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ON ERGODIC AVERAGES AND THE RANGE OF A CLOSED OPERATOR

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Abstract. For a γ-th order Cesàro mean bounded linear operator T on a Banach space X, we characterize the range $R(A)$ of the operator $A = T - I$, by using an A-ergodic net and its companion net which were introduced by Dotson and developed by Shaw. Similarly, if A is the generator of a γ-th order Cesàro mean bounded C_0-semigroup (or strongly continuous cosine operator function) of bounded linear operators on X, then we characterize the range $R(A)$.

1. INTRODUCTION

Let X be a Banach space and A be a (bounded or unbounded) closed operator in X with domain $D(A)$ and range $R(A)$. By using ergodic theory, many authors have studied the problem of solving the functional equation $Ax = y$ for a given $y \in X$. See, for example, Alonso, Hong and Obaya [1], Assani [2], Dotson [4-6], Krengel and Lin [10], Lin and Sine [12], Sato [14-18], Shaw [19, 20], and Shaw and Li [21]. In particular, Shaw [19] (see also Dotson [4] and Shaw-Li [21]) studied deeply the mean ergodic properties of an A-ergodic net $\{A_\alpha\}$ and its companion net $\{B_\alpha\}$ consisting of bounded linear operators on X, and applied them to the problem successfully. In this paper the author intends to adapt Shaw's method of study in order to obtain new results and generalize some known results in [12], [19] and [20]. In §2 some preliminary results are presented, which will be useful to understand the general situation. In §3 we consider a (bounded or unbounded) closed operator A. An A-ergodic net $\{A_\alpha\}$ and its companion net $\{B_\alpha\}$ are defined. (Our definition of an A-ergodic net is slightly different from that

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used in [21].) Some lemmas and mean ergodic properties of these nets are obtained, which will be used in later sections. In §4 we apply the results obtained in §3 to the problem of the form $(T-I)x = y$, where T is a bounded linear operator on X satisfying $\sup_{n\geq 0} \|C_n^\gamma(T)\| < \infty$ for some $\gamma > -1$, $C_n^\gamma(T)$ being the γ-th order Cesàro mean of the operator sequence $\{T^n\}_{n=0}^\infty$. In §5 [resp. §6] we study the problem of the form $Ax = y$, where A is the generator of a C_0-semigroup $\{T_t\}_{t\geq 0}$ [resp. a strongly continuous cosine operator function $\{C(t)\}_{t\geq 0}$] of bounded linear operators on X such that $\sup_{t>0} \|C_t^\gamma\| < \infty$ for some $\gamma \geq 0$, where C_t^γ denotes the Cesàro mean of order γ of $\{T_t\}_{t\geq 0}$ [resp. $\{C(t)\}_{t\geq 0}$], i.e., $C_t^0 = T_t$ [resp. $C_t^0 = C(t)$], and

$$C_t^\gamma = \frac{\gamma}{t^\gamma}\int_0^t (t-s)^{\gamma-1} T_s\, ds \quad \left[\text{resp.}\ \ C_t^\gamma = \frac{\gamma}{t^\gamma}\int_0^t (t-s)^{\gamma-1} C(s)\, ds\right] \ \ \text{for } \gamma > 0.$$

2. Preliminary Results

Let T be a bounded linear operator on a Banach space X. We first define its Cesàro means $C_n^\gamma(T)$ of order $\gamma \neq -1, -2, -3, \ldots$ as follows:

$$C_n^\gamma(T) = \begin{pmatrix} \gamma+n \\ n \end{pmatrix}^{-1} \sum_{k=0}^{n} \begin{pmatrix} \gamma-1+k \\ k \end{pmatrix} T^{n-k} \qquad (n \geq 0),$$

where $\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = 1$, and $\begin{pmatrix} \alpha \\ k \end{pmatrix} = \dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$ for $\alpha \in \boldsymbol{R}$ and $k \in \boldsymbol{N} = \{1, 2, \ldots\}$. In particular, we have $C_n^0(T) = T^n$ and $C_n^1(T) = (n+1)^{-1}\sum_{k=0}^n T^k$ for $n \geq 0$. In this paper we only consider the case $\gamma > -1$, because the other case $\gamma < -1$ is not interesting (cf. Chapter III of Zygmund [24]); and we mainly consider the case $\gamma \geq 0$, because a pathological result happens in the case $-1 < \gamma < 0$ (cf. Proposition 4.1 of Li-Sato-Shaw [11].) In the following we give straightforward sufficient conditions for $y \in \overline{R(T-I)}$ and $y \in R(T-I)$, and similarly for $y \in \overline{R(A)}$ and $y \in R(A)$, where A is a closed operator. These will become necessary conditoins when T [resp. A] satisfies some appropriate additional hypotheses; this will be considered in later sections.

Fact 1. Let $y \in X$ and the series $\sum_{k=0}^\infty r^k T^k y$ be summable for all r, with $0 < r < 1$. Let $0 < r_n < 1$ for all $n \geq 1$, and $r_n \uparrow 1$ as $n \to \infty$. Then the following hold:

(i) If weak-$\lim_n (1-r_n)\sum_{k=0}^\infty r_n^k T^k y = 0$, then $y \in \overline{R(T-I)}$.

(ii) If weak-$\lim_n \sum_{k=0}^\infty r_n^k T^k y = x_0$ for some $x_0 \in X$, then $x_0 \in \overline{R(T-I)}$ and $y = x_0 - Tx_0$.

Proof.

(i) We define $f(r) = \sum_{k=0}^{\infty} r^k T^k y$ for $0 \leq r < 1$. If $x^* \in X^*$ is such that $x^* = 0$ on $\overline{R(T-I)}$, then

$$\langle (1-r)f(r),\, x^*\rangle = (1-r)\sum_{k=0}^{\infty} r^k \langle T^k y,\, x^*\rangle = (1-r)\sum_{k=0}^{\infty} r^k \langle y,\, x^*\rangle = \langle y,\, x^*\rangle.$$

Since weak-$\lim_{n\to\infty} (1-r_n)f(r_n) = 0$ by hypothesis, we have

$$0 = \lim_n \langle (1-r_n)f(r_n),\, x^*\rangle = \langle y,\, x^*\rangle,$$

which implies $y \in \overline{R(T-I)}$ by the Hahn-Banach theorem.

(ii) Since $f(r_n)$ converges to x_0 weakly, there exists a constant $M > 0$ such that $\|f(r_n)\| \leq M$ for all $n \geq 1$. Then

$$\begin{aligned} f(r_n) - Tf(r_n) &= \textstyle\sum_{k=0}^{\infty} r_n^k T^k y - \sum_{k=0}^{\infty} r_n^k T^{k+1} y \\ &= y + (r_n - 1)\textstyle\sum_{k=0}^{\infty} r_n^k T^{k+1} y = y - (1-r_n)Tf(r_n), \end{aligned}$$

and hence

$$\|(f(r_n) - Tf(r_n)) - y\| \leq (1-r_n)\|T\|M \to 0$$

as $n \to \infty$. Thus, we have

$$x_0 - Tx_0 = \text{ weak-}\lim_n\, (f(r_n) - Tf(r_n)) = y.$$

On the other hand, since $\lim_n\, (1-r_n)\|\sum_{k=0}^{\infty} r_n^k T^k y\| = 0$, we have $y \in \overline{R(T-I)}$ by (i), whence $T^k y \in \overline{R(T-I)}$ for all $k \geq 0$. It follows that $f(r_n) \in \overline{R(T-I)}$ for all $n \geq 1$, and hence $x_0 = \text{weak-}\lim_n\, f(r_n) \in \overline{R(T-I)}$. This completes the proof.

Fact 2. Let $y \in X$. Then the following hold:

(i) If $\gamma > -1$ and weak-$\lim_n\, C_n^{\gamma}(T)y = 0$, then $y \in \overline{R(T-I)}$.

(ii) If $\gamma > 0$ and the set $\{nC_n^{\gamma}(T)y : n \geq 0\}$ is weakly sequentially compact, then there exists $x_0 \in \overline{R(T-I)}$ such that $y = x_0 - Tx_0$.

Proof.

(i) Let $x^* \in X^*$ be such that $x^* = 0$ on $\overline{R(T-I)}$. Since $\langle T^n y,\, x^*\rangle = \langle y,\, x^*\rangle$ for all $n \geq 0$, it follows that $\langle C_n^{\gamma}(T)y,\, x^*\rangle = \langle y,\, x^*\rangle$. This and the hypothesis of (i) imply $0 = \langle y,\, x^*\rangle$, so that $y \in \overline{R(T-I)}$ by the Hahn-Banach theorem.

(ii) From the hypothesis of (ii) it follows that the set

$$\left\{\frac{\gamma+n}{\gamma}C_n^\gamma(T)y : n \geq 0\right\}$$

is weakly sequentially compact, and $\sup_{n\geq 0}\,(\gamma+n)\gamma^{-1}\|C_n^\gamma(T)y\| < \infty$. Thus, letting

$$A_n^\alpha = \binom{\alpha+n}{n} \quad \text{and} \quad S_n^\alpha(T) = \sum_{k=0}^n A_{n-k}^{\alpha-1}T^k \quad \text{for } \alpha \in \boldsymbol{R} \text{ and } n \geq 0,$$

we have (cf. Chapter III of Zygmund [24]) that for $0 < r < 1$,

$$\sum_{n=0}^\infty r^nT^ny = (1-r)^\gamma(1-r)^{-\gamma}\sum_n r^nT^ny = (1-r)^\gamma(\sum_{n=0}^\infty A_n^{\gamma-1}r^n)\sum_{n=0}^\infty r^nT^ny$$

$$= (1-r)^\gamma\sum_{n=0}^\infty r^n(\sum_{k=0}^n A_{n-k}^{\gamma-1}T^ky) = (1-r)^\gamma\sum_{n=0}^\infty r^nS_n^\gamma(T)y$$

$$= (1-r)^\gamma\sum_{n=0}^\infty r^nA_n^{\gamma-1}\cdot\frac{A_n^\gamma}{A_n^{\gamma-1}}C_n^\gamma(T)y\,,$$

where

$$\frac{A_n^\gamma}{A_n^{\gamma-1}} = \frac{(\gamma+1)(\gamma+2)\ldots(\gamma+n)\,n!}{n!\,\gamma(\gamma+1)\ldots(\gamma+n-1)} = \frac{\gamma+n}{\gamma} \qquad (n \geq 0).$$

Since $(1-r)^\gamma\sum_{n=0}^\infty r^nA_n^{\gamma-1} = 1$ and $A_n^{\gamma-1} > 0$ for all $n \geq 0$, it follows from Theorem V.6.4 of [7] that the set $\{\sum_{n=0}^\infty r^nT^ny : 0 < r < 1\}$ is weakly sequentially compact. Hence we can apply (ii) of Fact 1 to complete the proof.

Example 1. There exists an example showing that (ii) of Fact 2 is not true if the hypothesis $\gamma > 0$ is replaced with $\gamma = 0$. In fact, there may exist $f \notin R(T-I)$ such that $\|nC_n^0(T)f\| \to 0$ as $n \to 0$ (and so $\{nC_n^0(T)f : n \geq 0\}$ is weakly sequentially compact). To see this, we first note that, by an elementary argument, there exists a sequence $\{p_n\}_{n=1}^\infty$ of positive real numbers (for example, $p_n = (n\log n)^{-1}$) such that

$$1 \geq p_n \downarrow 0 \quad \text{as } n \to \infty, \quad \lim_{n\to\infty} np_n = 0, \quad \text{and} \quad \sum_{n=1}^\infty p_n = \infty.$$

Define a measure μ on $\boldsymbol{Z}$ by

$$\mu(\{m\}) = \begin{cases} 1 & \text{if } m \leq 0, \\ p_m & \text{if } m \geq 1. \end{cases}$$

We consider $X = L_1(\boldsymbol{Z}, \mu)$, and define an operator $T : L_1(\boldsymbol{Z}, \mu) \to L_1(\boldsymbol{Z}, \mu)$ by

$$Tf(m) = f(m-1) \qquad (m \in \boldsymbol{Z}).$$

It is clear that $\|T\|_1 = 1$. If we set $f = \chi_{\{0\}}$, then $T^n f = \chi_{\{n\}}$ for $n \geq 0$, and

$$\|nC_n^0(T)f\|_1 = \|nT^n f\|_1 = \|n\chi_{\{n\}}\|_1 = np_n \to 0$$

as $n \to \infty$. We next prove that $f \notin R(T - I)$. Assume the contrary: $f = \chi_{\{0\}} = Tg - g$ for some $g \in L_1(\boldsymbol{Z}, \mu)$. Then, since $\chi_{\{0\}}(m) = g(m-1) - g(m)$ for all $m \in \boldsymbol{Z}$, it follows that $g(0) + 1 = g(-1) = g(-2) = \ldots$, and $g(0) = g(n)$ for all $n \geq 1$. Since $g \in L_1(\boldsymbol{Z}, \mu)$, we must have $0 = g(0) + 1 = g(-1) = g(-2) = \ldots$, and thus $-1 = g(0) = g(n)$ for all $n \geq 1$. But, this is a contradiction, because $\infty = \sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} |g(n)| \cdot \mu(\{n\}) \leq \|g\|_1 < \infty$.

Fact 3. Let A be a closed operator in X with domain $D(A)$ and range $R(A)$. Let $\rho(A)$ denote the resolvent set of A, and assume that $0 \neq \lambda_n \in \rho(A)$ for $n \geq 1$ and $\lim_{n\to\infty} \lambda_n = 0$. Let $y \in X$. Then the following hold:

(i) If weak-$\lim_{n\to\infty} \lambda_n(\lambda_n - A)^{-1}y = 0$, then $y \in \overline{R(A)}$.

(ii) If weak-$\lim_{n\to\infty}(\lambda_n - A)^{-1}y = x_0$ for some $x_0 \in X$, then $x_0 \in \overline{R(A)} \cap D(A)$ and $y = -Ax_0$.

Proof.

(i) Since $y - (y - \lambda_n(\lambda_n - A)^{-1}y) = \lambda_n(\lambda_n - A)^{-1}y \to 0$ weakly as $n \to \infty$, and since $y - \lambda_n(\lambda_n - A)^{-1}y = -A(\lambda_n - A)^{-1}y \in R(A)$, it follows that $y \in \overline{R(A)}$.

(ii) Since the hypothesis of (ii) implies $\sup_{n\geq 1} \|(\lambda_n - A)^{-1}y\| < \infty$, we obtain that $\|A(\lambda_n - A)^{-1}y + y\| = \|\lambda_n(\lambda_n - A)^{-1}y\| \to 0$ as $n \to \infty$. Thus, by the (weak) closedness of A, we see that $x_0 \in D(A)$ and $-y = Ax_0$. Furthermore,

$$x_0 = \text{weak-}\lim_{n\to\infty} (\lambda_n - A)^{-1}y = \text{weak-}\lim_{n\to\infty} (\lambda_n - A)^{-1}A(-x_0)$$

$$= \text{weak-}\lim_{n\to\infty} A(\lambda_n - A)^{-1}(-x_0).$$

Hence $x_0 \in \overline{R(A)} \cap D(A)$, and the proof is complete.

Example 2. There exists an example showing that the converses of (i) and (ii) of Facts 1, 2 and 3 fail to hold. To see this, we define a measure μ on $\boldsymbol{Z}$ by

$$\mu(\{m\}) = \begin{cases} 1 & \text{if } m \le 0, \\ (m+1)^2 & \text{if } m \ge 1. \end{cases}$$

We consider $X = L_1(\boldsymbol{Z}, \mu)$, and define, as in Example 1, an operator $T : L_1(\boldsymbol{Z}, \mu) \to L_1(\boldsymbol{Z}, \mu)$ by $Tf(m) = f(m-1)$ for $m \in \boldsymbol{Z}$. It follows that $\|T^n\|_1 = (n+1)^2$ for $n \ge 0$, and hence $r(T) = 1$, where $r(T)$ denotes the spectral radius of T. If we set $g = \chi_{\{0\}} - \chi_{\{-1\}}$, then $g = T\chi_{\{-1\}} - \chi_{\{-1\}} \in (T - I)L_1(\boldsymbol{Z}, \mu)$, and for $0 < r < 1$ we have

$$h_r(m) := \sum_{n=0}^{\infty} r^n T^n g(m) = \begin{cases} 0 & \text{if } m \le -2, \\ -1 & \text{if } m = -1, \\ r^m - r^{m+1} & \text{if } m \ge 0. \end{cases}$$

Therefore,

$$\begin{aligned} \|h_r\|_1 &= 1 + \sum_{m=0}^{\infty} (1-r) r^m \mu(\{m\}) = 1 + (1-r) \sum_{m=0}^{\infty} r^m (m+1)^2 \\ &> 1 + (1-r)2^{-1} \sum_{m=0}^{\infty} (m+2)(m+1) r^m = 1 + (1-r)^{-2}, \end{aligned}$$

so that $\lim_{r\uparrow 1} \|h_r\|_1 = \lim_{r\uparrow 1} \|\sum_{n=0}^{\infty} r^n T^n g\|_1 = \infty$, and furthermore

$$\lim_{r\uparrow 1} \|(1-r) \sum_{n=0}^{\infty} r^n T^n g\|_1 = \infty. \tag{1}$$

Next, let $A = T - I$. If $\lambda > 0$ then, clearly, $\lambda \in \rho(A)$, and

$$(\lambda - A)^{-1} = ((\lambda+1) - T)^{-1} = (\lambda+1)^{-1} \sum_{n=0}^{\infty} \frac{T^n}{(\lambda+1)^n}. \tag{2}$$

Thus, using the equality $(\lambda - A)^{-1} A g = \lambda(\lambda - A)^{-1} g - g$, we get by (1) and (2) that

$$\begin{aligned} \lim_{\lambda\downarrow 0} \|(\lambda - A)^{-1} A g\|_1 &\ge \lim_{\lambda\downarrow 0} \|\lambda(\lambda - A)^{-1} g\|_1 - \|g\|_1 \\ &= \lim_{\lambda\downarrow 0} \frac{\lambda}{\lambda+1} \left\| \sum_{n=0}^{\infty} \frac{T^n g}{(\lambda+1)^n} \right\|_1 - 2 = \infty. \end{aligned} \tag{3}$$

Since $g \in R(T - I)$, we see that the converses of (i) and (ii) of Facts 1, 2 and 3 fail to hold for $y = g$ and $y = -Ag = g - Tg$, respectively.

3. General Results on Closed Operators

In this section we consider a closed operator A in X with domain $D(A)$ and range $R(A)$. Let $\{A_\alpha\}$ be a net of bounded linear operators on X such that

(a) $\|A_\alpha\| \le M\ (< \infty)$ for all α,

(b) $R(A_\alpha - I) \subset \overline{R(A)}$ for all α, and $\lim_\alpha \|x - A_\alpha x\| = 0$ for all $x \in N(A) := \{x \in D(A) : Ax = 0\}$.

Let P denote the operator in X defined by

$$D(P) = \{x \in X : \lim_\alpha A_\alpha x \text{ exists}\}, \text{ and } Px = \lim_\alpha A_\alpha x \quad \text{for } x \in D(P).$$

By (a), $D(P)$ is a closed subset of X, and P is a closed operator in X. Furthermore, from (b) we see that

(4) $Px = x$ for all $x \in N(A)$, $N(A) \subset R(P) \cap D(P)$, and $N(P) \subset \overline{R(A)}$.

We consider the following conditions:

(M1) $P^2 = P$, $R(P) = N(A)$, and $N(P) = \overline{R(A)}$.
(M2) $D(P) = \{x \in X : \{A_\alpha x\}$ has a weak cluster point$\}$.

(M1) implies at once that $D(P) = N(A) \oplus \overline{R(A)}$. In [19], Shaw proved that (a), (b) and the following condition (c) imply (M1) and (M2). (We note that (c) does not follow from the above conditions (a), (b), (M1) and (M2). See Remark 6 below.)

(c) $\bigcup_\alpha R(A_\alpha) \subset D(A)$, weak-$\lim_\alpha AA_\alpha x = 0$ for all $x \in X$, and $\lim_\alpha \|A_\alpha Ax\| = 0$ for all $x \in D(A)$.

In many cases there exists another net $\{B_\alpha\}$ of bounded linear operators on X which is related to A and $\{A_\alpha\}$ in the following way:

(b′) $R(B_\alpha) \subset D(A)$ and $I - A_\alpha = AB_\alpha \supset B_\alpha A$ for all α.

It is direct that (b′) implies (b), and furthermore $AA_\alpha x = A_\alpha Ax$ for all $x \in D(A)$. As in Dotson [4] and Shaw-Li [21], we will call $\{A_\alpha\}$ *an A-ergodic net* and $\{B_\alpha\}$ *its companion net*.

We next consider the following conditions:

(w) There exists a weakly compact operator H on X such that $A_\alpha H = HA_\alpha$ for all α, and $R(H - I) \subset \overline{R(A)}$.

(w′) There exists a weakly compact operator H on X such that $B_\alpha H = HB_\alpha$ for all α, and $R(H-I) \subset R(A)$.

Lemma 1. $(a), (b), (M1), (M2)$ *and* (w) *imply* $D(P) = X$.

Proof. Let $x \in X$. Since H is weakly compact and $A_\alpha Hx = HA_\alpha x$ by (w), and since $\|A_\alpha x\| \le M\|x\|$ for all α by (a), it follows that $\{A_\alpha Hx\}$ has a weak cluster point. Hence $Hx \in D(P)$ by (M2), and then putting

$$x_h = PHx = \lim_\alpha A_\alpha Hx,$$

we have $x_h \in R(P) = N(A)$ by (M1). To see that $x - x_h \in \overline{R(A)}$, we notice from (b) and (w) that $R(I - A_\alpha) \subset \overline{R(A)}$ for all α, and $R(H-I) \subset \overline{R(A)}$. Thus, if $x^* \in X^*$ is such that $x^* = 0$ on $\overline{R(A)}$, then $\langle x_h, x^*\rangle = \lim_\alpha \langle A_\alpha Hx, x^*\rangle = \langle Hx, x^*\rangle = \langle x, x^*\rangle$, whence $\langle x - x_h, x^*\rangle = 0$. It follows that $x - x_h \in \overline{R(A)}$ by the Hahan-Banach theorem, and thus (M1) completes the proof.

Lastly we consider the condition

(d) $B_\alpha^* x^* = \varphi(\alpha)x^*$ for all $x^* \in X^*$ such that $x^* = 0$ on $\overline{R(A)}$, and $\lim_\alpha |\varphi(\alpha)| = \infty$.

Lemma 2. *Under the hypotheses (a), (b′), (M1), (w′) and (d), the condition* $y \in R(A)$ *is equivalent to* $\sup_\alpha \|B_\alpha y\| < \infty$.

Proof. Suppose $y = Ax$ for some $x \in D(A)$. Then $B_\alpha y = B_\alpha Ax = x - A_\alpha x$ for all α by (b′), and hence $\sup_\alpha \|B_\alpha y\| \le (1+M)\|x\|$ by (a).

Conversely, suppose $\sup_\alpha \|B_\alpha y\| < \infty$. Since H is weakly compact and $B_\alpha Hy = HB_\alpha y$ by (w′), it follows that $\{B_\alpha Hy\}$ has a weak cluster point $x_p \in X$. Let $\{B_\beta Hy\}$ be a subnet of $\{B_\alpha Hy\}$ with

$$x_p = \text{weak-}\lim_\beta B_\beta Hy. \tag{5}$$

If $x^* \in X^*$ is such that $x^* = 0$ on $\overline{R(A)}$, then

$$\langle x_p, x^*\rangle = \lim_\beta \langle B_\beta Hy, x^*\rangle = \lim_\beta \varphi(\beta)\langle Hy, x^*\rangle \quad \text{and} \quad \lim_\beta |\varphi(\beta)| = \infty,$$

by (d). It follows that $\langle Hy, x^*\rangle = 0$, and hence $Hy \in \overline{R(A)}$ by the Hahn-Banach theorem. Thus $PHy = 0$ by (M1). Furthermore, by (b′),

$$Hy = Hy - PHy = \lim_\beta (Hy - A_\beta Hy) = \lim_\beta AB_\beta Hy,$$

so that (5) and the (weak) closedness of A can be applied to infer that $x_p \in D(A)$ and $Hy = Ax_p$. Since $Hy - y = Ax$ for some $x \in D(A)$ by (w′), we conclude that $y = Hy - (Hy - y) = A(x_p - x)$, and this completes the proof.

Remark 1. Under the hypotheses of Lemma 2, the condition $\sup_\alpha \|B_\alpha y\| < \infty$ is equivalent to $\liminf_\alpha \|B_\alpha y\| < \infty$. Indeed, if $\liminf_\alpha \|B_\alpha y\| < \infty$, then there exists a subnet $\{B_\beta y\}$ of the net $\{B_\alpha y\}$ such that $\sup_\beta \|B_\beta y\| < \infty$. Then, clearly, $(A,\, A_\beta,\, B_\beta)$ satisfies conditions (a), (b′), (M1), (w′) and (d) with β in place of α. It follows from Lemma 2 that $y \in R(A)$, and hence $\sup_\alpha \|B_\alpha y\| < \infty$.

Fact 4. (Cf. Theorem 1 of [20].) Let $0 \neq \lambda_n \in \rho(A)$ for $n \geq 1$, and $\lim_n \lambda_n = 0$. If $\|\lambda_n(\lambda_n - A)^{-1}\| \leq M\ (< \infty)$ for all $n \geq 1$, and the operator $H = \sum_{j=1}^n a_j(b_j - A)^{-1}$, where $0 \neq b_j \in \rho(A)$ and $\sum_{j=1}^n a_j/b_j \neq 0$, is weakly compact, then

(i) $X = N(A) \oplus \overline{R(A)}$, and

(ii) $y \in R(A) \Leftrightarrow \sup_{n\geq 1} \|(\lambda_n - A)^{-1}y\| < \infty \Leftrightarrow \liminf_{n\to\infty} \|(\lambda_n - A)^{-1}y\| < \infty$.

Proof. Put $A_n = \lambda_n(\lambda_n - A)^{-1}$ and $B_n = -(\lambda_n - A)^{-1}$ for $n \geq 1$. By an elementary calculation it is known (cf. [19]) that $\{A_n\}$ is an A-ergodic sequence satisfying condition (c), and $\{B_n\}$ is its companion sequence satisfying condition (d). Letting $c = \sum_{j=1}^n a_j/b_j$, we then see that

$$c^{-1}H - I = \sum_{j=1}^n c^{-1}(a_j/b_j)[b_j(b_j - A)^{-1} - I] = \sum_{j=1}^n c^{-1}(a_j/b_j)A(b_j - A)^{-1},$$

so that $R(c^{-1}H - I) \subset R(A)$. It is clear that $HA_n = A_nH$ and $HB_n = B_nH$ for all $n \geq 1$. Thus, conditions (w) and (w′) hold with $c^{-1}H$ in place of H, and hence we can apply Lemmas 1 and 2 together with Remark 1 to complete the proof.

Remark 2. Suppose $0 \in \rho(A)$. Then, since $R(A) = X$, Fact 4 is trivial.

Fact 5. Let T be a bounded linear operator on X such that $r(T) \leq 1$. Let $0 < r_n < 1$ for all $n \geq 1$, and $\lim_n r_n = 1$. If the operators $A_n = (1 - r_n)\sum_{k=0}^\infty r_n^k T^k$ satisfy $\|A_n\| \leq M\ (< \infty)$ for all $n \geq 1$, and the operator $H = \sum_{k=1}^N a_k T^k$, where $\sum_{k=1}^N a_k \neq 0$, is weakly compact, then

(i) $X = N(T - I) \oplus \overline{R(T - I)}$, and

(ii) $y \in R(T-I) \Leftrightarrow \sup_{n\geq 1} \|\sum_{k=0}^\infty r_n^k T^k y\| < \infty \Leftrightarrow \liminf_{n\to\infty} \|\sum_{k=0}^\infty r_n^k T^k y\| < \infty$.

Proof. Putting $A = T - I$ and $\lambda_n = (1 - r_n)/r_n$ (hence $r_n = (\lambda_n + 1)^{-1}$ and $1 - r_n = \lambda_n(\lambda_n + 1)^{-1}$), we see, as in Example 2 (cf. (2)), that $A_n =$

$\lambda_n(\lambda_n - A)^{-1}$. It follows that $\{A_n\}$ is an A-ergodic sequence satisfying (c), and that $\{-(\lambda_n - A)^{-1}\} = \{-r_n \sum_{k=0}^{\infty} r_n^k T^k\}$ is its companion sequence satisfying (d). Furthermore, if we set $c = \sum_{k=1}^{N} a_k$, then

$$c^{-1}H - I = \sum_{k=1}^{N} c^{-1}a_k(T^k - I) = (T - I)\sum_{k=1}^{N} c^{-1}a_k(\sum_{j=0}^{k-1} T^j),$$

and so $R(c^{-1}H - I) \subset R(A)$. Thus, as in Fact 4, Lemmas 1 and 2 and Remark 1 can be applied to establish Fact 5.

The following result may be of independent interest in view of Theorem 2.3 of [9] and Theorem 3.3 of [21].

Fact 6. Let H be a weakly compact operator on X such that $R(H) \subset D(A), AH \supset HA$ and $R(H - I) \subset R(A)$. Then $\overline{A(U \cap D(A))} \subset R(A)$, where U is the closed unit ball of X.

Proof. Let $x_n \in U \cap D(A)$, $n = 1, 2, \ldots$, and $y \in X$ be such that $\lim_{n\to\infty} \|Ax_n - y\| = 0$. Since H is weakly compact, there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ and a point $x_p \in X$ such that $x_p = \text{weak-}\lim_j Hx_{n_j}$. Then we have $\lim_j AHx_{n_j} = \lim_j HAx_{n_j} = Hy$, since $\lim_{j\to\infty} \|Ax_{n_j} - y\| = 0$, and thus, from the (weak) closedness of A we see that $x_p \in D(A)$ and $Ax_p = Hy$. On the other hand, since $R(H - I) \subset R(A)$ by hypothesis, there exists $x \in D(A)$ such that $Hy - y = Ax$. Thus, $y = Hy - (Hy - y) = A(x_p - x)$, and this completes the proof.

Lemma 3. *Under the hypotheses (a), (b′), (M1) and (d) the following hold:*

(i) *If* $y \in A(\overline{R(A)} \cap D(A))$, *then* $\lim_\alpha B_\alpha y$ *exists.*

(ii) *If* $\{B_\alpha y\}$ *has a weak cluster point* $x \in X$, *then* $x = \lim_\alpha B_\alpha y$, $x \in \overline{R(A)} \cap D(A)$ *and* $y = Ax$.

Proof.

(i) Suppose $y = Ax$, where $x \in \overline{R(A)} \cap D(A)$. Then $B_\alpha y = B_\alpha Ax = x - A_\alpha x$ by (b′), and $\lim_\alpha B_\alpha y = x - \lim_\alpha A_\alpha x = x - Px = x$ by (M1), since $x \in \overline{R(A)}$.

(ii) Let $x^* \in X^*$ be such that $x^* = 0$ on $\overline{R(A)}$. Since $\langle B_\alpha y, x^*\rangle = \varphi(\alpha)\langle y, x^*\rangle$ and $\lim_\alpha |\varphi(\alpha)| = \infty$ by (d), the relation

$$0 = \liminf_\alpha |\langle B_\alpha y, x^*\rangle - \langle x, x^*\rangle| = \liminf_\alpha |\varphi(\alpha)\langle y, x^*\rangle - \langle x, x^*\rangle|$$

implies $\langle y, x^*\rangle = 0$. Consequently $y \in \overline{R(A)}$. Denoting by $\{B_\beta y\}$ a subnet of $\{B_\alpha y\}$ such that $x = \text{weak-}\lim_\beta B_\beta y$, and then using (b′), (M1) and the (weak) closedness of A, we see that $x \in D(A)$ and $y = y - Py = \lim_\beta (y - A_\beta y) = \text{weak-}\lim_\beta AB_\beta y = Ax$. Furthermore,

$$x = \text{weak-}\lim_\beta B_\beta y = \text{weak-}\lim_\beta B_\beta Ax = \text{weak-}\lim_\beta AB_\beta x \in \overline{R(A)},$$

so that $x \in \overline{R(A)} \cap D(A)$, and hence $\lim_\alpha B_\alpha y = x$ by the above proof of (i). This completes the proof.

From the proof of (i) of the above lemma we see that if $y = Ax$ for some $x \in \overline{R(A)} \cap D(A)$, then $x = \lim_\alpha B_\alpha y$; it follows that to every $y \in A(\overline{R(A)} \cap D(A))$ there corresponds a unique $x \in \overline{R(A)} \cap D(A)$ such that $y = Ax$.

Remark 3. Under the hypotheses (a), (b′), (M1) and (d), we have: $y \in A(D(P) \cap D(A)) \Leftrightarrow y \in A(\overline{R(A)} \cap D(A)) \Leftrightarrow \lim_\alpha B_\alpha y$ exists $\Leftrightarrow \{B_\alpha y\}$ has a weak cluster point, so that, in particular, if $D(A) \subset D(P)$, then $y \in A(\overline{R(A)} \cap D(A))$ is equivalent to $y \in R(A)$. Indeed, by Lemma 3 it suffices to show that $y \in A(D(P) \cap D(A))$ implies $y \in A(\overline{R(A)} \cap D(A))$. For this purpose, suppose $y = Ax$ for some $x \in D(P) \cap D(A)$. Since $D(P) = N(A) \oplus \overline{R(A)}$ by (M1), we then have $x = x_1 + x_2$ with $x_1 \in N(A)$ and $x_2 \in \overline{R(A)}$. Then it follows that $x_2 = x - x_1 \in D(A)$, and hence $y = Ax = Ax_1 + Ax_2 = Ax_2 \in A(\overline{R(A)} \cap D(A))$. This completes the proof.

4. The Range of the Operator $T - I$

In this section we consider a bounded linear operator T on X. Let $\gamma > -1$, and suppose the Cesàro means $C_n^\gamma(T)$ of order γ satisfy

$$\sup_{n \geq 0} \|C_n^\gamma(T)\| = M \ (< \infty). \tag{6}$$

It follows (cf. Chapter III of Zygmund [24]) that $r(T) \leq 1$, and that

$$\sup_{0<r<1} \|(1-r)\sum_{k=0}^{\infty} r^k T^k\| \leq \sup_{n \geq 0} \|C_n^\alpha(T)\| \leq M \tag{7}$$

for all α with $\gamma < \alpha < \infty$.

We use the following fundamental equation (this can be easily checked from Chapter III of Zygmund [24]):

$$(T - I)C_n^\alpha(T) = \frac{\alpha}{n+1}\,[C_{n+1}^{\alpha-1}(T) - I] \tag{8}$$

for $\alpha \neq 0, -1, -2, \ldots$, and $n \geq 0$. By this, if we set

(9) $$A = T - I, \quad A_n = C_{n+1}^{\gamma+1}(T), \quad \text{and} \quad B_n = \frac{-(n+1)}{\gamma+2}\, C_n^{\gamma+2}(T) \quad \text{for} \ \ n \geq 0,$$

then

(10) $$AB_n = B_n A = \frac{-(n+1)}{\gamma+2}(T-I)\, C_n^{\gamma+2}(T) = I - C_{n+1}^{\gamma+1}(T) = I - A_n,$$

and $\|A_n\| \leq M$ for all $n \geq 1$, by (7). Furthermore, using (6) and the relations

(11) $$A_n A = AA_n = (T-I)\, C_{n+1}^{\gamma+1}(T) = \frac{\gamma+1}{n+2}\,[C_{n+2}^{\gamma}(T) - I],$$

we get $\lim_{n\to\infty} \|A_n A\| = 0$. Thus $\{A_n\}$ is an A-ergodic sequence satisfying condition (c), and $\{B_n\}$ is its companion sequence. To see that $\{B_n\}$ satisfies condition (d), suppose $x^* \in X^*$ is such that $x^* = 0$ on $\overline{R(A)}$. Then, for $x \in X$ we have

(12) $$\langle B_n x,\, x^*\rangle = \left\langle \frac{-(n+1)}{\gamma+2}\, C_n^{\gamma+2}(T)x,\, x^* \right\rangle = \frac{-(n+1)}{\gamma+2}\langle x,\, x^*\rangle,$$

whence $B_n^* x^* = \varphi(n)x^*$, with $\varphi(n) = -(n+1)/(\gamma+2)$. This proves that $\{B_n\}$ satisfies (d). Therefore it follows from the preceding section that if we set

$$D(P) = \{x \in X : \lim_n\ A_n x \ \text{exists}\}, \quad \text{and} \quad Px = \lim_n\ A_n x \quad \text{for} \ \ x \in D(P),$$

then (M1) and (M2) hold with A_n in place of A_α.

Before continuing the argument the author thinks it would be appropriate to mention the necessity of introducing condition (6) for our discussion. As is well-known (cf. [8], [11]), if T is a *positive* linear operator on a *Banach lattice*, then $\sup_{n\geq 0} \|C_n^1(T)\| < \infty$ is equivalent to $\sup_{0<r<1} \|(1-r)\sum_{k=0}^{\infty} r^k T^k\| < \infty$. Of course, this equivalence is strongly due to the positivity of T, and if we do not assume the positivity of T, the situation is quite different. To see this, the following results (cf. [11]) would be interesting; and furthermore by (i) below it seems that the only case $\gamma \geq 0$ is of interest.

(i) There exists an example of a positive linear isometry T on an L_1-space such that $\sup_{n\geq 0} \|C_n^{\gamma}(T)\|_1 = \infty$ for all γ with $-1 < \gamma < 0$. (Here, we have $\sup_{n\geq 0} \|C_n^{\gamma}(T)\|_1 = 1$ for all $\gamma \geq 0$, because $\|T\|_1 = 1$.)

(ii) If $0 < \gamma < 1$, then there exists an example of positive linear operator T on an L_1-space such that $\sup_{n\geq 0} \|C_n^{\gamma}(T)\|_1 = \infty$, but $\sup_{n\geq 0} \|C_n^{\alpha}(T)\|_1 < \infty$ for all $\alpha > \gamma$.

(iii) If k is a positive integer, then there exists an example of a bounded linear operator T on X such that $\sup_{n\geq 0}\|C_n^k(T)\|<\infty$, but $\sup_{n\geq 0}\|C_n^\beta(T)\|=\infty$ for all β with $-1<\beta<k$.

(iv) There exists an example of a bounded linear operator T on X, with $r(T)=1$, such that $\sup_{n\geq 0}\ \|(1-r)\sum_{k=0}^\infty r^kT^k\|<\infty$, but $\|C_n^\beta(T)\|=\infty$ for all β with $-1<\beta<\infty$.

We continue the argument. The next result is a direct consequence of Lemma 3 and Remark 3 (see also (7) and the proof of Fact 5).

Theorem 1. (Cf. Theorems 2.3 and 2.4 of [19].) *Let $\gamma>-1$, and suppose (6) holds. Then the following conditions are equivalent:*

(i) $y\in(T-I)(D(P))$;

(ii) $y\in(T-I)(\overline{R(T-I)})$;

(iii) $\lim_n\ nC_n^{\gamma+2}y$ *exists;*

(iv) $\{nC_n^{\gamma+2}y\}$ *has a weak cluster point;*

(v) $\lim_{r\uparrow 1}\ \sum_{k=0}^\infty r^kT^ky$ *exists;*

(vi) $\{\sum_{k=0}^\infty r^kT^ky\}$ *has a weak cluster point as* $r\uparrow 1$.

Relating to the above theorem, the next result may be of independent interest.

Lemma 4. *If* $0<\alpha<\beta<\infty$ *and* $y\in X$, *then the following hold:*

(i) $\sup_{n\geq 0}\ \|(n+1)C_n^\alpha(T)y\|=M<\infty$ *implies*

$$\sup_{n\geq 0}\|(n+1)C_n^\beta(T)y\|\leq(\beta\vee 1)\frac{(\alpha+1)}{\alpha}M,$$

and

$$\sup_{0<r<1}\|\sum_{k=0}^\infty r^kT^ky\|\leq\frac{\alpha+1}{\alpha}M\,;$$

(ii) *strong [resp. weak]-* $\lim_{n\to\infty}\ nC_n^\alpha(T)y=x$ *implies*

$$\textit{strong [resp. weak]-}\lim_{n\to\infty}\ nC_n^\beta(T)y=\beta\alpha^{-1}x,$$

and

$$\textit{strong [resp. weak]-}\lim_{r\uparrow 1}\sum_{k=0}^\infty r^kT^ky=\alpha^{-1}x.$$

Proof. To see the first inequality of (i), we notice that $A_n^\alpha / A_n^{\alpha-1} = (\alpha+n)/\alpha$ for all $n \geq 0$. Thus we have, using the relation $C_n^\alpha(T) = S_n^\alpha(T)/A_n^\alpha$ (cf. §2), that

$$\begin{aligned}\left\| \frac{1}{A_n^{\alpha-1}} S_n^\alpha(T)y \right\| &= \left\| \frac{A_n^\alpha}{A_n^{\alpha-1}} C_n^\alpha(T)y \right\| = \left\| \frac{\alpha+n}{\alpha} C_n^\alpha(T)y \right\| \\ &= \frac{\alpha+n}{\alpha(n+1)} \|(n+1)C_n^\alpha(T)y\| \leq \frac{\alpha+n}{\alpha(n+1)} M \leq \frac{\alpha+1}{\alpha} M,\end{aligned}$$

and so, since (cf. Chapter III of [24])

$$\frac{1}{A_n^{\beta-1}} S_n^\beta(T)y = \frac{1}{A_n^{\beta-1}} \sum_{k=0}^n A_{n-k}^{\beta-\alpha-1} S_k^\alpha(T)y = \frac{1}{A_n^{\beta-1}} \sum_{k=0}^n A_{n-k}^{\beta-\alpha-1} A_k^{\alpha-1} \frac{S_k^\alpha(T)y}{A_k^{\alpha-1}},$$

and since

$$A_n^{\beta-1} = \sum_{k=0}^n A_{n-k}^{\beta-\alpha-1} A_k^{\alpha-1}, \quad \text{where} \quad A_{n-k}^{\beta-\alpha-1} > 0 \quad \text{and} \quad A_k^{\alpha-1} > 0,$$

we find that

$$\begin{aligned}\left\| \frac{\beta+n}{\beta} C_n^\beta(T)y \right\| &= \left\| \frac{A_n^\beta}{A_n^{\beta-1}} C_n^\beta(T)y \right\| = \left\| \frac{1}{A_n^{\beta-1}} S_n^\beta(T)y \right\| \\ &\leq \frac{1}{A_n^{\beta-1}} \sum_{k=0}^n A_{n-k}^{\beta-\alpha-1} A_k^{\alpha-1} \left\| \frac{S_k^\alpha(T)y}{A_k^{\alpha-1}} \right\| \leq \frac{\alpha+1}{\alpha} M\end{aligned}$$

for all $n \geq 0$. Then

$$\begin{aligned}\|(n+1)C_n^\beta(T)y\| &= \left\| \frac{\beta(n+1)}{\beta+n} \cdot \frac{\beta+n}{\beta} C_n^\beta(T)y \right\| \\ &\leq \frac{\beta(n+1)}{\beta+n} \cdot \frac{\alpha+1}{\alpha} M \leq (\beta \vee 1) \frac{\alpha+1}{\alpha} M.\end{aligned}$$

This proves the first inequality. The second inequality of (i) can be proved as in the proof of (ii) of Fact 2, and hence we may omit the details.

The proof of (ii) is similar to that of (i) with a slight modification, and hence we also omit the details.

Remark 4. Let $\gamma > -1$, and suppose (6) holds. Then we have the following:

(i) For every β with $\beta > \gamma$, we have $\lim_{n\to\infty} \|(T-I)C_n^\beta(T)\| = 0$ (cf. the proof of Lemma 1 of [3]); thus by putting

$$A_n' = C_{n+1}^\beta(T), \quad \text{and} \quad B_n' = \frac{-(n+1)}{\beta+1} C_n^{\beta+1}(T) \quad \text{for} \ \ n \geq 0,$$

we get another A-ergodic sequence $\{A'_n\}$ satisfying condition (c), and its companion sequence $\{B'_n\}$ satisfying condition (d). It follows that the condition $y \in (T-I)(D(P))$ is equivalent to the existence of the limit $\lim_{n\to\infty} nC_n^{\beta+1}(T)y$, with $\beta > \gamma$. Apparently, this condition is stronger than (iii) in Theorem 1, by Lemma 4. On the other hand, the condition $y \in (T-I)(D(P))$ does not imply the existence of the limit $\lim_{n\to\infty} nC_n^{\gamma+1}(T)y$, in general. In fact, if $\gamma = 0$, let T be a unitary operator on a Hilbert space H. Then T is mean ergodic, and $(n+1)C_n^1(T)(T-I)x = T^{n+1}x - x$ for $x \in H$. Here, the existence of the limit $\lim_{n\to\infty} T^{n+1}x$ cannot be expected, in general. As a counter-example for the case $\gamma = 1$, we can take a bounded linear operator T on a *reflexive* Banach space X such that $\sup_{n\geq 0} \|C_n^1(T)\| < \infty$, and also such that $\limsup_{n\to\infty} n^{-1}\|T^n x\| > 0$ for some $x \in X$ (cf. [8] or [11]). Then clearly $\lim_{n\to\infty} C_n^1(T)x$ fails to exist, but we have $D(P) = N(T-I) \oplus \overline{R(T-I)} = X$, since X is reflexive. By (8)

$$(n+1)C_n^2(T)(T-I)x = 2[C_{n+1}^1(T)x - x],$$

whence the limit $\lim_{n\to\infty} nC_n^2(T)y$, with $y = (T-I)x$, does not exist.

(ii) If $y = (T-I)x$ for some $x \in X$, then, by (8)

$$nC_n^{\gamma+1}(T)y = nC_n^{\gamma+1}(T)(T-I)x = n\frac{\gamma+1}{n+1}\,[C_{n+1}^{\gamma}(T)x - x],$$

and thus we have $\sup_{n\geq 0} \|nC_n^{\gamma+1}(T)y\| < \infty$. But here, if $0 < \epsilon < 1$, then we cannot obtain the inequality $\sup_{n\geq 0} \|nC_n^{\gamma+\epsilon}(T)y\| < \infty$, in general. In fact, if T is a bounded linear operator on X such that $\sup_{n\geq 0} \|C_n^k(T)\| < \infty$, where k is a positive integer, and also such that $\sup_{n\geq 0} \|C_n^\beta(T)\| = \infty$ for all β with $-1 < \beta < k$, then to each δ with $0 < \delta < 1$ there corresponds $x \in X$ so that $\sup_{n\geq 0} \|C_n^{k-\delta}(T)x\| = \infty$. Then, we must have $\sup_{n\geq 0} \|nC_n^{k+1-\delta}(T)(T-I)x\| = \infty$, by (8).

Theorem 2. (Cf. Corollary 2 of [12].) *Let $\gamma > -1$, and suppose (6) holds. If the operator $H = \sum_{k=1}^N a_k T^k$, where $\sum_{k=1}^N a_k \neq 0$, is weakly compact, then*

(i) $X = N(T-I) \oplus \overline{R(T-I)}$, *and*

(ii) $y \in R(T-I) \Leftrightarrow \sup_{n\geq 0} \|nC_n^{\gamma+1}(T)y\| < \infty \Leftrightarrow \liminf_{n\to\infty} \|nC_n^{\gamma+2}(T)y\| < \infty \Leftrightarrow \sup_{0<r<1} \|\sum_{k=0}^\infty r^k T^k y\| < \infty \Leftrightarrow \liminf_{r\uparrow 1} \|\sum_{k=0}^\infty r^k T^k y\| < \infty.$

Proof.

(i) By virtue of (7), this follows from (i) of Fact 5.

(ii) By (ii) of Remark 4 and (i) of Lemma 4, we see that

$$y \in R(T-I) \Rightarrow \sup_{n\geq 0} \|nC_n^{\gamma+1}(T)y\| < \infty \Rightarrow \liminf_{n\to\infty} \|nC_n^{\gamma+2}y\| < \infty;$$

and $\liminf_{n\to\infty} \|nC_n^{\gamma+2}y\| < \infty \Rightarrow y \in R(T-I)$ follows from Lemma 2 and Remark 1. The remaining parts follow immediately from Fact 5. This completes the proof.

Remark 5. It is interesting to note that the condition $\sup_{n\geq 0} \|nC_n^{\gamma+1}(T)y\| < \infty$ does not imply $y \in R(T-I)$, in general. Indeed, it is known (cf. [12]) that there exists an example of a mean ergodic linear contraction T on X such that $\sup_{n\geq 0} \|nC_n^1(T)y\| < \infty$ does not imply $y \in R(T-I)$. Clearly, T satisfies (6), with $\gamma = 0$ and $M = 1$, in this case. (See also Facts 4 and 5.)

Theorem 3. (Cf. Theorem 2.3 of [9].) *Let* $\gamma > -1$, *and suppose (6) holds. Then the following conditions are equivalent:*

(i) $\{y \in X : \sup_{n\geq 0} \|nC_n^{\gamma+1}(T)y\| < \infty\} = R(T-I)$;

(ii) $\overline{(T-I)U} \subset R(T-I)$, where $U = \{x \in X : \|x\| \leq 1\}$;

(iii) $\{y \in X : \liminf_{r\uparrow 1} \|\sum_{k=0}^{\infty} r^k T^k y\| < \infty\} = R(T-I)$.

Proof.

(i) $\Rightarrow$ (ii). For every $x \in U$ we have, by (8),

$$\|nC_n^{\gamma+1}(T)(T-I)x\| \leq (\gamma+1)\|C_{n+1}^{\gamma}(T)x - x\| \leq (\gamma+1)(M+1),$$

and so

$$\overline{(T-I)U} \subset \{y \in X : \sup_{n\geq 0} \|nC_n^{\gamma+1}(T)y\| \leq (\gamma+1)(M+1)\}.$$

Hence, (i) implies that $\overline{(T-I)U} \subset R(T-I)$.

(ii) $\Rightarrow$ (iii). Putting $A = T-I$ and $\lambda = (1-r)/r$ for $0 < r < 1$ (cf. the proof of Fact 5), we have

$$\lambda(\lambda - A)^{-1} = (1-r)\sum_{k=0}^{\infty} r^k T^k, \quad \text{and} \quad (\lambda - A)^{-1} = r\sum_{k=0}^{\infty} r^k T^k,$$

so that $\sup_{\lambda>0} \|\lambda(\lambda-A)^{-1}\| \leq M$ by (7), and hence $\{\lambda(\lambda-A)^{-1}\}_{\lambda>0}$ is an A-ergodic net satisfying (c), and $\{-(\lambda-A)^{-1}\}_{\lambda>0}$ is its companion net satisfying (d). In particular, we have that $\|A\lambda(\lambda-A)^{-1}\| = \|\lambda(I-\lambda(\lambda-A)^{-1})\| \leq \lambda(1+M) \downarrow 0$

as $\lambda \downarrow 0$. Thus we can apply Theorem 3.3 of Shaw-Li [21] to infer that (ii) is equivalent to

(iii)′ $\{y \in X : \sup_{0<r<1} \|\sum_{k=0}^{\infty} r^k T^k y\| < \infty\} = R(T - I)$,
which is also equivalent to (iii) (cf. Remark 1).

(iii) $\Rightarrow$ (i). By (i) of Lemma 4 and (ii) of Remark 4 we see that (iii)′ implies (i). This completes the proof.

5. The Range of the Generator A of a C_0-Semigroup of Operators

In this section we consider a C_0-semigroup $\{T_t\} = \{T_t\}_{t\geq 0}$ of bounded linear operators on X. Thus $T_0 = I$, $T_{t+s} = T_t T_s$ for $t,\, s \geq 0$, and for every $x \in X$ the mapping $t \mapsto T_t x$ is strongly continuous on $[0, \infty)$. Its generator A is defined by

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T_t x - x}{t} \text{ exists}\}, \quad \text{and } Ax = \lim_{t\downarrow 0} \frac{T_t x - x}{t} \quad \text{for } x \in D(A).$$

Then it is known (cf. e.g. [13]) that A is a densely defined closed operator in X. For $\alpha \geq 0$ we define the Cesàro means C_t^α, $t > 0$, of order α of $\{T_t\}$ by

$$C_t^\alpha = \begin{cases} T_t & \text{if } \alpha = 0, \\ \alpha t^{-\alpha} \displaystyle\int_0^t (t-s)^{\alpha-1} T_s \, ds & \text{if } \alpha > 0. \end{cases}$$

In particular, we have $C_t^1 x = t^{-1} \int_0^t T_s x \, ds$ for $x \in X$. By Fubini's theorem, and then by an induction argument on n we easily see that

(i) if $0 < \alpha,\, \beta < \infty$, then for every $x \in X$

$$C_t^{\alpha+\beta} x = \frac{\displaystyle\int_0^t (t-s)^{\beta-1} \left[\int_0^s (s-r)^{\alpha-1} T_r x \, ds\right] ds}{\displaystyle\int_0^t (t-s)^{\beta-1} \left[\int_0^s (s-r)^{\alpha-1} dr\right] ds}, \tag{13}$$

(ii) if $n \geq 1$ is an integer, then for every $x \in X$

$$C_t^n x = n!\, t^{-n} \int_0^t \left[\int_0^{s_1} \left(\int_0^{s_2} \left(\cdots \left(\int_0^{s_{n-1}} T_{s_n} x \, ds_n\right) \cdots\right) ds_3\right) ds_2\right] ds_1. \tag{14}$$

Furthermore, if $0 < \alpha < \beta < \infty$, then for every $x \in X$

$$\sup_{t>0} \|T_t x\| \geq \sup_{t>0} \|C_t^\alpha x\| \geq \sup_{t>0} \|C_t^\beta x\|. \tag{15}$$

Let $\gamma \geq 0$, and assume that

$$\sup_{t>0} \|C_t^\gamma\| = M \ (< \infty). \tag{16}$$

We then choose a positive integer n so that $\gamma \le n$. By (15) we have $\|C_t^n\| \le M$ for all $t > 0$. Thus, by (14), for $\lambda \in \boldsymbol{C}$ with $\Re\lambda > 0$ we can define a bounded linear operator $R(\lambda)$ on X by

$$
(17) \qquad R(\lambda)x = \lambda^n \int_0^\infty e^{-\lambda t} \left\{ \int_0^t \left[\int_0^{s_1} (\dots (\int_0^{s_{n-1}} T_{s_n} x \, ds_n) \dots) \, ds_2 \right] ds_1 \right\} dt \quad \text{for } x \in X.
$$

On the other hand, if $\Re\lambda > \max\{0, \omega_0\}$ (ω_0 denotes the growth order of $\{T_t\}$), then, using Fubini's theorem n times we have

$$
(\lambda - A)^{-1}x = \int_0^\infty e^{-\lambda t} T_t x \, dt = R(\lambda)x \qquad \text{for } x \in X,
$$

so that $(\lambda - A)^{-1} = R(\lambda)$. And, since the operator-valued function $\lambda \mapsto R(\lambda)$ is analytic on $\{\lambda : \Re\lambda > 0\}$, it follows by analytic continuation that

$$
(18) \qquad \{\lambda : \Re\lambda > 0\} \subset \rho(A), \text{ and } (\lambda - A)^{-1} = R(\lambda) \quad \text{for } \lambda \text{ with } \Re\lambda > 0.
$$

It also follows from (14) and (17) that for every $x \in X$

$$
(19) \qquad \sup_{\lambda > 0} \|\lambda(\lambda - A)^{-1}x\| = \sup_{\lambda > 0} \|\lambda R(\lambda)x\| \le \sup_{t > 0} \|C_t^n x\|,
$$

and thus

$$
(20) \qquad \sup_{\lambda > 0} \|\lambda(\lambda - A)^{-1}\| = \sup_{\lambda > 0} \|\lambda R(\lambda)\| \le M.
$$

We now use the fundamental equation (it is interesting to compare this with (8)):

$$
(21) \qquad C_t^{\alpha+1} A \subset A C_t^{\alpha+1} = \frac{\alpha + 1}{t} [C_t^\alpha - I] \qquad (t > 0, \ \alpha \ge 0),
$$

which is due to Shaw [19] for the special case $\alpha = 1$. This can be proved as follows. By (13), or using Fubini's theorem directly, we observe that

$$
\int_0^t (t - s)^\alpha T_s x \, ds = \begin{cases} \alpha \displaystyle\int_0^t (t - s)^{\alpha - 1} \left(\int_0^s T_r x \, dr \right) ds & \text{if } \alpha > 0, \\ \displaystyle\int_0^t T_s x \, ds & \text{if } \alpha = 0. \end{cases}
$$

Thus, if $x \in X$ and $\alpha > 0$, then by the closedness of A we see that $C_t^{\alpha+1} x \in D(A)$, and

$$
A C_t^{\alpha+1} x = \frac{\alpha + 1}{t^{\alpha+1}} A \int_0^t (t - s)^\alpha T_s x \, ds = \frac{\alpha(\alpha + 1)}{t^{\alpha+1}} \int_0^t (t - s)^{\alpha - 1} A \left(\int_0^s T_r x \, dr \right) ds
$$

$$= \frac{\alpha(\alpha+1)}{t^{\alpha+1}} \int_0^t (t-s)^{\alpha-1}(T_s x - x)\,ds = \frac{\alpha+1}{t}\,[C_t^\alpha - I]x.$$

By a similar calculation, if $x \in D(A)$ and $\alpha > 0$, then $C_t^{\alpha+1}Ax = (\alpha+1)t^{-1}\,[C_t^\alpha - I]x$. Hence, (21) holds for the case $\alpha > 0$. The special case $\alpha = 0$ is a basic property of a C_0-semigroup.

By (15) and (21), if we set

$$A_t = C_t^{\gamma+1}, \quad \text{and} \quad B_t = \frac{-t}{\gamma+2} C_t^{\gamma+2} \qquad \text{for } \; t > 0, \tag{22}$$

then

$$\begin{cases} \sup_{t>0} \|A_t\| \le M, \quad B_t A \subset AB_t = I - A_t, \\ A_t A \subset AA_t = \frac{\gamma+1}{t}[C_t^\gamma - I], \quad \text{and} \quad \lim_{t\to\infty} \|AA_t\| = 0. \end{cases} \tag{23}$$

Hence $\{A_t\}$ is an A-ergodic net satisfying condition (c), and $\{B_t\}$ is its companion net. It is also easily checked that $\{B_t\}$ satisfies condition (d). Now we define

$$D(P) = \{x \in X : \lim_{t\to\infty} A_t x \text{ exists}\}, \quad \text{and} \quad Px = \lim_{t\to\infty} A_t x \qquad \text{for } x \in D(P).$$

Then, (M1) and (M2) hold; and we have $N(A) = \{x \in X : T_t x = x \text{ for all } t > 0\}$ and $\overline{R(A)} = \overline{\{T_t x - x : x \in X,\, t > 0\}}$. The proof of the next result is similar to that of Theorem 1; hence we may omit the details.

Theorem 4. (Cf. Theorem 3.4 of [19].) *Let $\gamma \ge 0$, and suppose (16) holds. Then the following conditions are equivalent:*

(i) $y \in A(D(P) \cap D(A))$; (ii) $y \in A(\overline{R(A)} \cap D(A))$; (iii) $\lim_{t\to\infty} tC_t^{\gamma+2}y$ *exists;* (iv) $\{tC_t^{\gamma+2}y\}$ *has a weak cluster point as* $t \to \infty$; (v) $\lim_{\lambda\downarrow 0}(\lambda - A)^{-1}y$ *exists;* (vi) $\{(\lambda - A)^{-1}y\}$ *has a weak cluster point as* $\lambda \downarrow 0$.

The next result may be regarded as a continuous version of Lemma 4.

Lemma 5. *Let $\gamma \ge 0$, and suppose (16) holds. If $1 \le \alpha < \beta < \infty$ and $y \in X$, then the following hold:*

(i) $\sup_{t>0} \|tC_t^\alpha y\| = M < \infty$ *implies*

$$\sup_{t>0} \|tC_t^\beta y\| \le \beta\alpha^{-1}M, \quad \textit{and} \quad \sup_{\lambda>0} \|(\lambda - A)^{-1}y\| \le \alpha^{-1}M;$$

(ii) *strong [resp. weak]-* $\lim_{t\to\infty} tC_t^\alpha y = x$ *implies*

$$\textit{strong [resp. weak]-} \lim_{t\to\infty} tC_t^\beta y = \beta\alpha^{-1}x,$$

and

$$\textit{strong [resp. weak]-}\lim_{\lambda\downarrow 0}(\lambda - A)^{-1}y = \alpha^{-1}x.$$

Proof. To prove this lemma, we use the Beta function $B(p,q)$, where $p,q>0$. By the relations

$$\int_r^t (t-s)^{p-1}(s-r)^{q-1}\,ds = \int_0^{t-r}(t-(s+r))^{p-1}s^{q-1}\,ds$$
$$= (t-r)^{p+q-1}\int_0^1 (1-s)^{p-1}s^{q-1}\,ds = (t-r)^{p+q-1}B(p,q)$$

for $0 \le r \le t$, it follows that

$$B(\beta-\alpha,\alpha)\int_0^t (t-r)^{\beta-1}T_r y\,dr = \int_0^t\left(\int_r^t (t-s)^{\beta-\alpha-1}(s-r)^{\alpha-1}ds\right)T_r y\,dr$$
$$= \int_0^t (t-s)^{\beta-\alpha-1}\left(\int_0^s (s-r)^{\alpha-1}T_r y\,dr\right)ds \qquad \text{(by Fubini's theorem)}$$
$$= \begin{cases} \displaystyle\int_0^t (t-s)^{\beta-\alpha-1}\left(\int_0^s (s-r)^{\alpha-2}dr\right)\frac{\int_0^s (s-r)^{\alpha-1}T_r y\,dr}{\int_0^s (s-r)^{\alpha-2}dr}ds & \text{if } \alpha>1,\\ \displaystyle\int_0^t (t-s)^{\beta-2}\left(\int_0^s T_r y\,dr\right)ds & \text{if } \alpha=1. \end{cases}$$

Thus, if $\alpha>1$, then

$$tC_t^\beta y = \frac{\beta}{t^{\beta-1}}\int_0^t (t-r)^{\beta-1}T_r y\,dr$$
$$= \frac{\beta}{t^{\beta-1}B(\beta-\alpha,\alpha)}\int_0^t (t-s)^{\beta-\alpha-1}\left(\int_0^s (s-r)^{\alpha-2}dr\right)\frac{\int_0^s (s-r)^{\alpha-1}T_r y\,dr}{\int_0^s (s-r)^{\alpha-2}dr}ds.$$

Here we have, as above,

$$\int_0^t (t-s)^{\beta-\alpha-1}\left(\int_0^s (s-r)^{\alpha-2}dr\right)ds = B(\beta-\alpha,\alpha-1)\int_0^t (t-r)^{\beta-2}dr$$
$$= \frac{B(\beta-\alpha,\alpha-1)t^{\beta-1}}{(\beta-1)},$$

and

$$\frac{\left\|\int_0^s (s-r)^{\alpha-1} T_r y\, dr\right\|}{\int_0^s (s-r)^{\alpha-2} dr} = \frac{\alpha-1}{s^{\alpha-1}} \left\|\int_0^s (s-r)^{\alpha-1} T_r y\, dr\right\|$$

$$= \frac{\alpha-1}{\alpha} \|s C_s^\alpha y\| \leq \frac{\alpha-1}{\alpha} M.$$

It follows that

$$\|t C_t^\beta y\| \leq \frac{\beta}{t^{\beta-1} B(\beta-\alpha,\alpha)} \cdot \frac{B(\beta-\alpha,\alpha-1) t^{\beta-1}}{\beta-1} \cdot \frac{\alpha-1}{\alpha} M = \beta\alpha^{-1} M$$

for all $t > 0$. On the other hand, if $\alpha = 1$, then

$$t C_t^\beta y = \frac{\beta}{t^{\beta-1}} \int_0^t (t-r)^{\beta-1} T_r y\, dr = \frac{\beta}{t^{\beta-1} B(\beta-1,1)} \int_0^t (t-s)^{\beta-2} \left(\int_0^s T_r y\, dr\right) ds,$$

and

$$\left\|s C_s^1 y\right\| = \left\|\int_0^s T_r y\, dr\right\| \leq M.$$

Thus,

$$\left\|t C_t^\beta y\right\| \leq \frac{\beta}{t^{\beta-1} B(\beta-1,1)} \cdot \frac{t^{\beta-1}}{\beta-1} M = \beta M$$

for all $t > 0$. This proves the first inequality of (i).

To prove the second inequality of (i), we observe from (14), (17) and (18) that if $n \geq \max\{\gamma, \alpha\}$ and $\lambda > 0$, then

$$(\lambda - A)^{-1} y = \lambda^n \int_0^\infty e^{-\lambda t} \left\{\int_0^t \left[\int_0^{s_1} (\ldots(\int_0^{s_{n-1}} T_{s_n} y\, ds_n)\ldots)\, ds_2\right] ds_1\right\} dt$$

$$= \lambda^n \int_0^\infty e^{-\lambda t} \frac{t^n}{n!} C_t^n y\, dt.$$

Therefore,

$$\|(\lambda - A)^{-1} y\| \leq \lambda^n \int_0^\infty e^{-\lambda t} \frac{t^{n-1}}{n!} \|t C_t^n y\|\, dt \leq \lambda^n \int_0^\infty e^{-\lambda t} \frac{t^{n-1}}{n!} M'\, dt = \frac{M'}{n},$$

where $M' := \sup_{t>0} \|t C_t^n y\|$; we note that $M' \leq n\alpha^{-1} M < \infty$ by the first inequality of (i). This completes the proof of the second inequality of (i).

The proof of (ii) is similar to that of (i) with a slight modification, and hence we omit the details.

Theorem 5. *Let* $\gamma \geq 0$, *and suppose* (16) *holds. If the operator* $H = \int_\alpha^\beta a(t) T_t\, dt$, *where* $a(t)$ *is a real-valued continuous function on the interval* $[\alpha, \beta]$ *with* $\int_\alpha^\beta a(t)\, dt \neq 0$, *is weakly compact, then*

(i) $X = N(A) \oplus \overline{R(A)}$, *and*

(ii) $y \in R(A) \Leftrightarrow \sup_{t>0} \|tC_t^{\gamma+1}y\| < \infty \Leftrightarrow \liminf_{t\to\infty} \|tC_t^{\gamma+2}y\| < \infty$
$\Leftrightarrow \sup_{\lambda>0} \|(\lambda - A)^{-1}y\| < \infty \Leftrightarrow \liminf_{\lambda\downarrow 0} \|(\lambda - A)^{-1}y\| < \infty.$

Proof. We may assume that $\int_\alpha^\beta a(t)\,dt = 1$. Then, using the closedness of A, we have

$$(H - I)x = \int_\alpha^\beta a(t)[T_t x - x]\,dt = \int_\alpha^\beta a(t)A\left(\int_0^t T_s x\,ds\right)dt$$
$$= A\int_\alpha^\beta a(t)\left(\int_0^t T_s x\,ds\right)dt$$

for all $x \in X$, so that $R(H - I) \subset R(A)$. By this, together with the fact that $C_t^\eta H = HC_t^\eta$ and $(\lambda - A)^{-1}H = H(\lambda - A)^{-1}$ for all $\eta \geq 0$, $t > 0$ and $\lambda > 0$, we see that conditions (w) and (w′) hold for $(A, \{A_t\}, \{B_t\})$, and also for $(A, \{\lambda(\lambda - A)^{-1}\}, \{(\lambda - A)^{-1}\})$. Therefore, (i) follows from Lemma 1; and (ii) follows from (21) with $\alpha = \gamma$, Lemmas 5 and 2, and Remark 1.

Example 3. There exists an example of a mean ergodic C_0-semigroup $\{T_t\}$ of linear isometries on X such that $\sup_{t>0} \|tC_t^1 y\| < \infty$ does not imply $y \in R(A)$. To see this, let $X = C_0(\boldsymbol{R})$ be the space of all scalar-valued continuous functions f on the real line $\boldsymbol{R}$ such that $\lim_{s\to-\infty} f(s) = 0 = \lim_{s\to\infty} f(s)$. By the norm

$$\|f\| = \sup\{|f(s)| : s \in \boldsymbol{R}\}, \tag{24}$$

X becomes a Banach space. Let $\{T_t\} = \{T_t\}_{t\geq 0}$ be the C_0-semigroup of linear isometries on X defined by

$$T_t f(s) = f(t + s) \qquad \text{for } s \in \boldsymbol{R}. \tag{25}$$

It is immediate that $\lim_{b\to\infty} b^{-1}\|\int_0^b T_t f\,dt\| = 0$ for every $f \in X$. Thus $\{T_t\}$ is a mean ergodic semigroup. Let g be the function in X defined by

$$g(s) = \begin{cases} s & \text{if } 0 \leq s \leq 1, \\ 2 - s & \text{if } 1 \leq s \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then, a simple calculation shows that

$$\sup_{t>0} \|tC_t^1 g\| = \sup_{t>0} \left\|\int_0^t T_u g\,du\right\| \leq 1.$$

But we have $g \notin R(A)$. To see this, assume the contrary: $g = Ah$ for some $h \in D(A)$. Then, since $D(A) = \{f \in C_0(\boldsymbol{R}) : f' \in C_0(\boldsymbol{R})\}$, and since $Af = f'$ for $f \in D(A)$, we have $g = h'$. It follows that $h(0) - h(s) = \int_s^0 g(t)\,dt$ for all $s \leq 0$, so that $h(s) = h(0)$ for all $s \leq 0$, by the definition of g. Therefore $h(0) = \lim_{s\to-\infty} h(s) = 0$. This implies that for all $s \geq 2$

$$h(s) = h(0) + \int_0^s g(t)\,dt = \int_0^2 g(t)\,dt = 1,$$

which is a contradiction, because we must have $\lim_{s\to\infty} h(s) = 0$. On the other hand, it is clear that $\{T_t\}$ satisfies (16), with $\gamma = 0$ and $M = 1$. (Incidentally we remark that by putting $T = T_1$ we have $\sup_{n\geq 0} \|nC_n^1(T)g\| = \sup_{n\geq 0} \|\sum_{k=0}^n T^k g\| \leq 1$. But, we also have $g \notin R(T-I)$. Indeed, if $g = (T-I)h$ for some $h \in C_0(\boldsymbol{R})$, then $h(s) = -g(s) + Th(s) = -g(s) + h(1+s) \leq h(1+s)$ for all $s \in \boldsymbol{R}$, by the definition of g. Since $h \in C_0(\boldsymbol{R})$, this is possible only when $h = 0$ on $\boldsymbol{R}$. But, since $g \neq 0$, we must have $h \neq 0$, and hence this is impossible. Clearly, T is a mean ergodic linear isometry on $C_0(\boldsymbol{R})$. Cf. Remark 5.)

By virtue of (21) with $\alpha = \gamma$ and Lemma 5, the next theorem can be proved by essentially the same argument as that of Theorem 3, and hence we omit the details. (From Theorem 3.3 of [21] (see also Theorem 2.3 of [9]) we see that if $\sup_{\lambda>0} \|\lambda(\lambda-A)^{-1}\| < \infty$, then the condition $\overline{A(U \cap D(A))} \subset R(A)$ is equivalent to that $R(A)$ is an F_σ-set.)

Theorem 6. *Let* $\gamma \geq 0$, *and suppose* (16) *holds. Then the following conditions are equivalent:*

(i) $\{y \in X : \sup_{t>0} \|tC_t^{\gamma+1}y\| < \infty\} = R(A)$;

(ii) $\overline{A(U \cap D(A))} \subset R(A)$, *where* $U = \{x \in X : \|x\| \leq 1\}$;

(iii) $\{y \in X : \liminf_{\lambda\downarrow 0} \|(\lambda - A)^{-1}y\| < \infty\} = R(A)$.

6. The Range of the Generator A of a Strongly Continuous Cosine Operator Function

In this section we consider a strongly continuous cosine operator function $\{C(t)\} = \{C(t)\}_{t\geq 0}$. Thus, $C(0) = I$, $C(t+s) + C(t-s) = 2C(t)C(s)$ for $t \geq s \geq 0$, and $\lim_{s\to t} \|C(s)x - C(t)x\| = 0$ for $t \geq 0$ and $x \in X$. Its generator A is defined by

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{2[C(t)x - x]}{t^2} \text{ exists}\}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{2[C(t)x - x]}{t^2} \quad \text{for } x \in D(A).$$

It is known (cf. e.g. Sova [22]) that A is a densely defined closed operator in X. As in §5, we define the Cesàro means C_t^α, $t>0$, of order $\alpha \geq 0$ of the cosine operator function $\{C(t)\}$ by

$$C_t^\alpha = \begin{cases} C(t) & \text{if } \alpha = 0, \\ \alpha t^{-\alpha} \displaystyle\int_0^t (t-s)^{\alpha-1} C(s)\, ds & \text{if } \alpha > 0. \end{cases}$$

In particular, we have $C_t^1 x = t^{-1} \int_0^t C(s) x\, ds$ for $x \in X$.

Let $\gamma \geq 0$, and assume that (16) holds (i.e. $\sup_{t>0} \|C_t^\gamma\| = M < \infty$). As observed in §5, if n is a positive integer with $n \geq \gamma$, then, since $\|C_t^n\| \leq M$ for all $t>0$, we can define a bounded linear operator $R(\lambda^2)$ on X, for $\lambda \in \boldsymbol{C}$ with $\Re\lambda > 0$, by

$$R(\lambda^2)x = \lambda^{n-1} \int_0^\infty e^{-\lambda t} \left\{ \int_0^t \left[\int_0^{s_1} \left(\ldots \left(\int_0^{s_{n-1}} C(s_n) x\, ds_n \right) \ldots \right) ds_2 \right] ds_1 \right\} dt \tag{26}$$

for $x \in X$. Then we have, as in §5, that

$$\{\lambda^2 : \Re\lambda > 0\} \subset \rho(A), \quad \text{and} \quad (\lambda^2 - A)^{-1} = R(\lambda^2) \quad \text{for } \lambda \text{ with } \Re\lambda > 0 \tag{27}$$

(cf. e.g. [22], [23]). Since

$$\sup_{\lambda>0} \|\lambda(\lambda - A)^{-1} x\| = \sup_{\lambda>0} \|\lambda R(\lambda) x\| \leq \sup_{t>0} \|C_t^n x\| \tag{28}$$

for every $x \in X$, we have

$$\sup_{\lambda>0} \|\lambda(\lambda - A)^{-1}\| = \sup_{\lambda>0} \|\lambda R(\lambda)\| \leq M. \tag{29}$$

In this section the following equation is fundamental (cf. (21)):

$$C_t^{\alpha+2} A \subset A C_t^{\alpha+2} = \frac{(\alpha+2)(\alpha+1)}{t^2} [C_t^\alpha - I] \qquad (t>0,\ \alpha \geq 0), \tag{30}$$

which is due to Shaw [19] for the special case $\alpha = 2$. To prove this we first note by Fubini's theorem that

$$\int_0^t (t-s)^{\alpha+1} C(s) x\, ds = (\alpha+1) \int_0^t (t-s)^\alpha \left(\int_0^s C(r) x\, dr \right) ds$$

$$= \begin{cases} (\alpha+1)\alpha \displaystyle\int_0^t (t-s)^{\alpha-1} \left(\int_0^s (s-u) C(u) x\, du \right) ds & \text{if } \alpha > 0, \\ \displaystyle\int_0^t (t-s) C(s) x\, ds & \text{if } \alpha = 0. \end{cases}$$

Then, for $x \in X$ and $\alpha > 0$, we see by the closedness of A that $C_t^{\alpha+2}x \in D(A)$, and

$$\begin{aligned} AC_t^{\alpha+2}x &= \frac{\alpha+2}{t^{\alpha+2}} A \int_0^t (t-s)^{\alpha+1} C(s)x\, ds \\ &= \frac{(\alpha+2)(\alpha+1)\alpha}{t^{\alpha+2}} \int_0^t (t-s)^{\alpha-1} A(\int_0^s (s-u)C(u)x\, du)\, ds \\ &= \frac{(\alpha+2)(\alpha+1)\alpha}{t^{\alpha+2}} \int_0^t (t-s)^{\alpha-1} (C(s)x - x)\, ds \\ &= \frac{(\alpha+2)(\alpha+1)}{t^2} [C_t^\alpha - I]x, \end{aligned}$$

where the third equality comes from Lemma 2.14 of [22]. By a similar calculation, if $x \in D(A)$ and $\alpha > 0$, then $C_t^{\alpha+2}Ax = (\alpha+2)(\alpha+1)t^{-2}[C_t^\alpha - I]x$. This proves (30) for the case $\alpha > 0$. The special case $\alpha = 0$ is a basic property of a strongly continuous cosine operator function.

By (30), together with the fact that $\sup_{t>0} \|C(t)x\| \geq \sup_{t>0} \|C_t^\alpha x\| \geq \sup_{t>0} \|C_t^\beta x\|$ for $0 < \alpha < \beta < \infty$ and $x \in X$, it follows that if we set

$$(31) \qquad A_t = C_t^{\gamma+2}, \quad \text{and} \quad B_t = \frac{-t^2}{(\gamma+4)(\gamma+3)} C_t^{\gamma+4} \qquad \text{for } t > 0,$$

then

$$(32) \qquad \begin{cases} \sup_{t>0} \|A_t\| \leq M, \quad B_tA \subset AB_t = I - A_t, \\ A_tA \subset AA_t = \frac{(\gamma+2)(\gamma+1)}{t^2}[C_t^\gamma - I], \quad \text{and} \quad \lim_{t\to\infty} \|AA_t\| = 0. \end{cases}$$

Hence $\{A_t\}$ is an A-ergodic net satisfying condition (c), and $\{B_t\}$ is its companion net satisfying condition (d). Let

$$D(P) = \{x \in X : \lim_{t\to\infty} A_tx \text{ exists}\}, \quad \text{and} \quad Px = \lim_{t\to\infty} A_tx \qquad \text{for } x \in D(P).$$

Then, as in §5, (M1) and (M2) hold. The following are standard and easily checked: $N(A) = \{x \in X : C(t)x = x \text{ for all } t > 0\}$, and $\overline{R(A)} = \overline{\{C(t)x - x : x \in X,\ t > 0\}}$.

Using these we have the next result, which corresponds to Theorem 4. Since the proof is essentially the same as that of Theorem 1, we may omit it here.

Theorem 7. (Cf. Theorem 3.7 of [19].) *Let $\gamma \geq 0$, and suppose (16) holds. Then the following conditions are equivalent:*

(i) $y \in A(D(P) \cap D(A))$; (ii) $y \in A(\overline{R(A)} \cap D(A))$; (iii) $\lim_{t\to\infty} t^2 C_t^{\gamma+4}y$ *exists;* (iv) $\{t^2C_t^{\gamma+4}y\}$ *has a weak cluster point as* $t \to \infty$; (v) $\lim_{\lambda\downarrow 0}(\lambda - A)^{-1}y$ *exists;* (vi) $\{(\lambda - A)^{-1}y\}$ *has a weak cluster point as* $\lambda \downarrow 0$.

Lemma 6. *Let $\gamma \geq 0$, and suppose (16) holds. If $2 \leq \alpha < \beta < \infty$ and $y \in X$, then the following hold:*

(i) $\sup_{t>0} \|t^2 C_t^\alpha y\| = M < \infty$ *implies*

$$\sup_{t>0} \|t^2 C_t^\beta y\| \leq \frac{\beta(\beta-1)}{\alpha(\alpha-1)} M, \quad \textit{and} \quad \sup_{\lambda>0} \|(\lambda - A)^{-1} y\| \leq \frac{1}{\alpha(\alpha-1)} M\,;$$

(ii) *strong [resp. weak]*-$\lim_{t\to\infty} t^2 C_t^\alpha y = x$ *implies*

$$\textit{strong [resp. weak]-}\lim_{t\to\infty} t^2 C_t^\beta y = \frac{\beta(\beta-1)}{\alpha(\alpha-1)}\, x,$$

and

$$\textit{strong [resp. weak]-}\lim_{\lambda\downarrow 0} (\lambda - A)^{-1} y = \frac{1}{\alpha(\alpha-1)}\, x.$$

Proof. From the proof of Lemma 5 we see that

$$t^2 C_t^\beta y = \frac{\beta}{t^{\beta-2}} \cdot \frac{1}{B(\beta-\alpha,\alpha)} \int_0^t (t-s)^{\beta-\alpha-1} \left(\int_0^s (s-r)^{\alpha-1} C(r) y\, dr \right) ds. \tag{33}$$

Thus, if $\alpha > 2$, then

$$\begin{aligned} t^2 C_t^\beta y = \frac{\beta}{t^{\beta-2} B(\beta-\alpha,\alpha)} &\int_0^t (t-s)^{\beta-\alpha-1} \\ &\left(\int_0^s (s-r)^{\alpha-3}\, dr \right) \frac{\int_0^s (s-r)^{\alpha-1} C(r) y\, dr}{\int_0^s (s-r)^{\alpha-3}\, dr}\, ds. \end{aligned}$$

Here

$$\begin{aligned} \frac{\left\| \int_0^s (s-r)^{\alpha-1} C(r) y\, dr \right\|}{\int_0^s (s-r)^{\alpha-3}\, dr} &= \frac{\alpha-2}{s^{\alpha-2}} \left\| \int_0^s (s-r)^{\alpha-1} C(r) y\, dr \right\| \\ &= \frac{\alpha-2}{\alpha} \|s^2 C_s^\alpha y\| \leq \frac{\alpha-2}{\alpha} M\,, \end{aligned}$$

and

$$\begin{aligned} &\frac{\beta}{t^{\beta-2} B(\beta-\alpha,\alpha)} \int_0^t (t-s)^{\beta-\alpha-1} \left(\int_0^s (s-r)^{\alpha-3} dr \right) ds \\ &\quad = \frac{\beta}{t^{\beta-2} B(\beta-\alpha,\alpha)} \cdot B(\beta-\alpha, \alpha-2) \cdot \frac{t^{\beta-2}}{\beta-2} = \frac{\beta(\beta-1)}{(\alpha-1)(\alpha-2)}. \end{aligned}$$

Hence we have

$$\|t^2 C_t^\beta y\| \le \frac{\beta(\beta-1)}{(\alpha-1)(\alpha-2)} \cdot \frac{\alpha-2}{\alpha}\, M \;=\; \frac{\beta(\beta-1)}{(\alpha-1)\alpha}\, M$$

for all $t > 0$. On the other hand, if $\alpha = 2$, then

$$t^2 C_t^\beta y = \frac{\beta}{t^{\beta-2} B(\beta-2,2)} \int_0^t (t-s)^{\beta-3} \left(\int_0^s (s-r) C(r) y\, dr \right) ds,$$

and

$$\left\| \int_0^s (s-r) C(r) y\, dr \right\| = 2^{-1} \|s^2 C_s^2 y\| \le 2^{-1} M.$$

Thus,

$$\|t^2 C_t^\beta y\| \le \frac{\beta}{t^{\beta-2} B(\beta-2,2)} \cdot \frac{t^{\beta-2}}{\beta-2} \cdot \frac{1}{2} M = \frac{\beta(\beta-1)}{2}\, M.$$

This proves the first inequality of (i).

To prove the second inequality of (i), we note that if $n \ge \max\{\gamma, \alpha\}$ and $\lambda > 0$, then

$$\begin{aligned}
(\lambda^2 - A)^{-1} y &= R(\lambda^2) y \\
&= \lambda^{n-1} \int_0^\infty e^{-\lambda t} \left\{ \int_0^t \left[\int_0^{s_1} (\ldots (\int_0^{s_{n-1}} C(s_n) y\, ds_n) \ldots)\, ds_2 \right] ds_1 \right\} dt \\
&= \lambda^{n-1} \int_0^\infty e^{-\lambda t} \frac{t^n}{n!} C_t^n y\, dt \qquad\qquad (\text{cf. } (14)) \\
&= \frac{\lambda^{n-1}}{n!} \int_0^\infty e^{-\lambda t} t^{n-2} (t^2 C_t^n y)\, dt,
\end{aligned}$$

where $\|t^2 C_t^n y\| \le n(n-1)\alpha^{-1}(\alpha-1)^{-1} M$ for all $t > 0$ by the first inequality of (i), and

$$\frac{\lambda^{n-1}}{n!} \int_0^\infty e^{-\lambda t} t^{n-2}\, dt = \frac{1}{n!}\, \Gamma(n-1) = \frac{1}{n(n-1)}.$$

Thus we have

$$\|(\lambda^2 - A)^{-1} y\| \le \frac{1}{n(n-1)} \cdot \frac{n(n-1)M}{\alpha(\alpha-1)} = \frac{M}{\alpha(\alpha-1)},$$

which proves the second inequality of (i).

The proof of (ii) is similar to that of (i), and hence we may omit it.

The next two theorems correspond to Theorems 5 and 6, respectively.

Theorem 8. *Let $\gamma \ge 0$, and suppose (16) holds. If the operator $H = \int_\alpha^\beta a(t) C(t)\, dt$, where $a(t)$ is a real-valued continuous function on the interval $[\alpha, \beta]$ with $\int_\alpha^\beta a(t)\, dt \ne 0$, is weakly compact, then*

(i) $X = N(A) \oplus \overline{R(A)}$, and

(ii) $y \in R(A) \Leftrightarrow \sup_{t>0} \|t^2 C_t^{\gamma+2} y\| < \infty \Leftrightarrow \liminf_{t\to\infty} \|t^2 C_t^{\gamma+4} y\| < \infty$
$\Leftrightarrow \sup_{\lambda>0} \|(\lambda - A)^{-1} y\| < \infty \Leftrightarrow \liminf_{\lambda\downarrow 0} \|(\lambda - A)^{-1} y\| < \infty$.

Proof. As in the proof of Theorem 5 we may assume that $\int_\alpha^\beta a(t)\,dt = 1$. Then, using the closedness of A and Lemma 2.14 of [22], we find that for all $x \in X$

$$(H - I)x = \int_\alpha^\beta a(t)[C(t)x - x]\,dt = \int_\alpha^\beta a(t) A \left(\int_0^t (t-s) C(s) x\,ds \right) dt$$

$$= A \int_\alpha^\beta a(t) \int_0^t (t-s) C(s) x\,ds\,dt,$$

so that $R(H - I) \subset R(A)$. By this, together with (30) with $\alpha = \gamma$ and Lemma 6, the present theorem follows as in Theorem 5.

The proof of Theorem 9 below is essentially the same as that of Theorem 3, and so we may omit the details.

Theorem 9. *Let $\gamma \geq 0$, and suppose (16) holds. Then the following conditions are equivalent:*

(i) $\{y \in X : \sup_{t>0} \|t^2 C_t^{\gamma+2} y\| < \infty\} = R(A)$;

(ii) $\overline{A(U \cap D(A))} \subset R(A)$, where $U = \{x \in X : \|x\| \leq 1\}$;

(iii) $\{y \in X : \liminf_{\lambda\downarrow 0} \|(\lambda - A)^{-1} y\| < \infty\} = R(A)$.

Example 4. There exists an example of a mean ergodic cosine operator function $\{C(t)\}$ of linear contractions on X such that $\sup_{t>0} \|t^2 C_t^2 y\| < \infty$ does not imply $y \in R(A)$. To see this, as in Example 3, let $X = C_0(\boldsymbol{R})$ and $T_t f(s) = f(t+s)$ for t, $s \in \boldsymbol{R}$. Define

$$C(t) = (T_t + T_{-t})/2 \qquad (t \geq 0). \tag{34}$$

Then $\{C(t)\}$ becomes a strongly continuous cosine operator function on X with $\|C(t)\| \leq 1$ for all $t \geq 0$ (cf. [22]). It is clear that $\lim_{b\to\infty} b^{-1} \| \int_0^b C(t) f\,dt\| = 0$ for every $f \in X$. Thus $\{C(t)\}$ is mean ergodic. Define a function g in X by

$$g(s) = \begin{cases} s & \text{if } 0 \leq s \leq 1, \\ 2 - s & \text{if } 1 \leq s \leq 3, \\ s - 4 & \text{if } 3 \leq s \leq 4, \\ 0 & \text{otherwise.} \end{cases} \tag{35}$$

Then, for $t > 0$ and $u \in \boldsymbol{R}$, we have

$$t^2 C_t^2 g(u) = 2 \int_0^t \left(\int_0^s C(r) g(u)\, dr \right) ds = \int_0^t \left(\int_{u-s}^{u+s} g(r)\, dr \right) ds.$$

It follows from the definition of g that $|t^2 C_t^2 g(u)| \le 4$ for all $u \in \boldsymbol{R}$. Hence, $\sup_{t>0} \|t^2 C_t^2 g\| \le 4$. But we have $g \notin R(A)$. To see this, assume the contrary: $g = Ah$ for some $h \in D(A)$. Since $D(A) = \{f \in C_0(\boldsymbol{R}) : f'' \in C_0(\boldsymbol{R})\}$ and $Af = f''$ for $f \in D(A)$ (cf. e.g. Example 2.27 of [22]), we then have $g = Ah = h''$. Here $h' \in C_0(\boldsymbol{R})$ by Landau's inequality (cf. e.g. page 8 of [13]). On the other hand, by the relation $h'(u) - h'(0) = \int_0^u g(s)\, ds$ for $u \in \boldsymbol{R}$ and the definition of g, we have $h'(u) = h'(0)$ for all $u \ge 4$. It follows that $h'(0) = \lim_{u\to\infty} h'(u) = 0$. Hence, $h'(u) = h'(u) - h'(0) = \int_0^u g(s)\, ds = 0$ for all $u \le 0$, because $g = 0$ on $(-\infty, 0]$. Consequently we have: $h' = 0$ on $(-\infty, 0]$, $h' > 0$ on $(0, 4)$, and $h' = 0$ on $[4, \infty)$. Thus

$$h(u) - h(0) = \int_0^u h'(s)\, ds = 0 \qquad \text{for all } \ u < 0,$$

and since $\lim_{u\to-\infty} h(u) = 0$, we have $h = 0$ on $(-\infty, 0]$. Similarly, $h \in C_0(R)$ and $h' = 0$ on $[4, \infty)$ imply that $h = 0$ on $[4, \infty)$. Since $h' > 0$ on $(0, 4)$, this implies that $0 = h(u) - h(0) = \int_0^u h'(s)\, ds = \int_0^4 h'(s)\, ds > 0$ for all $u > 4$, a contradiction. It is clear that $\{C(t)\}$ satisfies (16), with $\gamma = 0$ and $M = 1$.

Remark 6. Let $\{C(t)\}$ be the same as in the above example. If we define $A_t = C_t^1$, and $B_t = -(t^2/6)C_t^3$ for $t > 0$, then $B_t A \subset AB_t = I - C_t^1 = I - A_t$, so that $\{A_t\}$ is an A-ergodic net with $\|A_t\| \le 1$, and $\{B_t\}$ is its companion net satisfying condition (d). We already observed that $Pf := \lim_{t\to\infty} A_t f = \lim_{t\to\infty} t^{-1} \int_0^t C(s) f\, ds = 0$ for all $f \in X = C_0(\boldsymbol{R})$. It follows that (M1) and (M2) hold. But here we would like to note that condition (c) does not hold for $\{A_t\}$. In fact, if g is the function in $C_0(\boldsymbol{R})$ defined by (35), then

$$A_t g(s) = \frac{1}{2t} \int_{s-t}^{s+t} g(r)\, dr \qquad (s \in \boldsymbol{R}).$$

Thus, $(A_t g)'(s) = (2t)^{-1}(g(s+t) - g(s-t))$ for all $s \in \boldsymbol{R}$. But, since g is not differentiable at the point $0 \in \boldsymbol{R}$ by its definition, we find that $A_t g \notin D(A)$ for all $t > 0$. Therefore, $R(A_t) \not\subset D(A)$ for all $t > 0$, and this implies that (c) does not hold for $\{A_t\}$.

7. Concluding Remarks

In this section we would like to mention the following two theorems without proof. By these theorems we may understand the necessity of introducing the boundedness condition (16) for C_0-semigroups and strongly continuous cosine operator functions in order to study our problem. The proofs of these theorems and more will appear in a forthcoming joint paper with Jeng-Chung Chen and Sen-Yen Shaw.

Theorem 10. *Let $0 < \gamma < 1$. Then there exists a C_0-semigroup $\{T_t\}$ [resp. a strongly continuous cosine operator function $\{C(t)\}$] of positive linear operators on an L_1-space such that $\sup_{t>0} \|C_t^\gamma\|_1 = \infty$, but $\sup_{t>0} \|C_t^\beta\|_1 < \infty$ for all $\beta > \gamma$.*

Theorem 11. *Let $k \geq 1$ be an integer. Then there exists a C_0-semigroup $\{T_t\}$ [resp. a strongly continuous cosine operator function $\{C(t)\}$] of bounded linear operators on X such that $\sup_{t>0} \|C_t^k\| < \infty$, but $\sup_{t>0} \|C_t^\alpha\| = \infty$ for all α, with $0 \leq \alpha < k$.*

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