) \geq 1\}$ . Clearly,  $D$  is a dominating set of the product  $G \square H$ . For each cell  $C_w^i = \pi_i \times \{w\}$  we introduce its *vertical neighborhood* as

$$V_w^i = \pi_i \times N_H[w].$$

We say that a cell  $C_w^i$  is **vertically  $k$ -undominated** if  $f(V_w^i) \leq k - 1$ , and **vertically  $k$ -dominated** otherwise. For  $i = 1, \dots, \gamma(G)$ , let  $\ell_i$  denote the number of vertically  $k$ -undominated cells in  $H_i$  and let  $f_i: V(H) \rightarrow \{0, 1, \dots, k\}$  be defined as follows: For each  $w \in V(H)$ , let  $f_i(w) = \min\{k, f(C_w^i)\}$  if the cell  $C_w^i$  is vertically  $k$ -dominated; otherwise, let  $f_i(w) = k$ . Then,  $f_i$  is a  $\{k\}$ -dominating function of  $H$ , and so

$$\gamma_{\{k\}}(H) \leq w(f_i) \leq f(H_i) + k\ell_i.$$

Thus, summing over all  $i$ ,

$$(3) \quad \gamma(G)\gamma_{\{k\}}(H) \leq f(V) + k \sum_{i=1}^{\gamma(G)} \ell_i.$$

If a cell  $C_w^i$  is vertically  $k$ -undominated, then, since  $f$  is a  $\{k\}$ -dominating function of  $G \square H$ ,  $C_w^i \subseteq N[D \cap V_w]$ . Hence each vertex in a vertically  $k$ -undominated

cell  $C_w^i$  is dominated by  $D \cap V_w$ . On the other hand, each vertex in a vertically  $k$ -dominated cell  $C_w^i$  is dominated by  $(u_i, w)$ . Hence if  $m_w$  denotes the number of vertically  $k$ -undominated cells in  $V_w$ , then  $\gamma(G) \leq |D \cap V_w| + (\gamma(G) - m_w)$ , or equivalently,  $m_w \leq |D \cap V_w|$ . Hence,

$$(4) \quad \sum_{i=1}^{\gamma(G)} \ell_i = \sum_{w \in V(H)} m_w \leq \sum_{w \in V(H)} |D \cap V_w| = |D| \leq f(V).$$

Thus, by Equations (3) and (4),

$$\gamma(G)\gamma_{\{k\}}(H) \leq (k + 1)f(V) = (k + 1)\gamma_{\{k\}}(G \square H).$$

The desired result now follows from Theorem 1. ■

For graphs  $G$  and  $H$ , let

$$\psi(G, H) = \min\left\{|V(H)| (k\gamma(G) - \gamma_{\{k\}}(G)), |V(G)| (k\gamma(H) - \gamma_{\{k\}}(H))\right\}.$$

**Theorem 7.** *For any graphs  $G$  and  $H$ ,*

$$\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq 2k\gamma_{\{k\}}(G \square H) + k\psi(G, H).$$

*Proof.* We shall follow the notation introduced in the proof of Theorem 6. For  $i = 1, \dots, \gamma(G)$  and for  $w \in V(H)$ , we define the *horizontal need*  $n(C_w^i)$  of the cell  $C_w^i$  to be  $k - f(V_w^i)$ . For  $j = 0, \dots, k - 1$ , let  $\ell_i^j$  denote the number of (vertically  $k$ -undominated) cells in  $H_i$  with  $f(V_w^i) = j$ . Let  $x$  denote the sum of the horizontal needs of all vertically  $k$ -undominated cells. Then,

$$x = \sum_{i=1}^{\gamma(G)} \sum_{j=0}^{k-1} \ell_i^j \cdot (k - j).$$

For  $i = 1, \dots, \gamma(G)$ , let  $g_i: V(H) \rightarrow \{0, 1, \dots, k\}$  be defined as follows: For each  $w \in V(H)$ , let  $g_i(w) = \min\{k, f(C_w^i)\}$  if the cell  $C_w^i$  is vertically  $k$ -dominated; otherwise, let  $g_i(w) = f(C_w^i) + n(C_w^i) = f(C_w^i) + k - f(V_w^i)$ . Then,  $g_i$  is a  $\{k\}$ -dominating function of  $H$ , and so

$$\gamma_{\{k\}}(H) \leq w(g_i) \leq f(H_i) + \sum_{j=0}^{k-1} \ell_i^j \cdot (k - j).$$

Thus, summing over all  $i$ ,

$$(5) \quad \gamma(G)\gamma_{\{k\}}(H) \leq f(V) + x.$$

If  $C_w^i$  is a vertically  $k$ -undominated cell with  $f(V_w^i) = j < k$ , then, since  $f$  is a  $\{k\}$ -dominating function of  $G \square H$ ,  $f(N[v] \cap V_w) \geq k - j$  for each  $v \in C_w^i$ . On the other hand, each vertex in  $\pi_i$  is dominated by  $\{u_i\}$  in  $G$ , and so by assigning an additional weight of  $f(V_w^i) = j$  to the vertex  $(u_i, w)$  in  $G \square H$ , we can guarantee that  $f(N[v] \cap V_w)$  is at least  $k$  for each  $v \in C_w^i$  with  $f(V_w^i) < k$ . More precisely, for  $i = 1, \dots, \gamma(G)$  and for  $w \in V(H)$ , let  $h_w: V(G) \rightarrow \{0, 1, \dots, k\}$  be defined as follows: For each  $v \in V(G)$ , let

$$h_w(v) = \begin{cases} \min\{k, f((v, w)) + j\} & \text{if } v = u_i \text{ and } f(V_w^i) = j < k \\ k & \text{if } v = u_i \text{ and } f(V_w^i) \geq k \\ f((v, w)) & \text{otherwise} \end{cases}$$

Then,  $h_w$  is a  $\{k\}$ -dominating function of  $G$ . Hence, for  $i = 1, \dots, \gamma(G)$  and for  $w \in V(H)$ , if  $m_w^j$  denotes the number of (vertically  $k$ -undominated) cells in  $V_w$  with  $f(V_w^i) = j$  for all  $j = 0, \dots, k - 1$ , then

$$\gamma_{\{k\}}(G) \leq h_w(G) \leq f(V_w) + k\left(\gamma(G) - \sum_{j=0}^{k-1} m_w^j\right) + \sum_{j=0}^{k-1} m_w^j \cdot j,$$

or equivalently,

$$\sum_{j=0}^{k-1} m_w^j \cdot (k - j) \leq f(V_w) + k\gamma(G) - \gamma_{\{k\}}(G).$$

Hence,

$$(6) \quad x = \sum_{w \in V(H)} \sum_{j=0}^{k-1} m_w^j \cdot (k - j) \leq f(V) + |V(H)|\left(k\gamma(G) - \gamma_{\{k\}}(G)\right).$$

Thus, by Equations (5) and (6),

$$\gamma(G)\gamma_{\{k\}}(H) \leq 2f(V) + |V(H)|\left(k\gamma(G) - \gamma_{\{k\}}(G)\right).$$

Hence, by Theorem 1 and since  $f(V) = \gamma_{\{k\}}(G \square H)$ , we have

$$(7) \quad \gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq 2k\gamma_{\{k\}}(G \square H) + k|V(H)|\left(k\gamma(G) - \gamma_{\{k\}}(G)\right).$$

Interchanging the roles of  $G$  and  $H$  shows that

$$(8) \quad \gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq 2k\gamma_{\{k\}}(G \square H) + k|V(G)|\left(k\gamma(H) - \gamma_{\{k\}}(H)\right).$$

The desired result now follows from Equations (7) and (8). ■

When  $k = 1$ , Theorems 6 and 7 simplify to  $\gamma(G)\gamma(H) \leq 2\gamma(G \square H)$  which is the result of Clark and Suen [4] (see Theorem 4).

Let us note that the version of Vizing's conjecture for  $\{k\}$ -domination, that is

$$\gamma_{\{k\}}(G \square H) \geq \gamma_{\{k\}}(G)\gamma_{\{k\}}(H)$$

is far from being true when  $k > 1$ . The simplest example is obtained by setting  $H = K_1$ . Then  $\gamma_{\{k\}}(G \square K_1) = \gamma_{\{k\}}(G)$  and  $\gamma_{\{k\}}(G)\gamma_{\{k\}}(K_1) = k\gamma_{\{k\}}(G)$ . For another (nontrivial) example note that  $\gamma_{\{2\}}(C_4) = 3$ , yet  $\gamma_{\{2\}}(C_4 \square C_4) \leq 8$ . This is in a contrast to the fact that  $\gamma_f(G \square H) \geq \gamma_f(G)\gamma_f(H)$  holds [6]. So in spite of seemingly strong connection in Theorem 2, integer domination and fractional domination behave quite independently. On the other hand, Theorem 2 implies that  $\gamma_{\{k\}}(G \square H) \geq k\gamma_f(G \square H)$  and hence

$$\gamma_{\{k\}}(G \square H) \geq k\gamma_f(G)\gamma_f(H)$$

holds for any  $k$  and any graphs  $G$  and  $H$ .

We also considered another possible generalization of Theorem 4

$$\gamma_{\{2\}}(G \square H) \geq \gamma(G)\gamma(H),$$

but we were unable to prove it in general. Note that it is weaker than Vizing's conjecture. Moreover, we cannot answer even the following much weaker question:

*Is there a natural number  $k$  such that for any pair of graphs  $G, H$*

$$\gamma_{\{k\}}(G \square H) \geq \gamma(G)\gamma(H)?$$

#### 4. INTEGER DOMINATION AND RELATED INVARIANTS

In this section we present a few results that relate  $\{k\}$ -domination number with some other parameters, notably domination number and  $k$ -tuple domination number. We also give an upper and a lower bound for the  $\{k\}$ -domination number of strong products of graphs.

The last question of the previous section would be trivial if we knew a lower bound for  $\gamma_{\{k\}}(G)$  expressed as a multiple of  $\gamma(G)$ . Furthermore, to evaluate the bound of Theorem 4, it would be necessary to bound the difference between  $k\gamma(G)$  and  $\gamma_{\{k\}}(G)$  for particular graphs. And so the following question is also interesting: given a natural number  $k$ , for which graphs is

$$(9) \quad k\gamma(G) = \gamma_{\{k\}}(G)?$$



We shall not study this for particular values of  $k$ , but try to give some answers for graphs for which (9) is achieved for every  $k$ . Note that these are precisely the graphs with equal domination and fractional domination number.

**Proposition 8.** *If  $P_2(G) = \gamma(G)$  then  $\gamma_{\{k\}}(G) = k\gamma(G)$  for every integer  $k \geq 1$ .*

*Proof.* Let  $S$  be the set with  $\gamma(G)$  vertices which are at pairwise distance at least 3. Let  $f: V(G) \rightarrow \{0, 1, \dots, k\}$  be a  $\{k\}$ -dominating function of  $G$ . Then for every  $v \in S$ ,  $f[v] \geq k$ . Since  $N[u] \cap N[v] = \emptyset$  for  $u, v \in S$  we derive that  $\gamma_{\{k\}}(G) \geq kP_2(G) = k\gamma(G)$ . The claim now follows from Theorem 1. ■

Next we present a sufficient condition for  $\gamma_f(G) < \gamma(G)$ , by requiring that the 2-packing number is less than domination number of a graph, and that it is domination-critical. Interestingly, this condition is sufficient for every  $k > 1$ .

**Proposition 9.** *If  $P_2(G) < \gamma(G)$  and  $G$  is domination-critical then  $\gamma_{\{k\}}(G) < k\gamma(G)$  for every  $k > 1$ .*

*Proof.* Let  $G$  be domination-critical and  $P_2(G) < \gamma(G)$  and let  $S$  be a minimum dominating set of  $G$ . Then there exists a vertex  $u \in V(G)$  such that  $|N[u] \cap S| \geq 2$ . Let  $S_1$  be a minimum dominating set for  $G - u$ . As  $G$  is domination-critical  $|S_1| = \gamma(G) - 1$ . Now, for each vertex  $v \in G$  set

$$f(v) = \begin{cases} k & \text{if } v \in S \cap S_1 \\ k - 1 & \text{if } v \in S \setminus S_1 \\ 1 & \text{if } v \in S_1 \setminus S \\ 0 & \text{otherwise} \end{cases}$$

Note that  $f$  is  $\{k\}$ -dominating function for  $G$ , and that  $w(f) = k\gamma(G) - 1$ , and so  $\gamma_{\{k\}}(G) \leq k\gamma(G) - 1$ . ■

In the next result we connect the integer domination and the  $k$ -tuple domination.

**Theorem 10.** *For any graph  $G$  and any  $k \geq 1$ ,  $\gamma_{\{k\}}(G) = \gamma_{\times k}(G \boxtimes K_k)$ .*

*Proof.* Let  $V(G) = \{v_1, \dots, v_n\}$  and  $V(K_k) = \{1, \dots, k\}$ .

Let  $\gamma_{\{k\}}(G) = s$  and let  $f$  be a minimum  $\{k\}$ -dominating function of  $G$  (of weight  $s$ ). Let  $S \subseteq V(G \boxtimes K_k)$  be defined by

$$S = \bigcup_{i=1}^n \{(v_i, j) \mid j = 1, \dots, f(v_i)\}.$$

Since  $f$  is a  $\{k\}$ -dominating function,  $f(v_i) \leq k$  ( $1 \leq i \leq n$ ), and so the set  $S$  is well-defined. Also,  $|S| = \sum_{i=1}^n f(v_i) = w(f) = s$ . Let  $(v_i, j)$  be an arbitrary vertex of  $G \boxtimes K_k$ . Since  $f[v_i] \geq k$ , we have that  $|N[(v_i, j)] \cap S| \geq k$ . Thus,  $S$  is a  $k$ -tuple dominating set of  $G \boxtimes K_k$ , and so  $\gamma_{\times k}(G \boxtimes K_k) \leq \gamma_{\{k\}}(G)$ .

Conversely, let  $\gamma_{\times k}(G \boxtimes K_k) = s$  and let  $S$  be a minimum  $k$ -tuple dominating set of  $G \boxtimes K_k$  (of size  $s$ ). Let  $g: V(G) \rightarrow \{0, 1, \dots, k\}$  be the function defined by  $f(v_i) = |S \cap \{(v_i, j) \mid j = 1, \dots, k\}|$ . Then it is easy to verify that  $f[v_i] \geq k$  holds for all  $i$ . Thus,  $f$  is a  $\{k\}$ -dominating function of weight  $s$ , and so  $\gamma_{\{k\}}(G) \leq \gamma_{\times k}(G \boxtimes K_k)$ . Consequently,  $\gamma_{\{k\}}(G) = \gamma_{\times k}(G \boxtimes K_k)$ . ■

Theorem 10 is of similar nature as the following result, see [11, Theorem 8.38]. For any graph  $G$  there exists an integer  $k$  such that  $\chi_f(G) = \chi(G \boxtimes K_k)/k$ . (This result is in [11] stated in terms of the lexicographic product, however, the lexicographic product of a graph  $G$  with  $K_k$  is isomorphic to  $G \boxtimes K_k$ .) We conclude the paper with the following bounds for the integer domination number of strong products of graphs.

**Theorem 11.** *For any  $k$  and any graphs  $G$  and  $H$ ,*

$$\begin{aligned} \max\{\gamma_{\{k\}}(G)P_2(H), P_2(G)\gamma_{\{k\}}(H)\} &\leq \gamma_{\{k\}}(G \boxtimes H) \\ &\leq \min\{\gamma_{\{k\}}(G)\gamma(H), \gamma(G)\gamma_{\{k\}}(H)\}. \end{aligned}$$

*Proof.* First let us prove the upper bound for  $\gamma_{\{k\}}(G \boxtimes H)$ . Let  $\gamma_{\{k\}}(G) = s$  and let  $f$  be a minimum  $\{k\}$ -dominating function of  $G$ . Let  $\gamma(H) = t$  and let  $X \subseteq V(H)$  be a minimum dominating set of  $H$ . Define  $\tilde{f}: V(G \boxtimes H) \rightarrow \{0, 1, \dots, s\}$  with

$$\tilde{f}(u, v) = \begin{cases} f(u) & \text{if } v \in X \\ 0 & \text{otherwise} \end{cases}$$

Let  $(u, v)$  be an arbitrary vertex of  $G \boxtimes H$  and let  $w$  be a vertex of  $X$  adjacent in  $H$  to  $v$ . Since  $f[u] = k$  and  $(u, v)$  is adjacent to all vertices  $\{(x, w) \mid ux \in E(G)\}$ , we have  $\tilde{f}[(u, v)] \geq k$ . Hence  $\tilde{f}$  is a  $\{k\}$ -dominating function of  $G \boxtimes H$ . Clearly, its weight is  $\gamma_{\{k\}}(G)|X| = \gamma_{\{k\}}(G)\gamma(H)$ . As the strong product is commutative, the result follows.

For the proof of the lower bound for  $\gamma_{\{k\}}(G \boxtimes H)$ , consider the family of subsets  $N[v_i]$ , where  $v_1, \dots, v_{P_2(G)}$  are vertices in  $G$  at pairwise distance at least three. These subsets are pairwise disjoint, as are the subsets  $N[v_i] \times V(H)$  in  $G \boxtimes H$ . If  $f$  is a minimum  $\{k\}$ -dominating function of  $G \boxtimes H$ , then clearly  $f(N[v_i] \times V(H)) \geq \gamma_{\{k\}}(H)$ . Hence,

$$\gamma_{\{k\}}(G \boxtimes H) = w(f) \geq \sum_{i=1}^{P_2(G)} f(N[v_i] \times V(H)) \geq \gamma_{\{k\}}(H)P_2(G).$$

The desired bound is obtained by reversing the roles of  $G$  and  $H$ . ■

Note that bounds are sharp, and can even coincide. For instance, as soon as the 2-packing number of one of the graphs, say  $H$ , equals its domination number, we have  $\gamma_{\{k\}}(G \boxtimes H) = \gamma_{\{k\}}(G)\gamma(H) = \gamma_{\{k\}}(G)P_2(H)$ .

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