TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 5, pp. 1139-1156, September 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

EIGENVALUE ESTIMATES OF THE BASIC DIRAC OPERATOR ON A RIEMANNIAN FOLIATION

Seoung Dal Jung and Yeon Soon Ko

Abstract. On a foliated Riemannian manifold with a transverse spin structure, we give a lower bound of the square for the eigenvalues of the basic Dirac operator in terms of the smallest eigenvalue of the basic Yamabe operator and of the norm of an appropriate endomorphism of the normal bundle Q of $\mathcal{F}$. We study the limiting case.

1. Introduction

The Dirac operator on a Riemannian spin manifold was studied by many authors([2,3,6,8,9,15,16,17]). In particular, the first sharp estimate for the eigenvalues λ of the Dirac operator D was given by Th. Friedrich ([6]). Using a suitable Riemannian spin connection, he proved the inequality

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R \tag{1.1}$$

on manifolds (M^n, g) with positive scalar curvature $R > 0$. The inequality (1.1) has been improved in several directions by many authors ([8,9,10,11,12,15,16,17]).

Recently, S. D. Jung([10]) proved the lower bound for the eigenvalues λ of the basic Dirac operator D_b on a foliated Riemannian manifold with a transverse spin structure, which is introduced by J.Brüning and F.W.Kamber([4]). Namely, let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and

Received August 16, 2004; accepted January 18, 2005.
Communicated by Shu-Cheng Chang.
2000 *Mathematics Subject Classification*: 53C12, 53C27, 57R30.
Key words and phrases: Minimal foliation, Basic Yamabe operator, Transversal Dirac operator.
This paper was supported by ABRL grant proj. No. R14-2002-003-01002-0 from KRF and No. R01-2003-000-10004-0 from KOSEF.

a bundle-like metric g_M. Let $S(\mathcal{F})$ be a foliated spinor bundle on M. Then the eigenvalue λ of the basic Dirac operator D_b satisfies the inequality

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M (\sigma^\nabla + |\kappa|^2), \tag{1.2}$$

where $q = codim\mathcal{F}$, σ^∇ is the transversal scalar curvature and κ is the mean curvature form of $\mathcal{F}$. In the limiting case, it is proved that $\mathcal{F}$ is minimal, transversally Einsteinian with constant transversal scalar curvature σ^∇.

In [12], S. D. Jung et al. improved (1.2) by the smallest eigenvalue of the basic Yamabe operator Y_b, which is defined by

$$Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^\nabla, \tag{1.3}$$

where Δ_B is the basic Laplacian acting on basic functions. In fact, the eigenvalue λ of D_b satisfies

$$\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf_M |\kappa|^2), \tag{1.4}$$

where μ_1 is the smallest eigenvalue of the basic Yamabe operator Y_b.

In this paper, we prove a new estimate for the eigenvalues of the basic Dirac operator by using the appropriate endomorphism on the normal bundle Q of $\mathcal{F}$. This paper is corresponding to Hijazi's one([9]) for the foliations.

The paper is organized as follows. In Section 2, we review the known facts on the foliated Riemannian manifold. In Section 3, we study the basic properties of the transversal Dirac operators of transversally conformally related metrics. In Section 4, we introduce the new connection with an appropriate endomorphism on Q and estimate the lower bound for the eigenvalues of the basic Dirac operator. Namely, let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and a bundle-like metric g_M such that κ is basic-harmonic. Then any eigenvalue λ of the basic Dirac operator corresponding to the eigenspinor $\Psi \in \Gamma S(\mathcal{F})$ satisfies

$$\lambda^2 \geq \frac{1}{4}(\mu_1 + \inf_M |\kappa|^2) + \inf_M |l_\Psi|^2, \tag{1.5}$$

where μ_1 is the first eigenvalue of the basic Yamabe operator Y_b of $\mathcal{F}$ and l_Ψ is an isomorphism on the normal bundle Q, which is defined in Section 4. In Section 5, we prove, in the limiting case that $\mathcal{F}$ is a minimal foliation.

Throughout this paper, we consider the bundle-like metric $\tilde{g}_M$ for $(M, \mathcal{F})$ such that the mean curvature form of $\mathcal{F}$ is basic-harmonic. The existence of the bundle-like metric g_M for $(M, \mathcal{F})$ such that κ is basic, i.e. $\kappa \in \Omega^1_B(\mathcal{F})$, is proved in

[5]. In [19, 20], it is proved that for any bundle-like metric g_M with $\kappa \in \Omega^1_B(\mathcal{F})$ there exists another bundle-like metric $\tilde{g}_M$ for which the mean curvature form is basic-harmonic.

2. Preliminaries and Known Facts

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$.

We recall the exact sequence

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0 \tag{2.1}$$

determined by the tangent bundle L and the normal bundle $Q = TM/L$ of $\mathcal{F}$([14]). The assumption of g_M to be a bundle-like metric means that the induced metric g_Q on the normal bundle $Q \cong L^\perp$ satisfies the holonomy invariance condition $\overset{\circ}{\nabla} g_Q = 0$, where $\overset{\circ}{\nabla}$ is the Bott connection in Q.

For a distinguished chart $\mathcal{U} \subset M$ the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a Riemannian submersion $f : \mathcal{U} \to \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of a model Riemannian manifold N.

For overlapping charts $U_\alpha \cap U_\beta$, the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on N are isometries. Further, we denote by ∇ the canonical connection of the normal bundle Q of $\mathcal{F}$. To show that such connections exist, consider a Riemannian metric g_M on M. Then TM splits orthogonally as

$$TM = L \oplus L^\perp.$$

This means that there is a bundle map $\sigma : Q \to L^\perp$ splitting the exact sequence (2.1), i.e., satisfying $\pi \circ \sigma = identity$. This metric g_M on TM is then a direct sum

$$g_M = g_L \oplus g_{L^\perp}.$$

With $g_Q = \sigma^* g_{L^\perp}$, the splitting map $\sigma : (Q, g_Q) \to (L^\perp, g_{L^\perp})$ is a metric isomorphism. Then the adapted connection ∇ in Q is defined by ([14])

$$\begin{cases} \nabla_X s = \overset{\circ}{\nabla}_X s = \pi([X, Y_s]) & \text{for } X \in \Gamma L, \\ \nabla_X s = \pi(\nabla^M_X Y_s) & \text{for } X \in \Gamma L^\perp, \end{cases} \tag{2.2}$$

where $s \in \Gamma Q$ and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $Q \cong L^\perp$. The connection ∇ is metric and torsion free. It corresponds to the Riemannian connection of the model space N. The curvature R^∇ of ∇ is defined by

$$R^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for } X,\ Y \in TM.$$

Its curvature R^∇ coincides with $\overset{\circ}{R}$ for $X, Y \in \Gamma L$, hence $R^\nabla(X,Y) = 0$ for $X, Y \in \Gamma L$. Since $i(X)R^\nabla = \theta(X)R^\nabla = 0$ ([13, 14, 22]), we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^∇ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla(s, E_a)E_a, \quad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a),$$

where $\{E_a\}_{a=1,\cdots,q}$ is an orthonormal basis of Q. $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, that is,

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot id \tag{2.3}$$

with constant transversal scalar curvature σ^∇. The *mean curvature form* κ for L is given by

$$\kappa(X) = g_Q\left(\sum_i \pi(\nabla^M_{E_i}E_i), X\right) \quad \text{for } X \in \Gamma Q, \tag{2.4}$$

where $\{E_i\}_{i=1,\cdots,p}$ is an orthonormal basis of L. The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

Let $\Omega^r_B(\mathcal{F})$ be the space of all *basic r-forms*, i.e.,

$$\Omega^r_B(\mathcal{F}) = \{\phi \in \Omega^r(M) |\ i(X)\phi = 0,\ \theta(X)\phi = 0,\ \text{for } X \in \Gamma L\}.$$

The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega^1_B(\mathcal{F})$. We already know that κ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric ([22]). Since the exterior derivative preserves the basic forms (that is, $\theta(X)d\phi = 0$ and $i(X)d\phi = 0$ for $\phi \in \Omega^r_B(\mathcal{F})$), the restriction $d_B = d|_{\Omega^*_B(\mathcal{F})}$ is well defined. Let δ_B be the formal adjoint operator of d_B. Then it is well-known([1,10]) that

$$d_B = \sum_a \theta_a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B), \tag{2.5}$$

where κ_B is the basic component of κ, $\{E_a\}$ is a local orthonormal basic frame in Q and $\{\theta_a\}$ is its g_Q-dual 1-form.

The *basic Laplacian* acting on $\Omega^*_B(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B. \tag{2.6}$$

If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian. In the more general case, the basic Laplacian and its spectrum provide information about the transverse geometry of $(M, \mathcal{F})$([21]).

3. Transversal Dirac Operators of Transversally Conformally Related Metrics

Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q, a bundle-like metric g_M with respect to $\mathcal{F}$ and a foliated spinor bundle $S(\mathcal{F})$([4,7,10]). Then the transversal Dirac operator D_{tr} is locally defined by

$$D_{tr}\Psi = \sum_a E_a \cdot \nabla_{E_a}\Psi - \frac{1}{2}\kappa \cdot \Psi \quad \text{for } \Psi \in \Gamma S(\mathcal{F}), \tag{3.1}$$

where $\{E_a\}$ is a local orthonormal basic frame of Q. We define the subspace $\Gamma_B(S(\mathcal{F}))$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$\Gamma_B(S(\mathcal{F})) = \{\Psi \in \Gamma S(\mathcal{F}) | \nabla_X \Psi = 0 \quad \text{for } X \in \Gamma L\}.$$

Trivially, we see that D_{tr} leaves $\Gamma_B(S(\mathcal{F}))$ invariant if and only if the foliation $\mathcal{F}$ is isoparametric, i.e., $\kappa \in \Omega^1_B(\mathcal{F})$. Let $D_b = D_{tr}|_{\Gamma_B(S(\mathcal{F}))} : \Gamma_B(S(\mathcal{F})) \to \Gamma_B(S(\mathcal{F}))$. This operator D_b is called the *basic Dirac operator* on (smooth) basic sections. On an isoparametric transverse spin foliation $\mathcal{F}$ with $\delta\kappa = 0$, it is well-known([7,10]) that

$$D^2_{tr}\Psi = \nabla^*_{tr}\nabla_{tr}\Psi + \frac{1}{4}K^\sigma \Psi, \tag{3.2}$$

where $K^\sigma = \sigma^\nabla + |\kappa|^2$ and

$$\nabla^*_{tr}\nabla_{tr}\Psi = -\sum_a \nabla^2_{E_a,E_a}\Psi + \nabla_\kappa \Psi. \tag{3.3}$$

The operator $\nabla^*_{tr}\nabla_{tr}$ is non-negative and formally self-adjoint ([10]) such that

$$\int_M < \nabla^*_{tr}\nabla_{tr}\Phi, \Psi > = \int_M < \nabla_{tr}\Phi, \nabla_{tr}\Psi > \tag{3.4}$$

for all $\Phi, \Psi \in \Gamma S(\mathcal{F})$. Moreover, the curvature transform R^S on $S(\mathcal{F})$ is given ([10,18]) as

$$R^S(X,Y)\Psi = \frac{1}{4}\sum_{a,b} g_Q(R^\nabla(X,Y)E_a, E_b)E_a \cdot E_b \cdot \Psi \quad \text{for } X, Y \in \Gamma TM. \tag{3.5}$$

Then we have the following lemma.

Lemma 3.1. ([10]) *On the foliated spinor bundle $S(\mathcal{F})$, we have the following equations*

$$\sum_{a<b} E_a \cdot E_b \cdot R^S(E_a, E_b)\Psi = \frac{1}{4}\sigma^\nabla \Psi, \tag{3.6}$$

(3.7) $$\sum_a E_a \cdot R^S(X, E_a)\Psi = -\frac{1}{2}\rho^\nabla(X)\cdot\Psi \quad \text{for } X \in \Gamma Q.$$

Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar g_Q = e^{2u} g_Q$. Let $\bar P_{so}(\mathcal{F})$ be the principal bundle of $\bar g_Q$-orthogonal frames. Locally, the section $\bar s$ of $\bar P_{so}(\mathcal{F})$ corresponding a section $s = (E_1, \cdots, E_q)$ of $P_{so}(\mathcal{F})$ is $\bar s = (\bar E_1, \cdots, \bar E_q)$, where $\bar E_a = e^{-u}E_a$ $(a = 1, \cdots, q)$. This isometry will be denoted by I_u. Thanks to the isomorphism I_u one can define a transverse spin structure $\bar P_{spin}(\mathcal{F})$ on $\mathcal{F}$ in such a way that the diagram

$$\begin{array}{ccc} P_{spin}(\mathcal{F}) & \xrightarrow{\tilde I_u} & \bar P_{spin}(\mathcal{F}) \\ \downarrow & & \downarrow \\ P_{so}(\mathcal{F}) & \xrightarrow{I_u} & \bar P_{so}(\mathcal{F}) \end{array}$$

commutes.

Let $\bar S(\mathcal{F})$ be the foliated spinor bundle associated with $\bar P_{spin}(\mathcal{F})$. For any section Ψ of $S(\mathcal{F})$, we write $\bar\Psi \equiv I_u\Psi$. If $<\ ,\ >_{g_Q}$ and $<\ ,\ >_{\bar g_Q}$ denote respectively the natural Hermitian metrics on $S(\mathcal{F})$ and $\bar S(\mathcal{F})$, then for any $\Phi,\ \Psi \in \Gamma S(\mathcal{F})$

(3.8) $$< \Phi, \Psi >_{g_Q} = < \bar\Phi, \bar\Psi >_{\bar g_Q},$$

and the Clifford multiplication in $\bar S(\mathcal{F})$ is given by

(3.9) $$\bar X \mathbin{\bar\cdot} \bar\Psi = \overline{X\cdot\Psi} \quad \text{for } X \in \Gamma Q.$$

Let $\bar\nabla$ be the metric and torsion free connection corresponding to $\bar g_Q$. Then we have ([12]) that for $X, Y \in \Gamma TM$,

(3.10) $$\bar\nabla_X \pi(Y) = \nabla_X\pi(Y) + X(u)\pi(Y) + Y(u)\pi(X) - g_Q(\pi(X), \pi(Y))grad_\nabla(u),$$

where $grad_\nabla(u) = \sum_a E_a(u)E_a$ is a transversal gradient of u and $X(u)$ is the Lie derivative of the function u in the direction of X. The formula (3.10) follows from that $\bar\nabla$ is the metric and torsion free connection with respect to $\bar g_Q$. From (3.10), we have the following proposition.

Proposition 3.2. ([12]) *The connections ∇ and $\bar\nabla$ acting repectively on the sections of $S(\mathcal{F})$ and $\bar S(\mathcal{F})$, are related, for any vector field X and any spinor field Ψ by*

(3.11) $$\bar\nabla_X\bar\Psi = \overline{\nabla_X\Psi} - \frac{1}{2}\overline{\pi(X)\cdot grad_\nabla(u)\cdot\Psi} - \frac{1}{2}g_Q(grad_\nabla(u), \pi(X))\bar\Psi.$$

Let $\bar{D}_{tr}$ be the transversal Dirac operator associated with the metric $\bar{g}_Q = e^{2u}g_Q$ and acting on the sections of the foliated spinor bundle $\bar{S}(\mathcal{F})$. Let $\{E_a\}$ be a local frame of $P_{so}(\mathcal{F})$ and $\{\bar{E}_a\}$ a local frame of $\bar{P}_{so}(\mathcal{F})$. Locally, $\bar{D}_{tr}$ is expressed by

(3.12) $$\bar{D}_{tr}\bar{\Psi} = \sum_a \bar{E}_a \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi} - \frac{1}{2}\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi},$$

where $\kappa_{\bar{g}}$ is the mean curvature form associated with $\bar{g}_Q$, which satisfies $\kappa_{\bar{g}} = e^{-2u}\kappa$.

Proposition 3.3. ([12]) *Let $\mathcal{F}$ be the transverse spin foliation of codimension q. Then the transverse Dirac operators D_{tr} and $\bar{D}_{tr}$ satisfy*

(3.13) $$\bar{D}_{tr}(e^{-\frac{q-1}{2}u}\bar{\Psi}) = e^{-\frac{q+1}{2}u}\overline{D_{tr}\Psi}$$

for any spinor field $\Psi \in S(\mathcal{F})$.

From Proposition 3.3, if $D_{tr}\Psi = 0$, then $\bar{D}_{tr}\bar{\Phi} = 0$, where $\Phi = e^{-\frac{q-1}{2}u}\Psi$, and conversely. Hence the dimension of the space of the foliated harmonic spinors is a transversally conformal invariant.

Assume the mean curvature form κ of $\mathcal{F}$ is basic- harmonic, i.e., $\kappa \in \Omega^1_B(\mathcal{F})$ and $\delta_B\kappa = 0$. Then we have the Lichnerowicz type formula ([12])

(3.14) $$\bar{D}^2_{tr}\bar{\Psi} = \bar{\nabla}^*_{tr}\bar{\nabla}_{tr}\bar{\Psi} + \frac{1}{4}K^{\bar{\nabla}}_{\sigma}\bar{\Psi},$$

where

(3.15) $$\bar{\nabla}^*_{tr}\bar{\nabla}_{tr}\bar{\Psi} = -\sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\sum \bar{\nabla}_{\bar{E}_a}\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi},$$

(3.16) $$K^{\bar{\nabla}}_{\sigma} = \sigma^{\bar{\nabla}} + |\bar{\kappa}|^2 + 2(q-2)\kappa_{\bar{g}}(u).$$

Moreover, it is well-known[12] that for all $\Phi, \Psi \in S(\mathcal{F})$

(3.17) $$\int_M < \bar{\nabla}^*_{tr}\bar{\nabla}_{tr}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} = \int_M < \bar{\nabla}_{tr}\bar{\Psi}, \bar{\nabla}_{tr}\bar{\Phi} >_{\bar{g}_Q},$$

where $< \bar{\nabla}_{tr}\bar{\Psi}, \bar{\nabla}_{tr}\bar{\Phi} >_{\bar{g}_Q} = \sum_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Phi} >_{\bar{g}_Q}$.

4. Eigenvalue Estimate of the Basic Dirac Operator

Let $(M, \tilde{g}_M, \mathcal{F}, S(\mathcal{F}))$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric $\tilde{g}_M$ such that κ is basic-harmonic.

Let l be a linear symmetric endomorphism of Q. For any tangent vector field X and any spinor field Ψ, we define the modified connection $\overset{l}{\nabla}$ on $S(\mathcal{F})$ by

$$\overset{l}{\nabla}_X \Psi = \nabla_X\Psi + l(\pi(X))\cdot\Psi. \tag{4.1}$$

It is easy to see that the connection $\overset{l}{\nabla}$ is metrical. By using the transversal divergence theorem ([23]), we have the following equation

$$\int_M <\overset{l}{\nabla}{}^*{}_{tr}\overset{l}{\nabla}_{tr}\Phi,\Psi>=\int_M <\overset{l}{\nabla}_{tr}\Phi,\overset{l}{\nabla}_{tr}\Psi> \tag{4.2}$$

for all $\Phi,\Psi\in\Gamma S(\mathcal{F})$. By a direct calculation, we have from (4.1) that for any linear symmetric endomorphism l of Q

$$|\overset{l}{\nabla}_{tr}\Psi|^2 = |\nabla_{tr}\Psi|^2 - 2Re\sum_a < l(E_a)\cdot\nabla_{E_a}\Psi,\Psi> + |l|^2|\Psi|^2. \tag{4.3}$$

We now show that for an appropriate choice of the symmetric endomorphism l, one gets a sharp estimate for the first eigenvalue of the basic Dirac operator on compact Riemannian manifolds. For this, we need the following: On the complement of the set of zeros of a spinor field $\Psi\in\Gamma_B S(\mathcal{F})$, we define for any tangent vector fields X and Y, the symmetric bilinear tensor F_Ψ (see [9]) by

$$F_\Psi(X,Y)=\frac{1}{2}Re<\pi(X)\cdot\nabla_Y\Psi+\pi(Y)\cdot\nabla_X\Psi,\Psi/|\Psi|^2> . \tag{4.4}$$

Since $<\kappa\cdot\Psi,\Psi>$ is pure imaginary, we have

$$\mathrm{tr}F_\Psi := \sum_a F_\Psi(E_a,E_a) = Re<D_b\Psi,\Psi/|\Psi|^2> . \tag{4.5}$$

Let l_Ψ be the symmetric endomorphism associated with F_Ψ. From (4.5), if $D_b\Psi = \lambda\Psi$, then

$$\mathrm{tr}\ l_\Psi = \lambda. \tag{4.6}$$

On the other hand, since l is a linear symmetric endomorphism, we have

$$Re\sum_a < l(E_a)\cdot\nabla_{E_a}\Psi,\Psi/|\Psi|^2>=\sum_a< l(E_a),l_\Psi(E_a)>=< l,l_\Psi> . \tag{4.7}$$

From (4.3) and (4.7), we have the following equation;

$$|\overset{l}{\nabla}_{tr}\Psi|^2 = |\nabla_{tr}\Psi|^2 - f(l)|\Psi|^2, \tag{4.8}$$

where $f(l) = 2 < l, l_\Psi > -|l|^2$. Note that $f(l)$ has maximum value $|l_\Psi|^2$ at $l = l_\Psi$ because l is a linear endomorphism. From (3.2), (3.4) and (4.8), we have that for any eigenspinor Ψ corresponding to an eigenvalue λ

$$\int_M |\overset{l_\Psi}{\nabla}_{tr}\Psi|^2 = \int_M \{\lambda^2 - (\frac{1}{4}K^\sigma + |l_\Psi|^2)\}|\Psi|^2. \tag{4.9}$$

Hence we have the following theorem.

Theorem 4.1. *Let $(M, \tilde g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric $\tilde g_M$. Then any eigenvalue λ of the basic Dirac operator D_b corresponding to the eigenspinor $\Psi \in \Gamma S(\mathcal{F})$ satisfies*

$$\lambda^2 \geq \inf_M(\frac{1}{4}K^\sigma + |l_\Psi|^2), \tag{4.10}$$

where $K^\sigma = \sigma^\nabla + |\kappa|^2$.
Moreover, by the Cauchy-Schwarz inequality we have

$$\frac{1}{q}(\mathrm{tr}l_\Psi)^2 \leq |l_\Psi|^2. \tag{4.11}$$

Hence we have the following corollary.

Corollary 4.2. (cf. [10]) *Under the same conditions as in Theorem 4.1, one has*

$$\lambda^2 \geq \frac{q}{4(q-1)}\inf_M K^\sigma. \tag{4.12}$$

On the other hand, we obtain that on the spinor bundle $\bar S(\mathcal{F})$ associated with the metric $\bar g_Q = e^{2u}g_Q$

$$F_{\bar\Phi}(\bar X, \bar Y) = e^{-u}F_\Phi(X,Y) = e^{-u}F_\Psi(X,Y), \quad X, Y \in \Gamma Q, \tag{4.13}$$

where $\Phi = e^{-\frac{q-1}{2}u}\Psi$. From (4.13), we have the following identity

$$l_{\bar\Phi} = e^{-u}l_\Phi = e^{-u}l_\Psi. \tag{4.14}$$

From (4.14), we have the following proposition (see [9] for the details).

Proposition 4.3. *The following relations hold:*

$$|l_{\bar\Phi}|^2_{\bar g_Q} = e^{-2u}|l_\Phi|^2 = e^{-2u}|l_\Psi|^2, \tag{4.15}$$

where $\Phi = e^{-\frac{q-1}{2}u}\Psi$.

Now, we introduce a connection $\overset{l}{\bar{\nabla}}$ on $\bar{S}(\mathcal{F})$, as

$$\overset{l}{\bar{\nabla}}_X \bar{\Psi} = \bar{\nabla}_X \bar{\Psi} + l(\pi(X)) \bar{\cdot} \bar{\Psi} \quad \text{for } X \in TM, \tag{4.16}$$

where l is a linear symmetric endomorphism on Q. Trivially, the connection $\overset{l}{\bar{\nabla}}$ is a metric connection and $\overset{l}{\bar{\nabla}}{}^*{}_{tr} \overset{l}{\bar{\nabla}}_{tr}$ is positive definite. By using the connection $\overset{l_{\bar{\Phi}}}{\bar{\nabla}}$, we can obtain the following equation

$$|\overset{l_{\bar{\Phi}}}{\bar{\nabla}}_{tr} \bar{\Phi}|^2_{\bar{g}_Q} = |\bar{\nabla}_{tr}\bar{\Phi}|^2 - |l_{\bar{\Phi}}|^2_{\bar{g}_Q} |\bar{\Phi}|^2_{\bar{g}_Q}. \tag{4.17}$$

Hence we have from (3.14), (3.17) and (4.17)

$$\begin{aligned} \int_M |\overset{l_{\bar{\Phi}}}{\bar{\nabla}}_{tr} \bar{\Phi}|^2_{\bar{g}_Q} = \int_M \Big\{ &< \bar{D}^2_{tr}\bar{\Phi}, \bar{\Phi} >_{\bar{g}_Q} - \frac{1}{4} < K^{\bar{\nabla}}_\sigma \bar{\Phi}, \bar{\Phi} >_{\bar{g}_Q} \\ &- |l_{\bar{\Phi}}|^2_{\bar{g}_Q} |\bar{\Phi}|^2_{\bar{g}_Q} \Big\}. \end{aligned} \tag{4.18}$$

Let $D_b\Psi = \lambda\Psi (\Psi \neq 0)$. From Proposition 3.3, we have

$$\bar{D}_b \bar{\Phi} = \lambda e^{-u} \bar{\Phi}, \tag{4.19}$$

where $\Phi = e^{-\frac{q-1}{2}u}\Psi$. From (4.15) and (4.18), we have that for any eigenspinor Ψ corresponding to the eigenvalue λ

$$\int_M |\overset{l_{\bar{\Phi}}}{\bar{\nabla}}_{tr} \bar{\Phi}|^2_{\bar{g}_Q} = \int_M e^{-2u} \{\lambda^2 - \frac{1}{4}(e^{2u} K^{\bar{\nabla}}_\sigma + 4|l_\Psi|^2)\} |\bar{\Phi}|^2_{\bar{g}_Q}, \tag{4.20}$$

where $\Phi = e^{-\frac{q-1}{2}u}\Psi$. Hence we have the following theorem.

Theorem 4.4. *Let $(M, \tilde{g}_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric $\tilde{g}_M$. Assume that $K^{\bar{\nabla}}_\sigma \geq 0$ for some transversally conformal metric $\bar{g}_Q = e^{2u} g_Q$. Then any eigenvalue λ of the basic Dirac operator D_b corresponding to the eigenspinor $\Psi \in \Gamma S(\mathcal{F})$ satisfies*

$$\lambda^2 \geq \frac{1}{4} \sup_u \inf_M (e^{2u} K^{\bar{\nabla}}_\sigma + 4|l_\Psi|^2), \tag{4.21}$$

where $K^{\bar{\nabla}}_\sigma = \sigma^{\bar{\nabla}} + |\bar{\kappa}|^2 + 2(q-2)\kappa_{\bar{g}}(u)$.

From (4.11), we have the following corollary.

Corollary 4.5. (cf. [12]) *Under the same assumption as in Theorem 4.4, we have*

$$\lambda^2 \geq \frac{q}{4(q-1)} \sup_u \inf_M (e^{2u} K_\sigma^{\bar\nabla}). \tag{4.22}$$

The transversal Ricci curvature $\rho^{\bar\nabla}$ of $\bar g_Q = e^{2u} g_Q$ and the transversal scalar curvature $\sigma^{\bar\nabla}$ of $\bar g_Q$ are related to the transversal Ricci curvature ρ^{∇} of g_Q and the transversal scalar curvature σ^{∇} of g_Q by the following lemma.

Lemma 4.6. ([12]) *On a Riemannian foliation $\mathcal{F}$, we have that for any $X \in Q$,*

$$\begin{aligned} e^{2u}\rho^{\bar\nabla}(X) &= \rho^{\nabla}(X) + (2-q)\nabla_X grad_\nabla(u) + (2-q)|grad_\nabla(u)|^2 X \\ &\quad + (q-2)X(u) grad_\nabla(u) + \{\Delta_B u - \kappa(u)\}X. \end{aligned} \tag{4.23}$$

$$e^{2u}\sigma^{\bar\nabla} = \sigma^{\nabla} + (q-1)(2-q)|grad_\nabla(u)|^2 + 2(q-1)\{\Delta_B u - \kappa(u)\}. \tag{4.24}$$

From (3.16) and (4.24), we have

$$e^{2u} K_\sigma^{\bar\nabla} = \sigma^{\nabla} + |\kappa|^2 + 2(q-1)\Delta_B u + (q-1)(2-q)|grad_\nabla(u)|^2 - 2\kappa(u). \tag{4.25}$$

On the other hand, for $q \geq 3$, if we choose the positive function h by $u = \frac{2}{q-2}\ln h$, then we have

$$\Delta_B u = \frac{2}{q-2}\{h^{-2}|grad_\nabla(h)|^2 + h^{-1}\Delta_B h\}, \tag{4.26}$$

$$|grad_\nabla(u)|^2 = (\frac{2}{q-2})^2 h^{-2} |grad_\nabla(h)|^2. \tag{4.27}$$

Hence we have

$$e^{2u} K_\sigma^{\bar\nabla} = h^{\frac{4}{q-2}} K_\sigma^{\bar\nabla} = h^{-1} Y_b h + |\kappa|^2 - \frac{4}{q-2} h^{-1}\kappa(h), \tag{4.28}$$

where

$$Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^{\nabla} \tag{4.29}$$

is a *basic Yamabe operator* of $\mathcal{F}$([12]). Now we put $\mathcal{K}_u = \{u \in \Omega_B^0(\mathcal{F}) | \kappa(u) = 0\}$. If we choose $u \in \mathcal{K}_u$, then $\kappa(h) = 0 = \kappa(u)$. From (4.25) and (4.28), we have

$$e^{2u} K_\sigma^{\bar\nabla} = K^\sigma + 2(q-1)\Delta_B u + (q-1)(2-q)|grad_\nabla(u)|^2 = h^{-1} Y_b h + |\kappa|^2, \tag{4.30}$$

where $K^\sigma = \sigma^{\nabla} + |\kappa|^2$. From (4.21) and (4.30), we have the following theorem.

Theorem 4.7. *Let $(M, \tilde g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and a bundle-like metric $\tilde g_M$. Then, any eigenvalue λ of the basic Dirac operator corresponding to the eigenspinor $\Psi \in \Gamma S(\mathcal{F})$ satisfies*

$$\lambda^2 \geq \frac{1}{4}(\mu_1 + \inf_M |\kappa|^2) + \inf_M |l_\Psi|^2, \tag{4.31}$$

where μ_1 is the first eigenvalue of the basic Yamabe operator Y_b of $\mathcal{F}$.

From (4.11), we have the following corollary.

Corollary 4.8. (cf.[12]) *Let $(M, \tilde g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and a bundle-like metric $\tilde g_M$. If the transversal scalar curvature satisfies $\sigma^\nabla \geq 0$, then any eigenvalue λ of D_b satisfies*

$$\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf |\kappa|^2). \tag{4.32}$$

5. The Limiting Case

In this section, we study the limiting foliations of (4.10) and (4.31). For any linear symmetric endomorphism l of Q, we define $Ric^l_\nabla : \Gamma Q \otimes S(\mathcal{F}) \to S(\mathcal{F})$ ([9, 10]) by

$$Ric^l_\nabla(X \otimes \Psi) = \sum_a E_a \cdot R^l(X, E_a)\Psi, \tag{5.1}$$

where R^l is the curvature tensor with respect to $\overset{l}{\nabla}$ defined by (4.1). Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$ for all a. By a long calculation, for $X \in \Gamma Q$ and $\Psi \in \Gamma S(\mathcal{F})$ we have that at x

$$Ric^l_\nabla(X \otimes \Psi) = -\frac{1}{2}\rho^\nabla(X)\cdot\Psi + \sum_a \{E_a \cdot dl(X, E_a) + E_a \cdot [l(X), l(E_a)]\}\cdot\Psi, \tag{5.2}$$

where $[X, Y] = X \cdot Y - Y \cdot X$ and $dl(X, Y) = (\nabla_X l)(Y) - (\nabla_Y l)(X)$. By the definition of Clifford multiplication, we have

$$\sum_a E_a \cdot dl(X, E_a) = \sum_a E_a \wedge dl(X, E_a) - [X(\mathrm{tr}\ l) - (div_\nabla l)(X)], \tag{5.3}$$

where $(div_\nabla l)(X) = \sum_a g_Q((\nabla_{E_a} l)(X), E_a)$. Since l is symmetric,

$$\sum_a E_a \cdot l(E_a) = \sum_a l(E_a) \cdot E_a, \tag{5.4}$$

and then

$$\sum_a E_a \cdot l(E_a) = -\mathrm{tr}\ l. \tag{5.5}$$

From (5.4) and (5.5), we have

$$\sum_a E_a \cdot [l(X), l(E_a)] \cdot \Psi = 2(\mathrm{tr}\ l) l(X) \cdot \Psi - 2l^2(X) \cdot \Psi. \tag{5.6}$$

From (5.2), (5.3) and (5.6), we have that

(5.7)

$$\begin{aligned} Ric^l_\nabla(X \otimes \Psi) &= -\tfrac{1}{2}\rho^\nabla(X) \cdot \Psi + \textstyle\sum_a (E_a \wedge dl(X, E_a)) \cdot \Psi - 2l^2(X) \cdot \Psi \\ &\quad + 2(\mathrm{tr}\ l) l(X) \cdot \Psi - \{X(\mathrm{tr}\ l) - (div_\nabla l)(X)\} \cdot \Psi. \end{aligned}$$

From (5.7), we have the following facts.

Proposition 5.1. *If M admits a non-zero spinor field $\Psi \in \Gamma S(\mathcal{F})$ with $\overset{l}{\nabla} \Psi = 0$, then $|\Psi|^2$ is constant and for any $X \in \Gamma Q$*

$$grad_\nabla(\mathrm{tr}\ l) = div_\nabla l, \quad (\mathrm{tr}\ l)^2 = \frac{1}{4}\sigma^\nabla + |l|^2, \tag{5.8}$$

$$\frac{1}{2}\rho^\nabla(X) \cdot \Psi = [\sum_a (E_a \wedge dl(X, E_a)) + 2(\mathrm{tr}\ l) l(X) - 2l^2(X)] \cdot \Psi. \tag{5.9}$$

Proof. Since $\overset{l}{\nabla}$ is a metric connection, the condition $\overset{l}{\nabla} \Psi = 0$ imply that $|\Psi|^2$ is constant. If $\overset{l}{\nabla}_X \Psi = 0$ for any $X \in \Gamma Q$, then $Ric^l_\nabla = 0$. Hence from (5.7), we have

$$\begin{aligned} &< \{X(\mathrm{tr}\ l) - (div_\nabla l)(X)\} \cdot \Psi, \Psi > \\ &=< \{-\frac{1}{2}\rho^\nabla(X) + \sum_a (E_a \wedge dl(X, E_a)) + 2(\mathrm{tr}\ l) l(X) - 2l^2(X)\} \cdot \Psi, \Psi > . \end{aligned}$$

The left-hand side is real, but the right-hand side is pure imaginary because $\sum_a < E_a \wedge dl(X, E_a) \cdot \Psi, \Psi >$ is pure imaginary. Therefore, both sides are zeros. Hence we get

$$X(\mathrm{tr}\ l) = (div_\nabla l)(X), \tag{5.10}$$

$$\frac{1}{2}\rho^{\nabla}(X)\cdot\Psi=[\sum_a(E_a\wedge dl(X,E_a))+2(\text{tr }l)l(X)-2l^2(X)]\cdot\Psi. \tag{5.11}$$

Hence the first equation in (5.8) and (5.9) are proved from (5.10) and (5.11), respectively. For the second equation in (5.8), Clifford multiplication of (5.11) with E_b and for $X=E_b$, gives

$$\begin{aligned}\frac{1}{2}\sum_b E_b\cdot\rho^{\nabla}(E_b)\cdot\Psi \;=\; &\sum_{a,b}E_b\cdot(E_a\wedge dl(E_b,E_a))\cdot\Psi+2(\text{tr }l)\sum_b E_b\cdot l(E_b)\cdot\Psi\\ &-2\sum_b E_b\cdot l^2(E_b)\cdot\Psi.\end{aligned}$$

From Lemma 3.1 and (5.5), we have

$$-\frac{1}{2}\sigma^{\nabla}\Psi=\sum_{a,b}E_a\cdot(E_b\wedge dl(E_a,E_b))\cdot\Psi-2(\text{tr }l)^2\Psi+2|l|^2\Psi. \tag{5.12}$$

On the other hand,

$$\begin{aligned}\sum_{a,b}E_a\cdot(E_b\wedge dl(E_a,E_b))\cdot\Psi \;=\; &(\sum_{a,b}E_a\wedge E_b\wedge dl(E_a,E_b))\cdot\Psi\\ &-\sum_{a,b}i(E_a)(E_b\wedge dl(E_a,E_b))\cdot\Psi.\end{aligned} \tag{5.13}$$

The first term of the right-hand side of (5.13) is zero since l is symmetric, and the last term is the Clifford multiplication of Ψ with a vector field, which gives an imaginary function when taking its scalar product with Ψ. Thus we have

$$-\frac{1}{2}\sigma^{\nabla}|\Psi|^2=-2(\text{tr }l)^2|\Psi|^2+2|l|^2|\Psi|^2.$$

Hence the proof of the second equation in (5.8) is completed. ■

Let Ψ_1 be the eigenspinor corresponding to the eigenvalue $\lambda_1^2=\frac{1}{4}\inf_M(K^\sigma+4|l_{\Psi_1}|^2$. From (4.9), we have

$$\overset{l_{\Psi_1}}{\nabla}\Psi_1=0,\quad K^\sigma=\text{constant},\quad |l_{\Psi_1}|=\text{constant}. \tag{5.14}$$

From Proposition 5.1, we know that $|\Psi_1|$ is constant and

$$\lambda_1^2=\frac{1}{4}\sigma^{\nabla}+|l_{\Psi_1}|^2. \tag{5.15}$$

From (5.15), the transversal scalar curvature σ^{∇} is constant and we have

$$\inf|\kappa|^2=0. \tag{5.16}$$

Since σ^∇ and $K^\sigma = \sigma^\nabla + |\kappa|^2$ are constant, $|\kappa|$ is constant and then $|\kappa| = \inf_M |\kappa| = 0$. This implies that $\mathcal{F}$ is minimal. Hence we have the following theorem.

Theorem 5.2. *Under the same assumption as in Theorem 4.1, if there exists an eigenspinor $\Psi(\neq 0)$ of the basic Dirac operator D_b for the eigenvalue $\lambda^2 = \frac{1}{4}\inf_M(K^\sigma + 4|l_\Psi|^2)$, then $|\Psi|$ is constant and $\mathcal{F}$ is minimal with the constant transversal scalar curvature $\sigma^\nabla = 4(\lambda^2 - |l_\Psi|^2)$.*

Next, we study the limiting case of (4.31). Similarly, we have

$$Ric_{\bar\nabla}^{l_{\bar\Phi}}(X\otimes\bar\Phi) = -\frac{1}{2}\rho^{\bar\nabla}(X)\bar\cdot\bar\Phi + \sum(\bar E_a\wedge dl_{\bar\Phi}(X,\bar E_a))\bar\cdot\bar\Phi - 2l_{\bar\Phi}^2(X)\bar\cdot\bar\Phi + 2(\mathrm{tr}l_{\bar\Phi})l_{\bar\Phi}(X)\bar\cdot\bar\Phi - \{X(\mathrm{tr}l_{\bar\Phi}) - (div_{\bar\nabla}l_{\bar\Phi})(X)\}\bar\cdot\bar\Phi. \tag{5.17}$$

From (5.17), we have the following proposition.

Proposition 5.3. *If M admits a non-zero spinor Ψ with $\overset{l_{\bar\Phi}}{\bar\nabla}\bar\Phi = 0$, where $\Phi = e^{-\frac{q-1}{2}u}\Psi$, then $|\Phi|$ is constant and*

$$\nabla_X\Psi = \tfrac{1}{2}\pi(X)\cdot grad_\nabla(u)\cdot\Psi + \tfrac{q}{2}X(u)\Psi - l_\Psi(\pi(X))\cdot\Psi,\quad X\in TM \tag{5.18}$$

$$X(\mathrm{tr}\ l_{\bar\Phi}) = (\mathrm{div}_{\bar\nabla}l_{\bar\Phi})(X),\quad (\mathrm{tr}\ l_{\bar\Phi})^2 = \tfrac{1}{4}\sigma^{\bar\nabla} + |l_{\bar\Phi}|^2. \tag{5.19}$$

Proof. Since $\overset{l_{\bar\Phi}}{\bar\nabla}$ is metrical, $|\Phi|$ is constant. Moreover, $\overset{l_{\bar\Phi}}{\bar\nabla}\bar\Phi = 0$ is equivalent to

$$\bar\nabla_X\bar\Phi + l_{\bar\Phi}(\pi(X))\bar\cdot\bar\Phi = 0. \tag{5.20}$$

From Proposition 3.2 and (5.20), we have that for $\Phi = e^{-\frac{q-1}{2}u}\Psi$

$$\nabla_X\Phi = \frac{1}{2}\pi(X)\cdot grad_\nabla(u)\cdot\Phi + \frac{1}{2}X(u)\Phi - l_\Psi(\pi(X))\cdot\Phi, \tag{5.21}$$

which gives (5.18). The proof of (5.19) is similar to the one in Proposition 5.1. ■

By a direct calculation together with (3.10), we obtain the following lemma.

Lemma 5.4. *For any vector field $X\in\Gamma Q$ and any isomorphism l, we have*

$$(div_{\bar\nabla}l)(X) = (div_\nabla l)(X) + q\ g_Q(l(X), grad_\nabla(u)) - X(u)\ \mathrm{tr}\ l, \tag{5.22}$$

where $\bar\nabla$ *is a Levi-Civita connection with respect to* $\bar g_Q = e^{2u} g_Q$.

On the other hand, we have that for any $\Phi = e^{-\frac{q-1}{2}u}\Psi$ and any $X \in \Gamma Q$

$$(div_\nabla l_{\bar\Phi})(X) = e^{-u}\{(div_\nabla l_\Psi)(X) - g_Q(l_\Psi(X), grad_\nabla(u)). \tag{5.23}$$

From (5.22) and (5.23), we have

$$\begin{aligned}&(div_{\bar\nabla} l_{\bar\Phi})(X)\\ &= e^{-u}\{(div_\nabla l_\Psi)(X) + (q-1)g_Q(l_\Psi(X), grad_\nabla(u)) - X(u)\ \mathrm{tr} l_\Psi\}.\end{aligned} \tag{5.24}$$

Comparing with (5.19) and (5.24), we have the following proposition.

Corollary 5.5. *If* M *admits a non-zero spinor* Ψ *with* $\overset{l_{\bar\Phi}}{\bar\nabla}\bar\Phi = 0$, *where* $\Phi = e^{-\frac{q-1}{2}u}\Psi$, *then for any* $X \in \Gamma Q$

$$X(tr\ l_\Psi) = (div_\nabla l_\Psi)(X) + (q-1)g_Q(l_\Psi(X), grad_\nabla(u)).$$

Let $D_b\Psi = \lambda\Psi$ with $\lambda^2 = \frac{1}{4}(\mu_1 + \inf_M|\kappa|^2) + \inf_M|l_\Psi|^2$. From (4.20) and (4.30), we have that for $\Phi = e^{-\frac{q-1}{2}u}\Psi$

$$\overset{l_{\bar\Phi}}{\bar\nabla}\bar\Phi = 0, \quad |\kappa| = \text{constant}, \quad |l_\Psi| = \text{constant}. \tag{5.25}$$

Since $|l_\Psi|$ is constant, we have from (4.15) and (5.19) that

$$\mu_1 + \inf_M|\kappa|^2 = e^{2u}\sigma^{\bar\nabla}. \tag{5.26}$$

From Lemma 4.6, we have that for $u \in \mathcal{K}_u$

$$\mu_1 + \inf_M|\kappa|^2 = \sigma^\nabla + 2(q-1)\Delta_B u + (q-1)(2-q)|grad_\nabla(u)|^2. \tag{5.27}$$

Let u be an eigenfunction of the basic Yamabe operator Y_b. Then from (4.30) we get

$$\inf_M|\kappa|^2 = 0. \tag{5.28}$$

Since $|\kappa|$ is constant, we have that $|\kappa| = 0$, i.e. $\mathcal{F}$ is minimal. Hence we have the following theorem.

Theorem 5.6. *Let* $(M, \tilde g_M, \mathcal{F})$ *be a compact Riemannian manifold with a transverse spin foliation* $\mathcal{F}$ *of codimension* $q \geq 3$ *and a bundle-like metric* $\tilde g_M$. *Assume that an eigenvalue* λ *of* D_b *corresponding to the eigenspinor* Ψ *satisfies*

$$\lambda^2 = \frac{1}{4}(\mu_1 + \inf_M|\kappa|^2) + \inf_M|l_\Psi|^2.$$

Then $|l_\Psi|$ *is constant and the foliation* $\mathcal{F}$ *is minimal. Moreover*

$$(div_\nabla l_\Psi)(X) = (1-q)g_Q(l_\Psi(X), grad_\nabla(u)) \tag{5.29}$$

for any $X \in \Gamma Q$.

Proof. The equation (5.29) is trivial from corollary 5.5. ■

REFERENCES

1. J. A. Alvarez López, The basic component of the mean curvature of Riemannian foliations, *Ann. Global Anal. Geom.*, **10** (1992), 179-194.
2. C. Bär, Real Killing spinors and holonomy, *Comm. Math. Phys.*, **154** (1993), 509-521
3. H. Baum, Th. Friedrich, R. Grunewald and I. Kath, *Twistors and Killing spinors on Riemannian manifolds,* Teubner, Leipzig/Stuttgart, 1991.
4. J. Brüning and F. W. Kamber, Vanishing theorems and index formulas for transversal Dirac operators, A. M. S. Meeting 845, *Special Session on operator theory and applications to Geometry*, Lawrence, K. A.; A. M. S. Abstracts, October 1988.
5. D. Dom´nguez, A tenseness theorem for Riemannian foliations, *C. R. Acad. Sci. Sér. I*, **320** (1995), 1331-1335.
6. Th. Friedrich, Der erste Eigenwert des Dirac operators einer kompakten,Riemannschen Mannigfaltigkeit nichtnegative skalarkrümmung, *Math. Nachr.*, **97** (1980), 117-146.
7. J. F. Glazebrook and F. W. Kamber, Transversal Dirac families in Riemannian foliations, *Comm. Math. Phys.*, **140** (1991), 217-240.
8. O. Hijazi, A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors, *Comm. Math. Phys.*, **104** (1986), 151-162.
9. O. Hijazi, Lower bounds for the eigenvalues of the Dirac operator, *J. Geom. Phys.*, **16** (1995), 27-38.
10. S. D. Jung, The first eigenvalue of the transversal Dirac operator, *J. Geom. Phys.*, **39** (2001), 253-264.
11. S. D. Jung and T. H. Kang, Lower bounds for the eigenvalue of the transversal Dirac operator on a Kähler foliation, *J. Geom. Phys.*, **45** (2003), 75-90.
12. S. D. Jung, B. H. Kim and J. S. Pak, Lower bounds for the eigenvalues of the basic Dirac operator on a Riemannian foliation, *J. Geom. Phys.*, **51** (2004), 166-182.
13. F. W. Kamber and Ph. Tondeur, *Foliated Bundles and Characteristic Classes,* Lecture Notes in Math. 493, Springer-Verlag, Berlin, 1975.
14. F. W. Kamber and Ph. Tondeur, Harmonic foliations *Proc. National Science Foundation Conference on Harmonic Maps*,, Tulane, Dec. 1980, Lecture Notes in Math. 949, Springer-Verlag, New York, 1982, pp. 87-121.

15. K.-D. Kirchberg, An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature, *Ann. Glob. Anal. Geom.*, **4** (1986), 291-326.

16. K.-D. Kirchberg, The first eigenvalue of the Dirac operator on Kähler manifolds, *J. Geom. Phys.*, **7** (1990), 449-468.

17. W. Kramer, U. Semmelmann and G. Weingart, Eigenvalue estimates for Dirac operator on quaternionic Kähler manifolds, *Math. Z.*, **230** (1999), 727-751.

18. H. B. Lawson, Jr. and M. L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, Princeton, New Jersey, 1989.

19. P. March, M. Min-Oo and E. A. Ruh, Mean curvature of Riemannian foliations, *Canad. Math. Bull.*, **39** (1996), 95-105.

20. A. Mason, An application of stochastic flows to Riemannian foliations, *Houston J. Math.*, **26** (2000), 481-515.

21. E. Park and K. Richardson, The basic Laplacian of a Riemannian foliation, *Amer. J. Math.*, **118** (1996), 1249-1275.

22. Ph. Tondeur, *Foliations on Riemannian Manifolds*, Springer-Verlag, New York, 1988.

23. S. Yorozu and T. Tanemura Green's theorem on a foliated Riemannian manifold and its applications, *Acta Math. Hungar.*, **56** (1990), 239-245.

Seoung Dal Jung and Yeon Soon Ko
Department of Mathematics,
Cheju National University,
Jeju 690-756, Korea
E-mail: sdjung@cheju.ac.kr
E-mail: mirewon@hanmail.net