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# THE $q$-LAGRANGE POLYNOMIALS IN SEVERAL VARIABLES

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**Abstract.** Recently, Chan, Chyan and Srivastava [1] introduced and systematically investigated the Lagrange polynomials in several variables. The main object of this paper is to study a basic (or $q-$) analogue of the Chan-Chyan-Srivastava multivariable polynomials, which we define here. Several results involving generating functions are presented in this paper for the multivariable $q-$Lagrange polynomials.

## 1. INTRODUCTION

The classical Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ are known to occur in certain problems in statistics (see, for details, [6, p. 267]; see also [6, p. 442, Eq. 8. 5 (17); p. 452, Problem 25] for its connections with the classical Jacobi polynomials). Recently, Chan, Chyan and Srivastava [1] introduced and systematically investigated various interesting properties of the following multivariable Lagrange polynomials defined by them:

$$g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,...,x_r) = \sum_{k_1+\ldots+k_r=n} (\alpha_1)_{k_1} \ldots (\alpha_r)_{k_r} \frac{x_1^{k_1}}{k_1!} \ldots \frac{x_r^{k_r}}{k_r!}, \tag{1.1}$$

which, for $r = 2$, would yield the classical case.

In this paper we first introduce a basic (or $q-$) analogue of the Chan-Chyan-Srivastava multivariable polynomials defined by $(1.1)$. To achieve this, we need some definitions and notations from the $q-$analysis. (see, for example, Jain and Srivastava [3, pp. 413-414]).

For a real or complex number $q$ $(|q| < 1)$, the number $(\lambda; q)_\mu$ is given by

$$(\lambda; q)_\mu := \frac{(\lambda; q)_\infty}{(\lambda q^\mu; q)_\infty}, \tag{1.2}$$

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where

$$(\lambda;q)_\infty := \prod_{k=0}^{\infty}(1-\lambda q^k),$$

and $\lambda$, $\mu$ are arbitrary parameters so that

$$(\lambda;q)_n := \begin{cases} 1 & , \text{ if } \quad n=0 \\ (1-\lambda)(1-\lambda q)...(1-\lambda q^{n-1}) & , \text{ if } \quad n=1,2,3,... \end{cases} \tag{1.3}$$

(see, for instance, [3, pp. 413-414]).

## 2. Construction of a Multivariable $q-$Lagrange Polynomials

In this section we construct a basic (or $q-$) analogue of the Chan-Chyan-Srivastava multivariable polynomials $g_{n,q}^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)$ given by (1.1), generated by

$$\prod_{i=1}^{r}\frac{1}{(x_i t;q)_{\alpha_i}} = \sum_{n=0}^{\infty} g_{n,q}^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)t^n\,, \tag{2.1}$$

where $|t| < \min\left\{|x_1|^{-1},...,|x_r|^{-1}\right\}$ .

Using the well-known $q-$binomial theorem (see, for instance, [5, p. 241-248], [3, p. 416]) and $(1.2)$, we get

$$\frac{1}{(x_i t;q)_{\alpha_i}} = \sum_{k=0}^{\infty}\frac{(q^{\alpha_i},q)_k}{(q,q)_k}(x_i t)^k; \qquad i=1,...,r.$$

Now, applying the well-known equality for double series (see [4]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k),$$

we conclude that

$$\prod_{i=1}^{r}\frac{1}{(x_i t;q)_{\alpha_i}} = \sum_{n=0}^{\infty}\left\{\sum_{k_1+...+k_r=n}(q^{\alpha_{k_1}},q)_{k_1}\cdots (q^{\alpha_{k_r}},q)_{k_r}\frac{x_1^{k_1}}{(q,q)_{k_1}}\cdots\frac{x_r^{k_r}}{(q,q)_{k_r}}\right\}t^n. \tag{2.2}$$

By $(2.1)$ and $(2.2)$ a basic (or $q-$) analogue of the Chan-Chyan-Srivastava multivariable polynomials are given explicitly by

$$g_{n,q}^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r) = \sum_{k_1+...+k_r=n}(q^{\alpha_1},q)_{k_1}\cdots(q^{\alpha_{k_r}},q)_{k_r}\frac{x_1^{k_1}}{(q,q)_{k_1}}\cdots\frac{x_r^{k_r}}{(q,q)_{k_r}}. \tag{2.3}$$

We note that, by setting $r = 2$, (2.1) immediately reduces to a basic (or $q-$) analogue of the classical Lagrange polynomials $g_{n,q}^{(\alpha,\beta)}(x,y)$ generated by

$$\frac{1}{(xt;q)_\alpha (yt;q)_\beta} = \sum_{n=0}^{\infty} g_{n,q}^{(\alpha,\beta)}(x,y)t^n, \tag{2.4}$$

where $|t| < \min\left\{|x|^{-1}, |y|^{-1}\right\}$.

## 3. Bilinear and Bilateral Generating Functions

In this section we derive several families of bilinear and bilateral generating functions for the basic (or $q-$) analogue of the Chan-Chyan-Srivastava multivariable polynomials given by $(2.3)$.

**Theorem 3.1.** *Corresponding to an identically non-vanishing function* $\Omega_{\mu}(y_1, ..., y_s)$ *of complex variables* $y_1, ..., y_s$ $(s \in \mathbb{N})$ *and of complex order* $\mu$*, let*

$$\Lambda_{\mu,n}(y_1, ..., y_s; z) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+nk}(y_1, ..., y_s) z^k$$

$$(a_k \neq 0\,, \quad n \in \mathbb{N})$$

*and*

$$\begin{aligned} &{}_q\Theta_{l,p}^{\mu,n}(x_1, ..., x_r; y_1, ..., y_s; z) : \\ &= \sum_{k=0}^{[l/p]} a_k g_{l-pk,q}^{(\alpha_1,...,\alpha_r)}(x_1, ..., x_r)\Omega_{\mu+nk}(y_1, ..., y_s) z^k, \end{aligned} \tag{3.1}$$

*where* $[\lambda]$ *denotes the integer part of* $\lambda \in \mathbb{R}$*. Then*

$$\sum_{l=0}^{\infty} {}_q\Theta_{l,p}^{\mu,n}(x_1, ..., x_r; y_1, ..., y_s; \frac{\eta}{t^p})t^l = \prod_{j=1}^{r} \frac{1}{(x_j t;q)_{\alpha_j}} \Lambda_{\mu,n}(y_1, ..., y_s; \eta)\,, \tag{3.2}$$

*provided that each member of* $(3.2)$ *exists.*

*Proof.* Take $z \to \dfrac{\eta}{t^p}$ in $(3.1)$ and sum from $l = 0$ to $\infty$ and also multiply by $t^l$. Thus we have

$$\begin{aligned} S \quad : \quad &= \sum_{l=0}^{\infty} {}_q\Theta_{l,p}^{\mu,n}(x_1, ..., x_r; y_1, ..., y_s; \frac{\eta}{t^p})t^l \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{[l/p]} a_k g_{l-pk,q}^{(\alpha_1,...,\alpha_r)}(x_1, ..., x_r)\Omega_{\mu+nk}(y_1, ..., y_s) \frac{\eta^k}{t^{pk}} t^l\,. \end{aligned}$$

Replacing $l$ by $l+pk$, we can write

$$\begin{aligned} S &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} a_k\, g_{l,q}^{(\alpha_1,\ldots,\alpha_r)}(x_1,...,x_r)\Omega_{\mu+nk}(y_1,...,y_s)\eta^k t^l \\ &= \sum_{l=0}^{\infty} g_{l,q}^{(\alpha_1,\ldots,\alpha_r)}(x_1,...,x_r)t^l \sum_{k=0}^{\infty} a_k\Omega_{\mu+nk}(y_1,...,y_s)\eta^k \\ &= \prod_{j=1}^{r}\frac{1}{(x_jt;q)_{\alpha_j}}\Lambda_{\mu,n}(y_1,...,y_s;\eta), \end{aligned}$$

which completes the proof. ■

To derive another family of bilateral generating functions for the polynomials $(2.3)$, we need the following lemma.

**Lemma 3.2.** *For the multivariable* $q-$*Lagrange polynomials, the following addition formula holds:*

$$\begin{aligned} & g_{n,q}^{(\alpha_1+\beta_1,\ldots,\alpha_r+\beta_r)}(x_1,...,x_r) \\ & = \sum_{k=0}^{n} g_{n-k,q}^{(\alpha_1,\ldots,\alpha_r)}(x_1,...,x_r) g_{k,q}^{(\beta_1,\ldots,\beta_r)}(x_1q^{\alpha_1},...,x_rq^{\alpha_r}). \end{aligned} \tag{3.3}$$

*Proof.* In $(2.1)$, by taking $\alpha_i \to \alpha_i+\beta_i$, we may write

$$\begin{aligned} \sum_{n=0}^{\infty} g_{n,q}^{(\alpha_1+\beta_1,\ldots,\alpha_r+\beta_r)}(x_1,...,x_r)t^n &= \prod_{j=1}^{r}\frac{1}{(x_jt;q)_{\alpha_j+\beta_j}} \\ &= \prod_{j=1}^{r}\frac{1}{(x_jt;q)_{\alpha_j}}\prod_{j=1}^{r}\frac{1}{(x_jtq^{\alpha_j};q)_{\beta_j}} \\ &= \sum_{n=0}^{\infty} g_{n,q}^{(\alpha_1,\ldots,\alpha_r)}(x_1,...,x_r)t^n \\ &\quad \times\sum_{k=0}^{\infty} g_{k,q}^{(\beta_1,\ldots,\beta_r)}(x_1q^{\alpha_1},...,x_rq^{\alpha_r})t^k \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} g_{n-k,q}^{(\alpha_1,\ldots,\alpha_r)}(x_1,...,x_r) \\ &\quad \times g_{k,q}^{(\beta_1,\ldots,\beta_r)}(x_1q^{\alpha_1},...,x_rq^{\alpha_r})\, t^n, \end{aligned}$$

which yields the addition formula $(3.3)$. ■

Next we derive the following result.

**Theorem 3.3.** *For a non-vanishing function* $\Omega_\mu(y_1, ..., y_s)$ *of complex variables* $y_1, ..., y_s$ $(s \in \mathbb{N})$ *and for* $p \in \mathbb{N}$, $\alpha = (\alpha_1, ..., \alpha_r)$, $\beta = (\beta_1, ..., \beta_r)$ *let*

$$\Lambda^{p,n}_{\mu,\alpha,\beta}(x_1, ..., x_r; y_1, ..., y_s; z) := \sum_{k=0}^{[n/p]} a_k g^{(\alpha_1+\beta_1,...,\alpha_r+\beta_r)}_{n-pk,q}(x_1, ..., x_r) \times\Omega_\mu(y_1, ..., y_s)z^k, \tag{3.4}$$

*where* $a_k \neq 0$ *and* $n \in \mathbb{N}$. *Then*

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l\, g^{(\alpha_1,...,\alpha_r)}_{n-k,q}(x_1, ..., x_r) g^{(\beta_1,...,\beta_r)}_{k-pl,q}(x_1q^{\alpha_1}, ..., x_rq^{\alpha_r})\Omega_\mu(y_1, ..., y_s)z^l = \Lambda^{p,n}_{\mu,\alpha,\beta}(x_1, ..., x_r; y_1, ..., y_s; z), \tag{3.5}$$

*provided that each member of* $(3.5)$ *exists.*

*Proof.* Let $T$ denote the left-hand side of the equality $(3.5)$. Then, by some simple calculations, we get

$$\begin{aligned} T :&= \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl} a_l\, g^{(\alpha_1,...,\alpha_r)}_{n-pl-k,q}(x_1, ..., x_r) g^{(\beta_1,...,\beta_r)}_{k,q}(x_1q^{\alpha_1}, ..., x_rq^{\alpha_r}) \\ &\quad \times\Omega_\mu(y_1, ..., y_s)z^l \\ &= \sum_{l=0}^{[n/p]} a_l\, \Omega_\mu(y_1, ..., y_s)z^l \\ &\quad \times \sum_{k=0}^{n-pl} g^{(\alpha_1,...,\alpha_r)}_{n-pl-k,q}(x_1, ..., x_r) g^{(\beta_1,...,\beta_r)}_{k,q}(x_1q^{\alpha_1}, ..., x_rq^{\alpha_r}). \end{aligned}$$

Now, using Lemma $3.2$ and the equality $(3.4)$, we find that

$$\begin{aligned} T &= \sum_{l=0}^{[n/p]} a_l\, \Omega_\mu(y_1, ..., y_s)z^l g^{(\alpha_1+\beta_1,...,\alpha_r+\beta_r)}_{n-pl,q}(x_1, ..., x_r) \\ &= \Lambda^{p,n}_{\mu,\alpha,\beta}(x_1, ..., x_r; y_1, ..., y_s; z), \end{aligned}$$

whence the result. ∎

For every suitable choice of the coefficients $a_k\,(k \in \mathbb{N}_0)$ , if the multivariable function $\Omega_\mu(y_1,...,y_s)$ $(s = 2,3,...)$ is expressed as an appropriate product of several simpler functions, the assertion of the above theorems can be applied in order to derive various families of multilinear and multilateral generating functions for the basic (or $q-$) analogue of the Chan-Chyan-Srivastava multivariable polynomials defined by $(2.1)$.

Finally, for the basic (or $q-$) analogue of the Chan-Chyan-Srivastava multivariable polynomials, we get

$$g_{n,q}^{(\alpha_1,...,\alpha_r)}(x,...,x) = \prod_{j=1}^{r} \frac{(q^{\alpha_j};q)_n}{(q;q)_n} x^n \qquad (n \in \mathbb{N}_0). \tag{3.6}$$

Indeed we have

$$\begin{aligned}\sum_{n=0}^{\infty} g_{n,q}^{(\alpha_1,...,\alpha_r)}(x,...,x)t^n &= \prod_{j=1}^{r} \frac{1}{(xt;q)_{\alpha_j}} \\ &= \prod_{j=1}^{r} \sum_{n=0}^{\infty} \frac{(q^{\alpha_j};q)_n}{(q;q)_n}(xt)^n \\ &= \sum_{n=0}^{\infty} \left[ \prod_{j=1}^{r} \frac{(q^{\alpha_j};q)_n}{(q;q)_n} x^n \right] t^n.\end{aligned}$$

This implies $(3.6)$ at once.

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