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FORWARDING INDICES OF CARTESIAN PRODUCT GRAPHS

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Abstract. For a given connected graph G of order n, a routing R is a set of $n(n-1)$ elementary paths specified for every ordered pair of vertices in G. The vertex-forwarding index $\xi(G)$ (the edge-forwarding index $\pi(G)$) of G is the maximum number of paths of R passing through any vertex (resp. edge) in G. In this paper we consider the vertex- and the edge- forwarding indices of the cartesian product of $k\,(\geq 2)$ graphs. As applications of our results, we determine the vertex- and the edge- forwarding indices of some well-known graphs, such as the n-dimensional generalized hypercube, the undirected toroidal graph, the directed toroidal graph and the cartesian product of the Petersen graphs.

1. Introduction

In general, we use a graph to model an interconnection network which consists of hardware and/or software entities that are interconnected to facilitate efficient computation and communications (see [9]).

A routing R of a connected graph G of order n is a set of $n(n-1)$ elementary paths $R(u,v)$ specified for all (ordered) pairs u,v of vertices of G. A routing R is said to be minimal if all the paths $R(u,v)$ of R are shortest paths from u to v, denoted by R_m. To measure the efficiency of a routing deterministically, Chung, Coffman, Reiman and Simon [5] introduced the concept of forwarding index of a routing.

The load of a vertex v (resp. an edge e) in a given routing R of $G=(V,E)$, denoted by $\xi(G,R,v)$ (resp. $\pi(G,R,e)$), is the number of paths of R going through v (resp. e), where v is not an end vertex. The parameters

$$\xi(G,R)=\max_{v\in V(G)}\xi(G,R,v)\quad\text{and}\quad\pi(G,R)=\max_{e\in E(G)}\pi(G,R,e)$$

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are defined as the vertex forwarding index and the edge forwarding index of G with respect to R, respectively; and the parameters

$$\xi(G) = \min_R \xi(G,R) \quad \text{and} \quad \pi(G) = \min_R \pi(G,R)$$

are defined as the vertex forwarding index and the edge forwarding index of G, respectively. Similarly, we can define the parameters

$$\xi_m(G) = \min_{R_m} \xi(G,R_m) \quad \text{and} \quad \pi_m(G) = \min_{R_m} \pi(G,R_m).$$

Clearly, $\xi(G) \leq \xi_m(G)$ and $\pi(G) \leq \pi_m(G)$. The equality however does not always hold. The original research of the forwarding indices is motivated by the problem of maximizing network capacity. Maximizing network capacity clearly reduces to minimizing vertex-forwarding index or edge-forwarding index of a routing. Thus, the forwarding index problem has been studied widely by many researchers (see, for example, [3-16]).

Although, determining the forwarding index problem has been shown to be NP-complete by Saad [14], the exact values of the forwarding index of many important classes of graphs have been determined (see, for example, [4, 8, 10, 15]).

Let $G_i = (V_i, E_i)$ be a connected graph with $|V_i| = n_i$ and $|E_i| = \varepsilon_i$ for $i = 1, 2, \cdots, k$. The cartesian product of $G_1, G_2, \cdots, G_k$, denoted by $G_1 \times G_2 \times \cdots \times G_k$, is the graph with the vertex-set $V_1 \times V_2 \times \cdots \times V_k$. Two vertices $(u_1, u_2, \cdots, u_k)$ and $(v_1, v_2, \cdots, v_k)$ are linked by an edge if and only if $(u_1, u_2, \cdots, u_k)$ and $(v_1, v_2, \cdots, v_k)$ differ exactly in one coordinate, say the ith, and there is an edge $u_i v_i \in E(G_i)$. Set

$$A(G_i) = \frac{1}{n_i} \sum_{u_i \in V_i} \left(\sum_{v_i \in V_i \setminus \{u_i\}} (d_{G_i}(u_i, v_i) - 1) \right), B(G_i) = \frac{1}{\varepsilon_i} \sum_{(u_i,\, v_i) \in V_i \times V_i} d_{G_i}(u_i, v_i).$$

For short, we will write ξ_i, π_i, A_i and B_i for $\xi(G_i), \pi(G_i), A(G_i)$ and $B(G_i)$, respectively, for $i = 1, 2, \cdots, k$. In this paper, we will give the following results.

(1) $\xi(G_1 \times G_2 \times \cdots \times G_k) = \sum_{i=1}^{k} n_1 n_2 \cdots n_{i-1} (\xi_i - 1) n_{i+1} \cdots n_k + (k-1) n_1 n_2 \cdots n_k + 1$ if $\xi_i = A_i$ for every $i = 1, 2, \cdots, k$.

(2) $\pi(G_1 \times G_2 \times \cdots \times G_k) = \max_{1 \leq i \leq k} \{n_1 n_2 \cdots n_{i-1} \pi_i n_{i+1} \cdots n_k\}$ if $\pi_i = B_i$ for every $i = 1, 2, \cdots, k$.

The proofs of the results are in Section 3. In Section 2, we will recall some known results to be used in our proofs. In Section 4, as applications of these results, we will determine the vertex-forwarding index and the edge-forwarding index of some well-known graphs.

2. Some Lemmas

Lemma 1. (Chung *et al.* [5]) *Let G be a simple connected graph of order n. Then*

(1) $A(G) \leq \xi(G) \leq \xi_m(G) \leq (n-1)(n-2)$, *and*

(2) *The equalities $\xi_G = \xi_m(G) = A(G)$ are true if and only if there exists a minimal routing in G which induces the same load on every vertex.*

Lemma 2. (Heydemann *et al.* [10]) *Let $G = (V, E)$ be a simple connected graph of order n. Then*

(1) $B(G) \leq \pi(G) \leq \pi_m(G) \leq \lfloor \frac{1}{2} n^2 \rfloor$, *and*

(2) *The equalities $\pi(G) = \pi_m(G) = B(G)$ are true if and only if there exists a minimal routing in G which induces the same load on every edge.*

Lemma 3. (Heydemann *et al.* [10]) *If G_1 and G_2 are two connected graphs of order n_1 and n_2, we have*

(1) $\xi(G_1 \times G_2) \leq n_1 \xi_2 + n_2 \xi_1 + (n_1 - 1)(n_2 - 1)$, *and*

(2) $\pi(G_1 \times G_2) \leq \max\{n_1\pi_2,\ n_2\pi_1\}$.

These inequalities are also valid for minimal routings. Moreover, the equality in (1) holds if both G_1 and G_2 are Cayley graphs.

3. Main Results

In this section, our aim is to give our main results on the vertex-forwarding index and the edge-forwarding index of the cartesian product $G_1 \times G_2 \times \cdots \times G_k$ for $k \geq 2$. In order to make our idea used in the proofs clear, we first consider a simple case of $k = 2$.

Lemma 4. *For each $i = 1, 2$, if G_i is a connected graph with order n_i, then*

(1) $\xi(G_1 \times G_2) \geq n_2 A_1 + n_1 A_2 + (n_1 - 1)(n_2 - 1)$,

(2) $\pi(G_1 \times G_2) \geq \max\{n_2 B_1,\ n_1 B_2\}$.

Proof. Let $U = V_1 \times V_2$ and let $(u_1, u_2),\ (v_1, v_2) \in U$, where $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$. Then, the distance $d_{G_1 \times G_2}((u_1, u_2), (v_1, v_2)) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)$. By Lemma 1, we have that

$$\xi(G_1 \times G_2) \geq \frac{1}{n_1 n_2} \sum_{(u_1,\, u_2) \in U} \sum_{(v_1,\, v_2) \in U \setminus \{(u_1,\, u_2)\}} (d_{G_1 \times G_2}((u_1,\ u_2),\ (v_1,\ v_2)) - 1)$$

$$
\begin{aligned}
&= \frac{1}{n_1 n_2} \sum_{(u_1,u_2)\in U} \sum_{(v_1,v_2)\in U\setminus\{(u_1,u_2)\}} (d_{G_1}(u_1,v_1) + d_{G_2}(u_2,v_2) - 1) \\
&= \frac{1}{n_1 n_2}\, n_2^2 \sum_{u_1\in V_1} \left(\sum_{v_1\in V_1\setminus\{u_1\}} (d_{G_1}(u_1,v_1) - 1) \right) \\
&\quad + \frac{1}{n_1 n_2}\, n_1^2 \sum_{u_2\in V_2} \left(\sum_{v_2\in V_2\setminus\{u_2\}} (d_{G_2}(u_2,v_2) - 1) \right) + (n_1 - 1)(n_2 - 1) \\
&= n_2 A_1 + n_1 A_2 + (n_1 - 1)(n_2 - 1)
\end{aligned}
$$

as desired, and the assertion (1) follows.

We now deduce the lower bound on $\pi(G_1 \times G_2)$ stated in (2). Suppose that R is a routing in $G_1 \times G_2$ such that $\pi(G_1 \times G_2) = \pi(G_1 \times G_2, R)$. Noting that for any $(u_1,u_2),(v_1,v_2) \in V_1 \times V_2$, the path $R((u_1,u_2),(v_1,v_2))$ defined by R has at least $d_{G_1}(u_1,v_1) + d_{G_2}(u_2,v_2)$ edges, we consider two cases.

We first consider that all loads induced by R on edges of the subgraph $\cup_{y\in V_2} G_1 \times \{y\}$. For every $y \in V_2$, use $l_y((u_1,u_2),(v_1,v_2))$ to denote the number of the edges in $R((u_1,u_2),(v_1,v_2))$ located in $G_1 \times \{y\}$. Then the sum $\sum_{y\in V_2} l_y((u_1,u_2),(v_1,v_2))$ of the loads induced by the path $R((u_1,u_2),(v_1,v_2))$ on edges of the subgraph $\cup_{y\in V_2} G_1 \times \{y\}$ is at least $d_{G_1}(u_1,v_1)$ for any $(u_2,v_2) \in V_2 \times V_2$, that is,

$$
\begin{aligned}
\sum_{(u_1,u_2),(v_1,v_2)\in U} \sum_{y\in V_2} l_y((u_1,u_2),(v_1,v_2)) &\geq \sum_{(u_2,v_2)\in V_2\times V_2} \left(\sum_{(u_1,v_1)\in V_1\times V_1} d_{G_1}(u_1,v_1) \right) \\
&= n_2^2 \sum_{(u_1,v_1)\in V_1\times V_1} d_{G_1}(u_1,v_1).
\end{aligned}
$$

Thus, the sum of the loads induced by R on edges of the subgraph $\cup_{y\in V_2} G_1 \times \{y\}$ satisfies the following inequality

$$
\begin{aligned}
\sum_{y\in V_2} \sum_{e\in G_1\times\{y\}} \pi(G_1 \times G_2, R, e) &= \sum_{(u_1,u_2),(v_1,v_2)\in U} \sum_{y\in V_2} l_y((u_1,u_2),(v_1,v_2)) \\
&\geq n_2^2 \sum_{(u_1,v_1)\in V_1\times V_1} d_{G_1}(u_1,v_1).
\end{aligned}
$$

Note that the maximum number of paths passing through one edge can not be less than the average number, we have that

$$
\pi(G_1 \times G_2) = \pi(G_1 \times G_2, R) \geq \frac{1}{n_2 \varepsilon_1}\, n_2^2 \left(\sum_{(u_1,v_1)\in V_1\times V_1} d_{G_1}(u_1,v_1) \right) = n_2 B_1.
$$

By considering the sum of the loads induced by R on edges of the subgraph $\cup_{x\in V_1}\{x\}\times G_2$, similarly, we can show that $\pi(G_1\times G_2)\geq n_1B_2$. Thus, we have that $\pi(G_1\times G_2)\geq \max\{n_2B_1,\ n_1B_2\}$, and the assertion (2) follows.

The proof is completed. ■

Combining Lemma 4 with Lemma 3, we obtain the following results immediately.

Theorem 1. *Let G_1 and G_2 be two connected graphs of order n_1 and n_2.*

(1) $\xi(G_1\times G_2)=n_1\xi_2+n_2\xi_1+(n_1-1)(n_2-1)$ *if* $\xi(G_i)=A(G_i)$ *for* $i=1,2$.

(2) $\pi(G_1\times G_2)=\max\{n_1\pi_2,\ n_2\pi_1\}$ *if* $\pi(G_i)=B(G_i)$ *for* $i=1,2$.

Theorem 2. *Let $G_1,G_2,\cdots,G_k$ be k connected graphs of order $n_1,n_2,\cdots,n_k$, respectively. Then*

(1) $\xi(G_1\times G_2\times\cdots\times G_k)=\sum_{i=1}^{k}n_1n_2\cdots n_{i-1}(\xi_i-1)n_{i+1}\cdots n_k+(k-1)n_1n_2\cdots n_k+1$ *if* $\xi_i=A_i$ *for every* $i=1,2,\cdots,k$.

(2) $\pi(G_1\times G_2\times\cdots\times G_k)=\max_{1\leq i\leq k}\{n_1n_2\cdots n_{i-1}\pi_in_{i+1}\cdots n_k\}$ *if* $\pi_i=B_i$ *for every* $i=1,2,\cdots,k$.

Proof. Let $G=G_1\times G_2\times\cdots\times G_k$ and $V=V(G)$. Then for any two vertices $x=(u_1,u_2,\cdots,u_k)$ and $y=(v_1,v_2,\cdots,v_k)$ in G, where $u_i,v_i\in V_i$ for each $i=1,2,\cdots,k$, the distance $d_G(x,y)=d_{G_1}(u_1,v_1)+d_{G_2}(u_2,v_2)+\cdots+d_{G_k}(u_k,v_k)$. By Lemma 1, we have that

$$
\begin{aligned}
\xi(G)\ \geq\ A(G) &= \frac{1}{n_1n_2\cdots n_k}\sum_{x\in V}\sum_{y\in V\setminus\{x\}}(d_G(x,\ y)-1)\\
&= \frac{1}{n_1n_2\cdots n_k}\sum_{x\in V}\sum_{y\in V\setminus\{x\}}\left(\sum_{i=1}^{k}d_{G_i}(u_i,\ v_i)-1\right)\\
&= \sum_{i=1}^{k}\frac{n_1^2n_2^2\cdots n_{i-1}^2n_{i+1}^2\cdots n_k^2}{n_1n_2\cdots n_k}\sum_{u_i\in V_i}\left(\sum_{v_i\in V_i\setminus\{u_i\}}(d_{G_i}(u_i,v_i)-1)\right)\\
&\quad +(k-1)n_1n_2\cdots n_k-\sum_{i=1}^{k}n_1n_2\cdots n_{i-1}n_{i+1}\cdots n_k+1\\
&= \sum_{i=1}^{k}\frac{n_1n_2\cdots n_{i-1}n_{i+1}\cdots n_k}{n_i}\ (n_iA_i)
\end{aligned}
$$

$$
\begin{aligned}
&+(k-1)n_1n_2\cdots n_k-\sum_{i=1}^{k}n_1n_2\cdots n_{i-1}n_{i+1}\cdots n_k+1\\
=&\sum_{i=1}^{k}n_1n_2\cdots n_{i-1}n_{i+1}\cdots n_k\ (A_i-1)+(k-1)n_1n_2\cdots n_k+1\\
=&\sum_{i=1}^{k}n_1n_2\cdots n_{i-1}(\xi_i-1)n_{i+1}\cdots n_k+(k-1)n_1n_2\cdots n_k+1
\end{aligned}
$$

On the other hand, we need to show that

$$\xi(G)\leq\sum_{i=1}^{k}n_1n_2\cdots n_{i-1}(\xi_i-1)n_{i+1}\cdots n_k+(k-1)n_1n_2\cdots n_k+1.$$

We proceed by induction on k. By Lemma 3, the inequality holds for $k=2$. Assume that the result is true for $k-1$ with $k>2$. Let $H=G_1\times G_2\times\cdots\times G_{k-1}$. By the induction hypothesis, we have that

$$\xi(H)\leq\sum_{i=1}^{k-1}n_1n_2\cdots n_{i-1}(\xi_i-1)n_{i+1}\cdots n_{k-1}+(k-2)n_1n_2\cdots n_{k-1}+1.$$

It follows from Lemma 3 that

$$
\begin{aligned}
\xi(G)&=\xi(H\times G_k)\\
&\leq n_k\xi(H)+n_1n_2\cdots n_{k-1}\xi_k+(n_1n_2\cdots n_{k-1}-1)(n_k-1)\\
&\leq\sum_{i=1}^{k}n_1n_2\cdots n_{i-1}(\xi_i-1)n_{i+1}\cdots n_k+(k-1)n_1n_2\cdots n_k+1
\end{aligned}
$$

as desired, and so the assertion (1) follows.

We now show the assertion (2). On the one hand, in the same idea as one used in the proof of Lemma 4, we can obtain that for each $i=1,2,\cdots,k$,

$$
\begin{aligned}
\pi(G)&\geq\frac{1}{n_1\cdots n_{i-1}\varepsilon_i n_{i+1}\cdots n_k}\sum_{(u_1,u_2,\cdots,u_k),(v_1,v_2,\cdots,v_k)\in V}d_{G_i}(u_i,v_i)\\
&\geq\frac{n_1^2\cdots n_{i-1}^2n_{i+1}^2\cdots n_k^2}{n_1\cdots n_{i-1}\varepsilon_i n_{i+1}\cdots n_k}\left(\sum_{(u_i,v_i)\in(V_i\times V_i)}d(u_i,v_i)\right)\\
&=n_1\cdots n_{i-1}B_in_{i+1}\cdots n_k\\
&=n_1\cdots n_{i-1}\pi_in_{i+1}\cdots n_k.
\end{aligned}
$$

On the other hand, we need to show that

$$\pi(G)\leq\max_{1\leq i\leq k}\{n_1n_2\cdots n_{i-1}\pi_in_{i+1}\cdots n_k\}.$$

We proceed by induction on $k \geq 2$. By Lemma 3, the inequality holds for $k = 2$. Assume that the result is true for $k-1$ with $k > 2$. Let $H = G_1 \times G_2 \times \cdots \times G_{k-1}$, the induction hypothesis implies that $\pi(H) \leq \max\limits_{1\leq i\leq k-1}\{n_1 n_2 \cdots n_{i-1}\pi_i n_{i+1}\cdots n_{k-1}\}$. It follows from Lemma 3 that

$$\begin{aligned}\pi(G) &= \pi(H \times G_k) \leq \max\{(n_1 n_2 \cdots n_{k-1})\ \pi_k,\ n_k \pi(H)\}\\ &\leq \max\{n_1 n_2 \cdots n_{k-1}\pi_k,\ \max_{1\leq i\leq k-1}\{n_1 n_2 \cdots n_{i-1}\pi_i n_{i+1}\cdots n_{k-1} n_k\}\}\\ &= \max_{1\leq i\leq k}\{n_1 n_2 \cdots n_{i-1}\pi_i n_{i+1}\cdots n_k\}\end{aligned}$$

as desired, and so the assertion (2) follows. ■

4. Applications

We first note that Gauyacq [7] introduces a class of vertex-transitive graphs which contains Cayley graphs, called quasi-Cayley graphs, and proves $\xi(G) = A(G)$ for any quasi-Cayley graph G. Thus, the conclusion (1) in Theorem 2 is valid for quasi-Cayley graphs. However, we have not yet known whether $\pi(G) = B(G)$ for any quasi-Cayley graph G. We also note that Solé [16] constructed a class of graphs, called orbital regular graphs, which satisfy $\pi(G) = B(G)$. Thus, the conclusion (2) in Theorem 2 is valid for orbital regular graphs. However, we have not yet known whether $\xi(G) = A(G)$ for any orbital regular graph G. In this section, we determine the vertex-forwarding index and the edge-forwarding index of some well-known graphs as applications of Theorem 2.

Example 1. The n-dimensional generalized hypercube, proposed by Bhuyan and Agrawal [1] and denoted by $Q(d_1, d_2, \cdots, d_n)$, where $d_i \geq 2$ is an integer for each $i = 1, 2, \cdots, n$, is defined as the cartesian products $K_{d_1} \times K_{d_2} \times \cdots \times K_{d_n}$. If $d_1 = d_2 = \cdots = d_n = d \geq 2$, then $Q(d, d, \cdots, d)$ is called the d-ary n-dimensional cube, denoted by $Q_n(d)$. It is clear that $Q_n(2)$ is Q_n.

It is clear that $\xi(K_d) = 0 = A(K_d)$ and $\pi(K_d) = 2 = B(K_d)$. By Theorem 2, we have that

$$\begin{aligned}\xi(Q(d_1, d_2, \cdots, d_n)) &= -\sum_{i=1}^{n} d_1 d_2 \cdots d_{i-1} d_{i+1}\cdots d_n + (n-1)d_1 d_2 \cdots d_n + 1,\\ \pi(Q(d_1, d_2, \cdots, d_n)) &= \max_{1\leq i\leq n}\{d_1 d_2 \cdots d_{i-1} 2 d_{i+1}\cdots d_n\}.\end{aligned}$$

In particular,

$$\xi(Q_n(d)) = ((d-1)n - d)d^{n-1} + 1, \quad \text{and} \quad \pi(Q_n(d)) = 2d^{n-1}.$$

For the n-dimensional hypercube Q_n,

$$\xi(Q_n) = (n-2)2^{n-1} + 1 \quad \text{and} \quad \pi(Q_n) = 2^n.$$

The last result has also been obtained by Heydemann et al. [10].

Example 2. The cartesian product $C_{d_1} \times C_{d_2} \times \cdots \times C_{d_n}$ of n undirected cycles $C_{d_1}, C_{d_2}, \cdots, C_{d_n}$ of order $d_1, d_2, \cdots, d_n$, $d_i \geq 3, i = 1, 2, \cdots, n$, is the undirected toroidal graph, denoted by $C(d_1, d_2, \cdots, d_n)$. A special case of $d_1 = d_2 = \cdots = d_n = d$, the $C(d, d, \cdots, d)$, denoted by $C_n(d)$, is also called a d-ary n-cube in the literature (see, for example, Bose et al. [2]) or generalized n-cube (see, for example, Heydemann et al. [10]).

It is easy to be verify (see, for example, Heydemann et al. [10]) that

$$\xi(C_d) = \left\lfloor \frac{(d-2)^2}{4} \right\rfloor = \frac{1}{d} \sum_{u \in V} \sum_{v \neq u} (d_{C_d}(u,v) - 1) = A(C_d),$$

and

$$\pi(C_d) = \left\lfloor \frac{d^2}{4} \right\rfloor = \frac{1}{d} \sum_{(u,v) \in V \times V} d_{C_d}(u,v) = B(C_d).$$

Therefore, by Theorem 2, we have that

$$\begin{aligned}
\xi(C(d_1, d_2, \cdots, d_n)) &= \sum_{i=1}^{n} d_1 d_2 \cdots d_{i-1} (\xi_i - 1) d_{i+1} \cdots d_n + (n-1) d_1 d_2 \cdots d_n + 1 \\
&= \sum_{i=1}^{n} d_1 d_2 \cdots d_{i-1} \left(\left\lfloor \frac{(d_i-2)^2}{4} \right\rfloor - 1 \right) d_{i+1} \cdots d_n \\
&\quad + (n-1) d_1 d_2 \cdots d_n + 1 \\
&= \sum_{i=1}^{n} d_1 d_2 \cdots d_{i-1} \left\lfloor \frac{d_i^2}{4} \right\rfloor d_{i+1} \cdots d_n - d_1 d_2 \cdots d_n + 1,
\end{aligned}$$

and

$$\begin{aligned}
\pi(C(d_1, d_2, \cdots, d_n)) &= \max_{1 \leq i \leq k} \{ d_1 d_2 \cdots d_{i-1} \pi_i d_{i+1} \cdots d_n \} \\
&= \max_{1 \leq i \leq k} \left\{ d_1 d_2 \cdots d_{i-1} \left\lfloor \frac{d_i^2}{4} \right\rfloor d_{i+1} \cdots d_n \right\}.
\end{aligned}$$

It follows that for the undirected toroidal mesh $C(d_1, d_2, \cdots, d_n)$,

$$\begin{aligned}
\xi(C(d_1, d_2, \cdots, d_n)) &= \sum_{i=1}^{n} d_1 d_2 \cdots d_{i-1} \left\lfloor \frac{d_i^2}{4} \right\rfloor d_{i+1} \cdots d_n - d_1 d_2 \cdots d_n + 1; \\
\pi(C(d_1, d_2, \cdots, d_n)) &= \max_{1 \leq i \leq n} \left\{ d_1 d_2 \cdots d_{i-1} \left\lfloor \frac{d_i^2}{4} \right\rfloor d_{i+1} \cdots d_n \right\}.
\end{aligned}$$

In particular,

$$\xi(C_n(d)) = nd^{n-1}\left\lfloor \frac{1}{4}d^2 \right\rfloor - (d^n - 1), \quad \text{and} \quad \pi(C_n(d)) = d^{n-1}\left\lfloor \frac{1}{4}d^2 \right\rfloor.$$

The last result has also been obtained by Heydemann et al. [10].

Example 3. Note that Theorem 2 is also valid for the cartesian product of strongly connected digraphs. Use $\overrightarrow{C}(d_1, d_2, \cdots, d_n)$ to denoted the cartesian product $\overrightarrow{C}_{d_1} \times \overrightarrow{C}_{d_2} \times \cdots \times \overrightarrow{C}_{d_n}$ of n directed cycles $\overrightarrow{C}_{d_1}, \overrightarrow{C}_{d_2}, \cdots, \overrightarrow{C}_{d_n}$ of order $d_1, d_2, \cdots, d_n$, $d_i \geq 3$ for each $i = 1, 2, \cdots, n$, which is called the directed toroidal graph. Set $\overrightarrow{C}_n(d) = \overrightarrow{C}(d, d, \cdots, d)$. It is easy to be verified that

$$\xi(\overrightarrow{C}_d) = \frac{(d-2)(d-1)}{2} = A(\overrightarrow{C}_d), \quad \text{and} \quad \pi(\overrightarrow{C}_d) = \frac{d(d-1)}{2} = B(\overrightarrow{C}_d).$$

By Theorem 2, we have that

$$\begin{aligned} \xi(\overrightarrow{C}(d_1, d_2, \cdots, d_n)) &= \frac{1}{2}\left(\sum_{i=1}^{n}(d_i - 3)\right) d_1 d_2 \cdots d_n + (n-1) d_1 d_2 \cdots d_n + 1; \\ \pi(\overrightarrow{C}(d_1, d_2, \cdots, d_n)) &= \frac{1}{2}\max_{1 \leq i \leq n}\{d_1 \cdots d_{i-1} d_i (d_i - 1) d_{i+1} \cdots d_n\}. \end{aligned}$$

In particular,

$$\xi(\overrightarrow{C}_n(d)) = \frac{n}{2}\, d^n(d-1) - d^n + 1, \quad \text{and} \quad \pi(\overrightarrow{C}_n(d)) = \frac{1}{2}\, d^n(d-1).$$

Example 4 Let $P = (V, E)$ be the Petersen graph. Note that P is vertex-transitive, $|V| = 10$, $|E| = 15$ and the shortest path between two distinct vertices is unique. It is easy to be determined that $\xi_m(P) = 6$ and $\pi_m(P) = 10$. We now compute $A(P)$ and $B(P)$. Since the diameter of P is two and from any given vertex three vertices can be reached in a distance of one and six vertices can be reached in a distance of two, thus,

$$A(P) = \frac{1}{|V|}\sum_{u \in V}\left(\sum_{v \in V \setminus \{u\}}(d_P(u, v) - 1)\right) = \frac{1}{10} \cdot 10 \cdot 6 = 6,$$

$$B(P) = \frac{1}{|E|}\sum_{(u,v) \in V \times V} d_P(u, v) = \frac{1}{15} \cdot 10 \cdot (3 + 6 \cdot 2) = 10.$$

Thus, we have that $6 = A(P) \leq \xi(P) \leq \xi_m(P) = 6$ by Lemma 1, and $10 = B(P) \leq \pi(P) \leq \pi_m(P) = 10$ by Lemma 2.

Let G be the cartesian product of n Petersen graphs. Then, by Theorem 2, we obtain that

$$\xi(G) = 10^{n-1}(15n-10)+1, \quad \text{and} \quad \pi(G) = 10^n.$$

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