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TOWARD THE POINCARÉ CONJECTURE

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1. n-MANIFOLDS

Let n be a positive integer. An n-manifold M is a Hausdorff topological space with a countable base of open sets such that M is locally the Euclidean space $\mathbb{R}^n$, that is, for each $x \in M$ there exists an open neighborhood U of x and a homeomorphism $U \overset{\varphi}{\to} V$ from U onto an open subset V of $\mathbb{R}^n$. A compact n-manifold is an n-manifold which is compact as a topological space. It is clear that any n-manifold is locally path-connected and so is a path-connected space if it is a connected space. It is also clear that a compact n-manifold has only a finite number of path components, each of these being a compact path-connected n-manifold.

$\mathbb{R}^n$ and any of its non-empty open subsets are examples of n-manifolds. These are not compact manifolds. Let

$$S^n = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} |\ \|x\| = \sqrt{x_1^2 + \cdots + x_{n+1}^2} = r > 0\},$$

called the n-sphere (of radius r). S^n is a path-connected compact n-manifold (note that $n \geq 1$). It is easy to see that any two n-spheres of different radii are homeomorphic.

Theorem 1.1. *Any path connected compact* 1*-manifold is homeomorphic to* S^1.

We refer to Milnor [10] for a proof of this fact.

All path connected compact 2-manifolds are also known (up to homeomorphisms). Call such a 2-manifold a surface. In Section 2 we describe these surfaces from both topological viewpoint and geometric viewpoint. In section 3 we begin with the statement of the Poincaré Conjecture which is about the topology and geometry of 3-manifolds related to the 3-sphere S^3. We will also describe some geometric aspects of compact 3-manifolds including Thurston Geometrization Conjecture on such manifolds of which the Poincaré Conjecture is a special case, and finally make some comments on a recent progress about these conjectures.

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2. SURFACES

We already know S^2 (say, of radius 1) is a surface. Let $P^2 = S^2/x \sim -x$, the space obtained from S^2 by identifying each point $x = (x_1, x_2, x_3) \in S^2$ with its antipodal point $-x = (-x_1, -x_2, -x_3)$. Consider the closed disk $D^2 = \{y = (y_1, y_2) | \ \|y\| = \sqrt{y_1^2 + y_2^2} \leq 1\}$ in $\mathbb{R}^2$ with boundary $\partial D^2 = S^1$. Up to a homeomorphism, P^2 can also be thought as the space obtained from D^2 by identifying each point y on the boundary S^1 with the antipodal point $-y$ as shown in Figure 1. P^2 is known as the 2-dimensional real projective space.

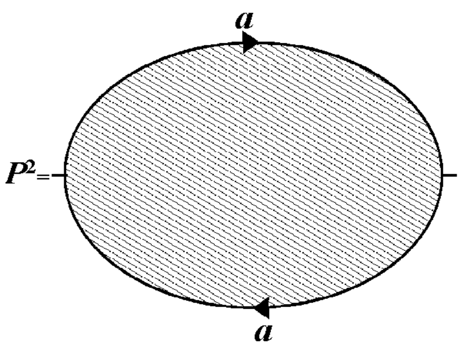

Fig. 1.

There are many ways to show S^2 and P^2 are different surfaces, that is, they are not homeomorphic. One of these, which perhaps is the most intuitive way, is the following. Puncture an open disk D_1 from S^2 and puncture an open disk D_2 (say, an open disk of radius $\frac{1}{2}$ with center the origin in Figure 1) from P^2. $S^2 - D_1$ is a closed disk while $P^2 - D_2$ is the Mӧbius band, and they are not homeomorphic since $S^2 - D_1$ is orientable while $P^2 - D_2$ is not. So S^2 and P^2 are not homeomorphic. We will give another way to distinguish P^2 from S^2 later.

The projective space P^2 can also be thought as the orbit space $S^2/\mathbb{Z}_2$ of the group action $\mathbb{Z}_2 = \{1, T\} \times S^2 \xrightarrow{\varphi} S^2$ given by $\varphi(1, x) = x$ and $\varphi(T, x) = -x$. In general, given a finite group G and an n-manifold M. We say G acts freely on M if there is a continuous function $G \times M \xrightarrow{\varphi} M$ (regard $G \times M$ as the disjoint union $\underbrace{M \cup M \cup \cdots \cup M}_{k}$ if $|G| = k$), and we write $\varphi(g, x) = gx$, such that

(1) $1x = x$ for all $x \in M$ where 1 is the identity of G,

(2) $(gh)x = g(hx)$ for all $g, h \in G$, $x \in M$,

(3) $gx = x$ for some $x \in M \Rightarrow g = 1$.

"freely" comes from condition (3) which says that if $g \neq 1$, then g moves every point of M. The orbit space M/G of such a group action is the quotient space of M consisting of all equivalence classes defined by the equivalence relation "$\sim$" on M by $x \sim y$ iff $gx = y$ for some $g \in G$. Then M/G is an n-manifold. The following is easy to prove from Linear algebra.

Proposition 2.1. *Let $G \times S^2 \xrightarrow{\varphi} S^2$ be a finite group free action such that $\langle gx, gy\rangle = \langle x, y\rangle$ for all $g \in G$ and $x, y \in S^2$ where $\langle , \rangle$ denotes the usual inner product on $\mathbb{R}^3$. Then either $|G| = 1$ or $|G| = 2$ and, in the latter case, $G \times S^2 \xrightarrow{\varphi} S^2$ is the group action $\mathbb{Z}_2 \times S^2 \xrightarrow{\varphi} S^2$ described above.*

One can actually show from Proposition 2.1 that P^2 is the only surface which is a quotient space of S^2 (besides S^2 itself of course).

The next simplest surface one can think of is the product space $T = S^1 \times S^1$, called the 2-dimensional torus (in general, the m-dimensional torus for $m \geq 2$ is $\underbrace{S^1 \times \cdots \times S^1}_{m}$). It can be embedded in $\mathbb{R}^3$ with a shape which looks like a donut or a tire as shown in Figure 2 below.

S^1 can be thought as the space obtained from the closed interval $[0, 1]$ by identifying the end points 0 and 1 to a point. Then $S^1 \times S^1$ is the space obtained from a square in $\mathbb{R}^2$ by identifying the 4 edges in pairs as shown in Figure 3. We use the symbol $aba^{-1}b^{-1}$ to denote these pairings of the 4 edges of the square. By this representation of the torus T we denote T as $T = S(aba^{-1}b^{-1})$.

It turns out that, from the surfaces S^2, P^2 and T above, one can construct all the surfaces (up to homeomorphisms) by an operation called the "connected sum" which is described as follows.

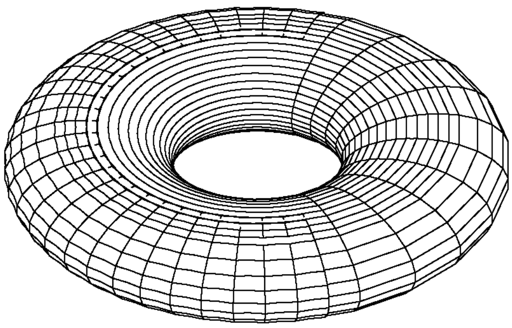

Fig. 2.

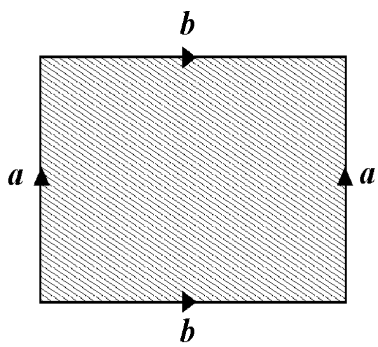

Fig. 3.

Given two path-connected n-manifolds M_1 and M_2. Let

$$D^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | \ \|x\| = \sqrt{x_1^2 + \cdots + x_n^2} \leq 1\},$$

the n-dimensional closed unit disk in $\mathbb{R}^n$ with boundary

$$\partial D^n = S^{n-1} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | \ \|x\| = \sqrt{x_1^2 + \cdots + x_n^2} = 1\},$$

the $(n-1)$-sphere of radius 1. Take any embeddings $D^n \xrightarrow{\varphi_i} M_i$, $i = 1, 2$. (From the definition of an n-manifold, such embeddings exist.) Let $\widetilde{D}^n = D^n - S^{n-1}$, the open unit disk in $\mathbb{R}^n$. Let $\overline{M}_i = M_i - \varphi_i(D^n)$ and let $\widetilde{M}_i = M_i - \varphi_i(\widetilde{D}^n)$, $i = 1, 2$. Clearly, $\widetilde{M}_i = \overline{M}_i \cup \varphi_i(S^{n-1})$ which is a disjoint union. Note that $\varphi_i(S^{n-1})$ is homeomorphic to S^{n-1} via $S^{n-1} \xrightarrow{\varphi_i} \varphi_i(S^{n-1})$, $i = 1, 2$. Consider the homeomorphism $\varphi_1(S^{n-1}) \xrightarrow{\varphi_2\varphi_1^{-1}} \varphi_2(S^{n-1})$. Let $M_1 \sharp M_2$ denote the quotient space of the disjoint union $\widetilde{M}_1 \cup \widetilde{M}_2$ by identifying each point $x \in \varphi_1(S^{n-1}) \subset \widetilde{M}_1$ with the corresponding point $\varphi_2\varphi_1^{-1}(x) \in \varphi_2(S^{n-1}) \subset \widetilde{M}_2$. It is not difficult to show the following.

Proposition 2.2.

(1) *$M_1 \sharp M_2$ is also a path-connected n-manifold and the topological type of $M_1 \sharp M_2$ is independent of the chosen embeddings $D^n \xrightarrow{\phi_i} M_i$, $i = 1, 2$.*

(2) *The operation "$\sharp$" is commutative and associative, that is, $M_1 \sharp M_2 \cong M_2 \sharp M_1$ and if M_3 is another path-connected n-manifold, then $(M_1 \sharp M_2) \sharp M_3 \cong M_1 \sharp (M_2 \sharp M_3)$.*

$M_1 \sharp M_2$ is called the connected sum of the manifolds M_1 and M_2. From Proposition 2.2 (2) we see if M_i, $i = 1, 2, \ldots, k$, are path-connected n-manifolds then, up to homeomorphisms, there is a well defined n-manifold $M_1 \sharp M_2 \sharp \cdots \sharp M_k$ which will be called the connected sum of the manifolds M_i, $i = 1, \ldots, k$. The following is easy to see.

Proposition 2.3. *$M \sharp S^n \cong S^n \sharp M \cong M$ for any path-connected n-manifold M.*

Applying the connected sum operation "$\sharp$" to the surfaces S^2, P^2 and T we obtain some new surfaces P^2_n and T_n described as follows.

First we note by Proposition 2.3 that it suffices to consider the connected sums $M_1 \sharp M_2 \sharp \cdots \sharp M_n$ with each M_i being either P^2 or T. Secondly, we refer to Massey [9] for a proof of the following result.

Proposition 2.4. $P^2 \sharp T \cong P^2 \sharp P^2 \sharp P^2$.

From this result we see the new surfaces obtained from P^2 and T by applying the connected sum operation in a finite number of times are of two types:

$$P_n^2 = \underbrace{P^2 \sharp \cdots \sharp P^2}_{n},\ n \geq 1 \quad \text{with} \quad P_1^2 = P^2$$

$$\text{and} \quad T_n = \underbrace{T \sharp \cdots \sharp T}_{n},\ n \geq 1 \quad \text{with} \quad T_1 = T.$$

In order to get a concrete picture of these new surfaces we recall from Figure 1 that $P_1^2 = P^2$ is the space

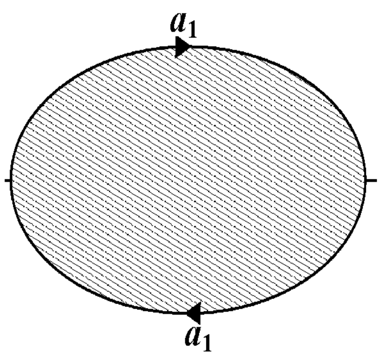

Fig. 4.

that is, the space obtained from the unit closed disk D^2 by identifying the two edges of equal length on the boundary $S^1 = \partial D^2$ as shown. We shall use the symbol $S(a_1 a_1 = a_1^2)$ to denote $P^2 = P_1^2$; so $P_1^2 = S(a_1^2)$. Similarly, from Figure 3, we see the torus $T_1 = T$ can be thought as the space

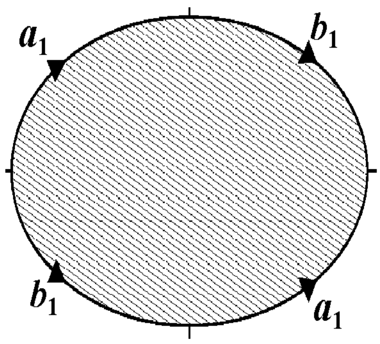

Fig. 5.

that is, the space obtained from D^2 by identifying in 2 pairs of the 4 edges of equal length on $S^1 = \partial D^2$ as shown, and we recall that, using this representation, T is thus denoted by $T = T_1 = S(a_1 b_1 a_1^{-1} b_1^{-1})$. It is easy to see, from the way that the connected sums are performed, that the new surfaces $P^2_n = P^2\sharp\cdots\sharp P^2$ and $T_n = T^2\sharp\cdots\sharp T^2$ for $n \geq 2$ have similar representations as quotient spaces of D^2. Namely, for P^2_n, if the boundary S^1 of D^2 is divided into $2n$ edges of equal length and identify these edges in n pairs according to the rule $a_1a_1a_2a_2\ldots a_na_n = a_1^2a_2^2\ldots a_n^2$ (generalizing the case $n = 1$) then the resulting quotient space of D^2 is P^2_n. Similarly, for T_n, if the boundary of D^2 is divided into $4n$ edges of equal length and identify these edges in $2n$ pairs according to the rule $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\ldots a_nb_na_n^{-1}b_n^{-1}$ (generalizing the case $n = 1$) then the resulting quotient space of D^2 is T_n. For example, P^2_2 and T_2 are the spaces

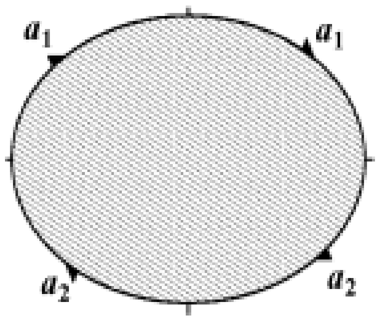

Fig. 6.

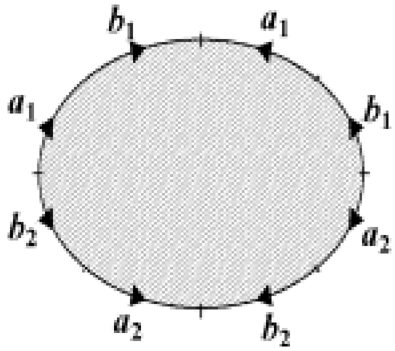

Fig. 7.

respectively. We still refer to [9] for proofs why the connected sums P^2_n and T_n (for $n \geq 2$) can be represented by such quotient spaces of the closed unit disk D^2. Again we use $S(a_1^2\ldots a_n^2)$ and $S(a_1b_1a_1^{-1}b_1^{-1}\ldots a_nb_na_n^{-1}b_n^{-1})$ to denote P^2_n and T_n respectively. We also refer to [9] for an intuitive proof of the following theorem which was already known at the end of the nineteenth century.

Theorem 2.5. *Up to homeomorphisms, the only surfaces are S^2, P^2, T and P^2_n, T_n for $n \geq 2$.*

Except for S^2 and P^2, the surfaces T, P^2_n and T_n for $n \geq 2$ are orbit spaces $\mathbb{R}^2/G$ for appropriate infinite discrete groups G respectively.

In order to describe these, we need to expand our notion of free group actions $G \times M \xrightarrow{\varphi} M$, discussed earlier, to that in which the groups G are infinite and discrete where M is a path-connected n-manifold. "discrete" means that G has the discrete topology, that is, every point of G is open. If G is infinite and discrete then, in addition to the three conditions (1), (2) and (3) on page 2 for the free group action $G \times M \xrightarrow{\varphi} M$ (and we still write gx to denote $\varphi(g, x)$), we also need the following condition in order for the orbit space M/G to be an n-manifold.

(4) For any two points x, y in M not in the same orbit (means $\nexists g \in G$ with $gx = y$) there are open neighborhoods U of x and V of y such that $U \cap GV = \varnothing$ where $GV = \{gv | g \in G, v \in V\}$.

In such a case, we say G acts properly discontinuously on M.

Now we describe T, P_n^2 and T_n for $n \geq 2$ as orbit spaces $\mathbb{R}^2/G$ as follows. First we describe this for $T = S^1 \times S^1$. Consider the set of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ as the infinite cyclic group in the usual way. $\mathbb{Z}$ acts on $\mathbb{R} = \mathbb{R}^1$ by $\varphi(m, x) = m + x$ for $m \in \mathbb{Z}$, $x \in \mathbb{R}^1$, that is, a translation action. It is easy to see that (4) is satisfied for this action and that the resulting orbit space $\mathbb{R}/\mathbb{Z}$ is S^1. From this one sees that $T = S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$ where $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R}^2$ by $\varphi((m, n), (x, y)) = (m + x, n + y)$ which is also a translation action. This translations group action can be seen more clearly from the representation $S(a_1 b_1 a_1^{-1} b_1^{-1})$ for T as follows. Recall that $T = S(a_1 b_1 a_1^{-1} b_1^{-1})$ is the space

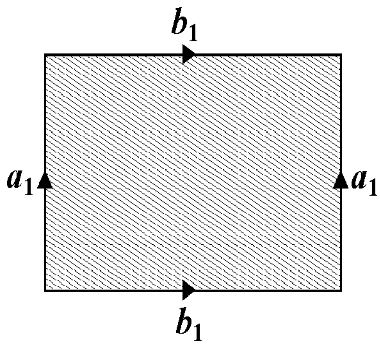

Fig. 8.

that is, the space obtained from a square in $\mathbb{R}^2$ with the 4 edges identified in 2 pairs as shown in the figure. We may take this square to be the square in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. In $\mathbb{R}^2$, a_1 corresponds to the translation $\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$ given by $A(x, y) = (x, y + 1)$ and b_1 corresponds to the translation $\mathbb{R}^2 \xrightarrow{B} \mathbb{R}^2$ given by $B(x, y) = (x + 1, y)$. These translations are isometries of the Euclidean space $\mathbb{R}^2$ relative to the usual metric on $\mathbb{R}^2$, that is, the usual inner product on $\mathbb{R}^2$. We use $ds^2 = dx_1^2 + dx_2^2$ to denote this standard metric. In what follows when we write ds^2 we will mean that it stands for the standard metric for $\mathbb{R}^2$. Let G be the group of isometries of $\mathbb{R}^2$ generated by A and B (relative to ds^2). Clearly, G is isomorphic to the free abelian group $\mathbb{Z} \times \mathbb{Z}$ on the generators A and B. So T can be considered as the orbit space $\mathbb{R}^2/G$ where $G = \mathbb{Z} \times \mathbb{Z}$ is the group of all the isometries $(x, y) \to (m + x, n + y)$ of $\mathbb{R}^2$. Finally from theoretic group theory, the group $G = \mathbb{Z} \times \mathbb{Z}$ can also be understood from the representation $T = S(a_1 b_1 a_1^{-1} b_1^{-1})$ as follows. Let F be the free group (not free abelian group) on two generators, also denoted by a_1 and b_1. Let N be the smallest normal subgroup of F containing the commutator $[a_1, b_1] = a_1b_1a_1^{-1}b_1^{-1} \in F$. Then it is easy to see that the quotient group F/N is isomorphic to the free abelian group $G = \mathbb{Z} \times \mathbb{Z}$. Thus T is the orbit space $\frac{\mathbb{R}^2}{F/N}$. This outcome of the group $G = \mathbb{Z} \times \mathbb{Z}$ from the representation $T = S(a_1b_1a_1^{-1}b_1^{-1})$ will be a model for treating the remaining surfaces P_n^2 and T_n for $n \geq 2$ as orbit spaces $\mathbb{R}^2/G$ for appropriate G that follow.

We will discuss the surface $P_2^2 = P^2 \sharp P^2$ and the surfaces P_n^2 for $n \geq 3$ and T_m for $m \geq 2$ separately because P_2^2 has a similar situation as that for the torus T above while P_n^2 for $n \geq 3$ and T_m for $m \geq 2$ do not. Just what "similar situation" is will be clear in a moment.

It can be shown (see [9]) that P_2^2 is homeomorphic to the Klein bottle K which is the space

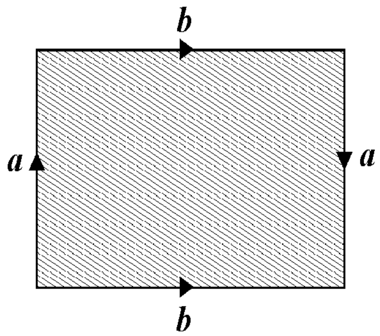

Fig. 9.

that is, the space obtained from a square in $\mathbb{R}^2$ by identifying the 4 edges in 2 pairs as shown in the figure. We may take this square to be the square in $\mathbb{R}^2$ with the 4 vertices $(\pm\frac{1}{2}, \pm\frac{1}{2})$. Then a corresponds to the translation $\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$ given by $A(x, y) = (x, y + 1)$. But now b corresponds to the map $\mathbb{R}^2 \xrightarrow{B} \mathbb{R}^2$ which is the composite

$$B : (x, y) \xrightarrow{B_1} (x + 1, y) \xrightarrow{R} (x + 1, -y)$$

with B_1 a translation and R a reflection. So A and B are also isometries of $\mathbb{R}^2$ relative to ds^2. It is easy to see that there is the relation $AB = BA^{-1}$, or equivalently, the relation $ABAB^{-1} = 1$ ($1 = id_{\mathbb{R}^2}$). Let G_2 be the group of isometries of $\mathbb{R}^2$ generated by A and B relative to ds^2; so G_2 is a group having two generators A and B subject only to the relation $ABAB^{-1} = 1$. Then, analogus to $T \cong \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$ above, P_2^2 is homeomorphic to the orbit space $\mathbb{R}^2/G_2$. Note that, from the representation $P_2^2 = K = S(abab^{-1})$ above, if we let F_2 be the free group on the generators a and b and consider the smallest normal subgroup N of F_2 containing the element $abab^{-1} \in F_2$ then the quotient group F_2/N is isomorphic to G_2. Thus F_2/N acts on $\mathbb{R}^2$ as a group of isometries (relative to ds^2) and $P_2^2 \cong \frac{\mathbb{R}^2}{F_2/N}$.

By "P_2^2 has a similar situation as that for T" that was mentioned earlier, we mean that both P_2^2 and T are orbit spaces $\mathbb{R}^2/G$ for appropriate groups G of isometries of $\mathbb{R}^2$ relative to ds^2.

We proceed to consider the surfaces

$$P_n^2 = S(a_1^2 a_2^2 \dots a_n^2) \text{ for } n \geq 3$$

and

$$T_m = S(a_1 b_1 a_1^{-1} b_1^{-1} \dots a_m b_m a_m^{-1} b_m) \text{ for } m \geq 2.$$

Let F_n (resp. $\overline{F}_m$) be the free group on n generators $a_1, \dots, a_n$ (resp. on $2m$ generators $a_1, \dots, a_m, b_1, \dots, b_m$). Let $N \subset F_n$ (resp. $\overline{N} \subset \overline{F}_m$) be the smallest normal subgroup containing $a_1^2 a_2^2 \dots a_n^2 \in F_n$ (resp. the smallest normal subgroup containing the product of commutators $[a_1, b_1][a_2, b_2] \dots [a_m, b_m] = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_m b_m a_m^{-1} b_m^{-1} \in \overline{F}_m$). Let $G_n = F_n/N$ and $\overline{G}_m = \overline{F}_m/\overline{N}$. So G_n is the group generated by n elements $a_1, a_2, \dots, a_n$ subject only to the relation $a_1^2 a_2^2 \dots a_n^2 = 1$ and $\overline{G}_m$ is the group generated by $2m$ elements $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m$ subject only to the relation $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_m b_m a_m^{-1} b_m^{-1} = 1$. We refer to the references [2] and [15] for what we are going to say about the topology and geometry of the surfaces P_n^2 for $n \geq 3$ and T_m for $m \geq 2$ via the groups $G_n = F_n/N$ and $\overline{G}_m = \overline{F}_m/\overline{N}$ respectively.

One can show that there are (free) group actions $G_n \times \mathbb{R}^2 \xrightarrow{\varphi_n} \mathbb{R}^2$ and $\overline{G}_m \times \mathbb{R}^2 \xrightarrow{\overline{\varphi}_m} \mathbb{R}^2$ such that $P_n^2 \cong \mathbb{R}^2/G_n$ and $T_m \cong \mathbb{R}^2/\overline{G}_m$. For $n \geq 3$ and $m \geq 2$ these groups G_n and $\overline{G}_m$ are not groups of isometries of $\mathbb{R}^2$ relative to the standard metric ds^2. The reason is the following. Let M be a surface obtained from a regular $2k$-gon Q in $\mathbb{R}^2$ by identifying the $2k$ edges of Q in k pairs such that the $2k$ vertices of Q are identified to a single point in M. If $k \geq 3$ then the angle sum of the polygon Q is $2k(\pi - \frac{2\pi}{2k}) = 2k\pi - 2\pi$ which is bigger than $2\pi = 360°$! Nevertheless, one can construct a new metric $\overline{ds^2}$ on $\mathbb{R}^2$, and this was invented in 19th century primarily by Gauss, Bolyai and Lobachevskii, such that

(*) at each point $x \in \mathbb{R}^2$ the metric $\overline{ds^2}$ is defined at the tangent space $T_x(\mathbb{R}^2) \cong \mathbb{R}^2$ with any geodesic triangle $\triangle_x$ around x having angle sum $S(\triangle_x)$ less that $180°$ and, when x moves away from the origin, $S(\triangle_x)$ tends to zero.

The new metric $\overline{ds^2}$ is described as follows. For each point $x \in \mathbb{R}^2$ the metric

$\overline{ds^2}$ on the tangent space $T_x(\mathbb{R}^2) = \mathbb{R}^2$ is given by

$$(**) \qquad \overline{ds^2} = \frac{4}{(1-r^2)^2}(dx_1^2 + dx_2^2) \quad \text{where } r = \frac{\|x\|}{1+\|x\|}.$$

$\mathbb{R}^2$ with this new metric $\overline{ds^2}$ is called the 2-dimensional hyperbolic space and is denoted by $\mathbb{H}^2$. Since $\mathbb{H}^2$ has the property in (*), any surface M obtained from a regular $2k$-gon Q in $\mathbb{R}^2$ as described above is embedded in $\mathbb{H}^2$ in a very nice way. In particular, this holds for the surfaces $P_n^2 \cong \mathbb{R}^2/G_n = \mathbb{H}^2/G_n$ for $n \geq 3$ and $T_m \cong \mathbb{R}^2/\overline{G}_m = \mathbb{H}^2/\overline{G}_m$ for $m \geq 2$, and in these cases, the groups G_n and $\overline{G}_m$ are transformation groups of isometries of $\mathbb{H}^2$, that is, the transformation groups of $\mathbb{R}^2$ relative to the new metric $\overline{ds^2}$.

The hyperbolic space $\mathbb{H}^2$, with metric $\overline{ds^2}$ given by (**), is well known to have constant curvature (Gaussian curvature) which is -1 (see reference [2]). Passing to orbit spaces we see $P_n^2 = \mathbb{H}^2/G_n$ and $T_m = \mathbb{H}^2/\overline{G}_m$ (for $n \geq 3$, $m \geq 2$) also have negative constant curvature -1. It is also known that $\mathbb{R}^2$ with the standard metric ds^2 has zero constant curvature. So the orbit spaces $T = \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$ and $P_2^2 = \mathbb{R}^2/G_2$ discussed earlier also have zero constant curvature. Finally, we also recall that the unit 2-sphere S^2 in $\mathbb{R}^3$ with the metric induced from the standard metric $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$ on $\mathbb{R}^3$ has constant curvature $+1$. So $P^2 = S^2/\mathbb{Z}_2$ also has constant curvature $+1$. We summarize all surfaces in terms of this geometry as follows:

(1) S^2 and $P^2 = S^2/\mathbb{Z}_2$ have constant curvature $+1$,

(2) $T = \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$ and $P_2^2 = \mathbb{R}^2/G_2$ have zero constant curvature,

(3) $P_n^2 = \mathbb{H}^2/G_n$ for $n \geq 3$ and $T_m = \mathbb{H}^2/\overline{G}_m$ have constant curvature -1.

By Theorem 2.5, the surfaces in (1), (2), (3) above exhaust all the possibilities of surfaces up to homeomorphisms. How can one tell they are different topological types one another? To answer this question we have to consider the fundamental groups of these surfaces. We refer to [9] for the basic theory on the fundamental group $\pi_1(X)$ of any path-connected topological space X. These include Proposition 2.6, Example 2.7 and Theorem 2.8 that follow. In the remainder of this section all spaces to be considered are assumed to be path-connected spaces.

For each space X there is associated a group $\pi_1(X)$, called the fundamental group of X. The association "$X \to \pi_1(X)$" has the following properties, called the functorial properties. For any continuous function $X \xrightarrow{f} Y$ of spaces, there is a natural induced homomorphism $\pi_1(X) \xrightarrow{f_*} \pi_1(Y)$ such that:

(1) $\pi_1(X) \xrightarrow{f_*=id} \pi_1(X)$ if $X \xrightarrow{f} X$ is the identify map.

(2) For any composite $X \xrightarrow{f} Y \xrightarrow{g} Z$ of spaces, the composite $\pi_1(X) \xrightarrow{f_*} \pi_1(Y) \xrightarrow{g_*} \pi_1(Z)$ is equal to $\pi_1(X) \xrightarrow{(gf)_*} \pi_1(Y)$.

From these properties one easily proves the following property.

Proposition 2.6. *If $X \xrightarrow{f} Y$ is a homeomorphism then the induced homomorphism $\pi_1(X) \xrightarrow{f_*} \pi_1(Y)$ is an isomorphism.*

Thus the fundamental group $\pi_1(X)$ is a topological invariant of X.

A space X is said to be simply connected if $\pi_1(X) = *$, that is, if its fundamental group $\pi_1(X)$ is the trivial group. We have the following examples of simply connected spaces.

Example 2.7.

(1) $\mathbb{R}^n$ for $n \geq 1$ and any convex subset of $\mathbb{R}^n$ are simply connected.

(2) The n-sphere S^n is simply connected for all $n \geq 2$.

A basic theorem in the theory of the fundamental groups is the following.

Theorem 2.8. *Let X be a path-connected space. Suppose U and V are path-connected open subsets of X such that $X = U \cup V$ and $U \cap V \neq \varnothing$. If U and V are simply connected and if $U \cup V$ is path-connected then X is simply connected.*

From this theorem one easily infers that the n-sphere S^n for $n \geq 2$ are simply connected as given in Example 2.7 (2). Indeed, the open subsets

$$U = \{x = (x_1, \ldots, x_{n+1}) \in S^n | x_{n+1} > -\frac{1}{2}\}$$
$$V = \{x = (x_1, \ldots, x_{n+1}) \in S^n | x_{n+1} < \frac{1}{2}\}$$

of S^n satisfy the conditions in the theorem, so S^n is simply connected for $n \geq 2$. This no longer is true for S^1. Indeed, we have $\pi_1(S^1) \cong \mathbb{Z}$, the infinite cyclic group of integers. To see this, we recall that S^1 is the orbit space $\mathbb{R}^1/\mathbb{Z}$. Then $\pi_1(S^1) \cong \mathbb{Z}$ follows from the following theorem which is also a basic theorem in the theory of π_1.

Theorem 2.9. *Suppose Y is a simply connected space and suppose $G \times Y \xrightarrow{\varphi} Y$ is a discrete group action (properly discontinuously). Consider the orbit space Y/G which is clearly a path-connected space. Then $\pi_1(Y/G) \cong G$.*

Now consider the surfaces

$$S^2,\ P^2 = S^2/\mathbb{Z}_2,\ T = \mathbb{R}^2/\mathbb{Z}\times\mathbb{Z},\ P_2^2 = \mathbb{R}^2/G_2,$$
$$P_n^2 = \mathbb{H}^2/G_n = \mathbb{R}^2/G_n \text{ for } n \geq 3, \text{ and}$$
$$T_m = \mathbb{H}^2/\overline{G}_m = \mathbb{R}^2/\overline{G}_m \text{ for } m \geq 2$$
$$\text{in (1), (2) and (3) above.}$$

Since S^2 and $\mathbb{R}^2$ are simply connected, from Theorem 2.9 we infer $\pi_1(S^2) = 0$, $\pi_1(P^2) = \mathbb{Z}_2$, $\pi_1(T) = \mathbb{Z}\times\mathbb{Z}$, $\pi_1(P_n^2) = G_n$ for $n \geq 2$ and $\pi_1(T_m) = \overline{G}_m$ for $m \geq 2$. From Proposition 2.6 we see that in order to prove these surfaces are of different topological types it suffices to show $\mathbb{Z}_2$, $\mathbb{Z}\times\mathbb{Z}$, G_n for $n \geq 2$ and $\overline{G}_m$ for $m \geq 2$ are non-isomorphic groups one another. It suffices to compare the groups $\mathbb{Z}\times\mathbb{Z}$, G_n and $\overline{G}_m$ one another since these are infinite groups while $\mathbb{Z}_2$ is a group of order 2.

For any group H let $[H,H]$ denote the commutator subgroup of H, that is, the subgroup generated by all the commutators $[g,h] = ghg^{-1}h^{-1}$ in H. It is easy to see $[H,H]$ is a normal subgroup of H. The quotient group $H/[H,H]$ is an abelian group, and this is easy to see. We shall denote $H/[H,H]$ by $A(H)$ ("A" means abelianization). If H is finitely generated then $A(H)$ is a finitely generated abelian group and so is isomorphic to a finite direct sum with each factor either the infinite cyclic group $\mathbb{Z}$ or a finite cyclic group $\mathbb{Z}_k$, by the fundamental theorem on finitely generated abelian groups. Finally note that if $H \xrightarrow{f} G$ is a group isomorphism then it induces a group isomorphism $A(H) \xrightarrow{\overline{f}} A(G)$. Thus if $A(H)$ and $A(G)$ are non-isomorphic groups then H and G are non-isomorphic groups.

Let G_1 denote $\mathbb{Z}\times\mathbb{Z}$. Clearly $A(G_1) \cong G_1 = \mathbb{Z}\times\mathbb{Z}$ since $\mathbb{Z}\times\mathbb{Z}$ is abelian. We recall that

> G_2 is a group on two generators a and b subject only to the relation $abab^{-1} = 1$,
> G_n, for each $n \geq 3$, is a group on n generators $a_1, a_2, \ldots, a_n$ subject only to the relation $a_1^2a_2^2\ldots a_n^2 = 1$,

and

> $\overline{G}_m$, for each $m \geq 2$, is a group on $2m$ generators $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$ subject only to the relation $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\ldots a_mb_ma_m^{-1}b_m^{-1} = 1$.

It is not difficult to show from these structures that

$$A(G_n) \cong \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{n-1}\oplus\mathbb{Z}_2 \text{ for } n \geq 2,$$

$$A(\overline{G}_m) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2m} \text{ for } m \geq 2.$$

Since these groups together with $A(G_1) = \mathbb{Z} \oplus \mathbb{Z}$ are non-isomorphic groups one another it follows that G_1, G_n for $n \geq 2$ and $\overline{G}_m$ for $m \geq 2$ are non-isomorphic groups one another. This completes the proof that the surfaces S^2, $P^2 = S^2/\mathbb{Z}_2$, $P_n^2 = \mathbb{R}^2/G_n$ for $n \geq 2$ and $T_m = \mathbb{R}^2/\overline{G}_m$ for $m \geq 2$ are of different topological types one another.

In concluding this section we remark that the concept of the fundamental groups of path-connected spaces was invented by Poincaré at the end of 19th century. This invention is intimated to his famous conjecture on a fundamental question about the topology of the 3-sphere S^3 in terms of the fundamental group, and this is described in the next section.

3. The Poincaré Conjecture

In this section any 3-manifold is meant a path-connected compact 3-manifold.

In 1904 Poincaré made the following conjecture.

Poincaré Conjecture 3.1. Any simply connected 3-manifold is homeomorphic to the 3-sphere S^3.

If all (compact) 3-manifolds can be classified, like the 2-dimensional case in the previous section, then one can immediately conclude whether the conjecture is true or false. Despite many efforts by many prominent mathematicians in the passing nearly 100 years, this famous conjecture remains open, not even to mention the classification of all 3-manifolds. However, recent progress work by Perelman, based on Hamilton's work in the 80's and 90's, might end this situation, not only solving the Poincaré conjecture in the affirmative but also classifying all 3-manifolds. We refer to three excellent survey articles on this progress which are the References [1], [8] and [11]. Here we just describe some topological and geometric aspects of the problem relating to this conjecture, largely from the viewpoint of Thurston Geometrization Conjecture. In particular, we will give a minor touch on Perelman and Hamilton's ideas.

We begin by giving some examples of 3-manifolds. The 3-sphere S^3 (of radius 1) that appears in **3.1** is of course the most prominent example of a 3-manifold. Analogous to the 2-dimensional case, we can also consider the orbit space $S^3/\mathbb{Z}_2 = P^3$, the 3-dimensional projective space where $\mathbb{Z}_2 = \{1, T\}$ acts on S^3 also by $T(x) = -x$. Unlike 2-dimensional case, there are some other orbit spaces of the form S^3/G besides P^3. This will be discussed later. From all the surfaces in section 2, one can construct a family of 3-manifolds as follows. For each surface M, the product $M \times S^1$ is an example of a 3-manifold. These manifolds are different (non-homeomorphic) for different M's. Also, they are different from orbit spaces of the form S^3/G, in particular, different from S^3 and P^3. These are easy to see (just look at their fundamental groups). These product 3-manifolds include $S^2 \times S^1$, $P^2 \times S^1$, $T^3 = T \times S^1 = S^1 \times S^1 \times S^1$, the 3-dimensional torus, etc. Analogous to $T = S^1 \times S^1 = \mathbb{R}^2/S^1 \times S^1$, T^3 is the orbit space $\mathbb{R}^3/\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. There are many other examples and this will be discussed below.

From the geometric analysis of surfaces in §2, it is natural to consider, in 3-dimensional case, three particular families of such manifolds. Namely, the families of 3-manifolds of constant curvatures $+1$, 0, and -1 respectively. In general, such manifolds are said to have constant curvature which may not be equal to any of these values, but when rescaled in Riemannian metric, can be adjusted to have $+1$, 0 and -1 as the constant curvature values respectively. Also, here we assume that any 3-manifold is a smooth (C^∞) manifold, and this actually is a theorem by the works of Moise, Munkres, Hirsh and Smale in the 50's and 60's. Similar to the 2-dimensional case, 3-manifolds of constant curvature $+1$ are orbit spaces of the form S^3/G_1, those of constant curvature 0 are orbit spaces of the form $\mathbb{R}^3/G_2$ and those of constant curvature -1 are orbit spaces of the form $\mathbb{H}^3/G_3$. Again this is a classical theorem which essentially is due to Killing and Hopf. Here G_1, G_2, $\mathbb{H}^3$ and G_3 are described as follows. G_1 is any finite subgroup of the orthogonal group $O(4)$ which is the symmetry group of S^3 in $\mathbb{R}^4$ relative to the standard metric ds^2 (G_1 has to be finite since S^3 is compact and S^3/G_1 is a 3-manifold). G_2 is any properly discontinuous group of isometries of $\mathbb{R}^3$ relative to the standard metric ds^2 so that the orbit space $\mathbb{R}^3/G_2$ is a compact 3-manifold. $\mathbb{H}^3$ is the 3-dimensional hyperbolic space which is $\mathbb{R}^3$ with the new metric $\overline{ds^2}$ given by

$$\overline{ds^2} = \frac{4}{(1-r^2)^2} ds^2$$

analogous to the 2-dimensional case $\mathbb{H}^2$ in §2, and finally, G_3 is any properly discontinuous group of isometries of $\mathbb{H}^3$ relative to $\overline{ds^2}$ so that the orbit space $\mathbb{H}^3/G_3$ is also a compact 3-manifold. Spaces of the form S^3/G_1 are called elliptic 3-manifolds, those of the form $\mathbb{R}^3/G_2$ are called 3-dimensional Euclidean manifolds and those of the form $\mathbb{H}^3/G_3$ are called 3-dimensional hyperbolic manifolds. Classification of elliptic 3-manifolds and that of 3-dimensional Euclidean manifolds are completely understood. See Thurston [15] for an excellent geometric treatise on these facts. The classification of 3-dimensional hyperbolic manifolds is not completely known yet and is still an active area of research.

We describe at least one example of an interesting elliptic 3-manifold S^3/D^* where D^* is a group of order 120, and so S^3/D^* is different from P^3. This example is of historical in connection with the Poincaré conjecture and is known as the Poincaré 3-manifold.

In order to describe this 3-manifold we recall that the skew-field $\mathbb{H}$ of quaternions is a 4-dimensional vector space over $\mathbb{R}$ on the standard basis 1, i, j and k. Elements in $\mathbb{H}$, called quaternions, are written as $q = a + bi + cj + dk$ with $a, b, c, d \in \mathbb{R}$. $\mathbb{H}$ is an algebra over $\mathbb{R}$ with multiplication determined by the rules $i^2 = j^2 = k^2 = -1 = -1 + 0i + 0j + 0k$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$, and with $1 = 1 + 0i + 0j + 0k$ as the identity of the multiplication. $\mathbb{H}$ is a normed linear algebra over $\mathbb{R}$ with norm $\| \ \|$ given by

$$\|q = a + bi + cj + dk\| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

so that $\|q_1q_2\| = \|q_1\|\|q_2\|$ for all $q_1, q_2 \in \mathbb{H}$. For each $q = a + bi + cj + dk \neq 0$, which is equivalent to $\|q\| > 0$, $q^{-1} = \frac{\overline{q}}{\|q\|^2} = \frac{1}{\|q\|^2}(a - bi - cj - dk)$ is the multiplicative inverse of q so that $q^{-1}q = qq^{-1} = 1$. This shows $\mathbb{H}$ is a skew-field, noting that the multiplication in $\mathbb{H}$ is not commutative ($ij \neq ji$, for example). $S^3 \subset \mathbb{R}^4 \cong \mathbb{H}$ can be considered as the subset of $\mathbb{H}$ consisting of all quaternions $q = a + bi + cj + dk$ with $\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2} = 1$, equivalently, $\|q\|^2 = a^2 + b^2 + c^2 + d^2 = 1$. The multiplication in $\mathbb{H}$ induces a multiplication

$$\begin{gathered} S^3 \times S^3 \to S^3 \\ (q_1,\ q_2) \to q_1q_2 \end{gathered}$$

in S^3 since $\|q_1q_2\| = \|q_1\|\|q_2\| = 1$. Also for each $q \in S^3$ its inverse q^{-1} also lies in S^3 since $1 = \|1 = qq^{-1}\| = \|q\|\|q^{-1}\|$ and $\|q\| = 1$ implies $\|q^{-1}\| = 1$. So S^3 is a topological group under this multiplication which is clearly continuous in the variables q_1 and q_2. Actually S^3 is a Lie group.

Now consider, for each $q \in S^3$, the conjugate map

$$(*) \qquad \begin{gathered} S^3 \overset{\varphi(q)}{\to} S^3 \\ x \to qxq^{-1} \end{gathered}$$

which, not only is a group isomorphism (this is easy to see), but is also an isometry as $\|\varphi(q)(x) = qxq^{-1}\| = \|q\|\|x\|\|q^{-1}\| = 1$. This defines a map $S^3 \overset{\varphi}{\to} O(4)$, the symmetry group of S^3. We recall that the rotation group $SO(4)$ is one of the two path-component of $O(4)$ that contains the identity $I_4 \in O(4)$. Since S^3 is path-connected and $\varphi(1) = I_4$ it follows that φ is $S^3 \overset{\varphi}{\to} SO(4)$. $S^3 \overset{\varphi}{\to} SO(4)$ is a group homomorphism since

$$\varphi(g_1g_2)(x) = g_1g_2x(g_1g_2)^{-1} = g_1g_2xg_2^{-1}g_1^{-1} = g_1(\varphi(g_2)(x))g_1^{-1} = \varphi(g_1)\varphi(g_2)(x).$$

It is easy to see that the group S^3 has center $\{\pm 1\}$. From this one sees that $ker\varphi = \{\pm 1\}$. Consider the rotation group $SO(3)$ which is a subgroup of $SO(4)$

via the embedding

$$SO(3) \to SO(4)$$
$$A \to \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Since $\varphi(1) = I_4$, it follows that $S^3 \xrightarrow{\varphi} SO(4)$ maps S^3 into $SO(3)$. So φ is actually a map $S^3 \xrightarrow{\varphi} SO(3)$ which is still a group homomorphism with $ker\varphi = \{\pm 1\}$. We refer to 2.7.4 and 2.7.7 in [15] for the fact that $S^3 \xrightarrow{\varphi} SO(3)$ is onto. We conclude that the conjugate map in (*) gives rise to

(**) a group epimorphism $S^3 \xrightarrow{\varphi} SO(3)$ with $ker\varphi = \{\pm 1\}$.

Proposition 3.2. (1) *For any finite group $G \subset SO(3)$, the inverse image $G^* = \varphi^{-1}(G)$ is a finite subgroup of S^3 with $|G^*| = 2|G|$ where φ is as in $(**)$.* (2) *Any finite subgroup H of S^3 is a group of isometries of S^3 via the (free) group action $H \times S^3 \xrightarrow{\psi} S^3$ given by $\psi(h, x) = hx$.*

Proof. (1) is a standard result in group theory. To see (2) note that $\|\psi(h, x)\| = \|hx\| = \|h\|\|x\| = 1$ for any $h \in H \subset S^3$ and any $x \in S^3$ and note that $hx = x$ for some $x \in S^3$ implies $h = 1$. ■

Finite subgroups of $SO(3)$ are completely known, and for this we refer to [15] or [17]. By Proposition 3.2 we thus see that all finite subgroups of S^3 can be classified and that each such a finite group H gives rise to an elliptic 3-manifold S^3/H. We should remark, however, that not every elliptic 3-manifold S^3/Γ arises this way; as we have already noted earlier, S^3/Γ is an elliptic 3-manifold if and only if Γ is a finite subgroup of $O(4)$. All such finite subgroups are also known, again see Thurston [15].

The Poincaré 3-manifold is a particular example S^3/D^* of Proposition 3.2. $D \subset SO(3)$ is the symmetry group of the dodecahedron as shown in the figure. D is called the dodecahedral group and $D^* = \varphi^{-1}(D)$ is called the binary dodecahedral group. It can be shown that D has order 60 and is isomorphic to the alternating subgroup A_5 of the symmetric group S_5 on 5 letters and that D^* is isomorphic to the special linear group $SL(2, \mathbb{Z}_5)$, that is, the multiplicative group of 2×2 matrices of determinant 1 with coefficients in the field $\mathbb{Z}_5$. It can also be shown that $SL(2, \mathbb{Z}_5)$ is a perfect group, that is, it is equal to its commutator subgroup $[SL(2, \mathbb{Z}_5), SL(2, \mathbb{Z}_5)]$. For all of these, see [17]. We conclude that the binary dodecahedral group D^* is perfect, that is, $D^* = [D^*, D^*]$.

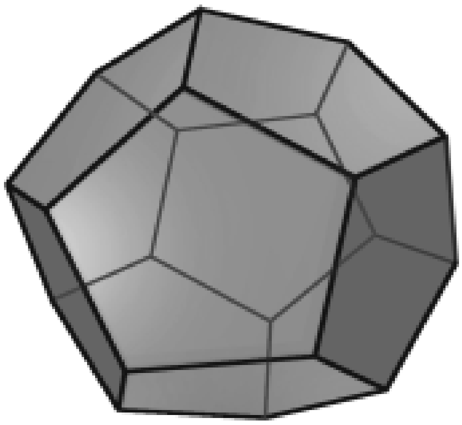

Fig. 10.

For this paragraph and the next we assume the readers are familiar with two notions in basic algebraic topology. First, for any path-connected compact orientable n-manifold M, the integral homology groups $H_j(M)$ and cohomology groups $H^k(M)$ satisfy $H_0(M) = H_n(M) = \mathbb{Z}$ and $H^{n-i}(M) \cong H_i(M)$ for $0 \leq i \leq n$. Secondly, for any path-connected topological space X, if $\pi_1 = \pi_1(X)$ is the fundamental group then the first integral homology $H_1(X)$ is the abelianization of π_1, that is, $H_1(X) \cong A(\pi_1) = \pi_1/[\pi_1, \pi_1]$.

S^3 is well known to be an orientable 3-manifold, and since $\pi_1(S^3) = 0$, it follows that $H_1(S^3) = 0$ and this implies $H_0(S^3) = H_3(S^3) = \mathbb{Z}$ and $H_i(S^3) = 0$ for $i = 1, 2$. Any orientable 3-manifold M with $H_0(M) = H_3(M) = \mathbb{Z}$ and $H_i(M) = 0$, $i = 1, 2$, is called a homology 3-sphere. It is easy to see that any elliptic 3-manifold S^3/Γ is orientable (see [15]). In particular, S^3/D^* is orientable. Now, since $\pi_1(S^3/D^*) = D^*$ (by Theorem 2.9) and $D^* = [D^*, D^*]$, it follows that $H_1(S^3/D^*) = 0$. So the Poincaré 3-manifold S^3/D^* is a homology 3-sphere.

The original version of Poincaré conjecture (around 1900) is that any 3-manifold which is a homology 3-sphere is homeomorphic to the 3-sphere S^3. Later on Poincaré found an example of a homology 3-sphere M^3, which about a quarter century later was recognized by Threlfall and Seifert to be the dodecahedral space S^3/D^* and is definitely not homeomorphic to S^3. So he corrected his conjecture to the version stated in **Poincaré Conjecture 3.1** at the outset of this section. Thus the Poincaré 3-manifold S^3/D^* is the starting point in the history of the Poincaré conjecture.

And the Poincaré dodecahedral space S^3/D^* is not just the beginning history of the Poincaré conjecture, it is still an important space not just in geometry and topology but also in cosmology! For this see [16].

We already noted that the following three families of 3-manifolds

(1) elliptic 3-manifolds S^3/G_1,
(2) 3-dimensional Eulidean manifolds $\mathbb{R}^3/G_2$ and
(3) 3-dimensional hyperbolic manifolds $\mathbb{H}^3/G_3$

are precisely the 3-manifolds of constant curvatures $+1$, 0 and -1 respectively. Unlike the 2-dimensional case in which these types of 2-manifolds exhaust all possibilities of surfaces, there are 3-manifolds which do not belong to one of the types 1, 2 and 3. For example, $S^2 \times S^1$ and $P^2 \times S^1$ are such examples, and this is roughly seen as follows. Since S^2 is simply connected, the fundamental group $\pi_1(S^2 \times S^1)$ is isomorphic to $\pi_1(S^1) = \mathbb{Z}$ (in general, $\pi_1(X \times Y) \cong \pi_1(X) \times \pi(Y)$). Since elliptic 3-manifolds have finite fundamental groups it follows that $S^2 \times S^1$ can not be an elliptic 3-manifold. In some sense, the infinite cyclic group $\mathbb{Z}$ is a "one-dimensional" object, namely, it has rank 1 as a free abelian group. So $\mathbb{R}^2/\mathbb{Z}$ is not a compact 3-manifold no matter what action of $\mathbb{Z}$ on $\mathbb{R}^3$ is. This implies $S^2 \times S^1$ is neither a 3-dimensional Euclidean manifold $\mathbb{R}^3/G_2$ nor a 3-dimensional hyperbolic manifold $\mathbb{H}^2/G_3$. Similar proof applies to $P^2 \times S^1$ since P^2 is an elliptic surface (this remark is crucial as, for example, $T \times S^1 = S^1 \times S^1 \times S^1 = \mathbb{R}^3/\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is a 3-dimensional Euclidean manifold).

Thurston calls each of the types 1, 2 and 3 above a geometric structure with $(O(4), S^3)$, $(\Gamma_1, \mathbb{R}^3)$ and $(\Gamma_2, \mathbb{H}^3)$ as the model geometries called elliptic model, Euclidean model and hyperbolic model respectively. Here Γ_1 (resp. Γ_2) is the full group of isometries of $\mathbb{R}^3$ (resp. $\mathbb{H}^3$) relative to ds^2 (resp. $\overline{ds^2}$). In late 1970's, based on previous works by many outstanding geometers and topologists on this subject, he proposed five other geometries and conjectured that any 3-manifold (recall that this means a compact 3-manifold) can be completely described in terms of these total 8 geometries. This conjecture will be described more precisely below. We refer to Thurston [15] for these extra 5 geometries which will not be described here. We just remark that $S^2 \times S^1$ and $\mathbb{H}^2 \times S^1$ are representatives for two of these geometries. We also remark that these extra 5 geometries are all well understood in the sense that their classifications are completely known as in the cases of elliptic and Euclidean geometries. So among the total 8 geometries proposed by Thurston only the hyperbolic geometry still remains to be classified (this we already remarked earlier).

Thurston conjecture can be stated as follows.

Thurston Geometrization Conjecture 3.3. Any 3-manifold can be decomposed in an essential unique way by disjoint embedded 2-spheres (via connected sum) and tori T into pieces each one having one of these 8 geometries.

The "decomposition" referred to in the statement above comes from a classical result on the standard two-stage decomposition of any closed 3-manifold M. The first decomposition is the connected sum decomposition of M into irreducible 3-manifolds (a 3-manifold is irreducible if any embedded 2-sphere bounds a 3-ball.) The second decomposition is the Jaco–Shalen–Johannson torus decomposition, which says that any irreducible 3-manifold N has a canonical minimal collection of disjointly embedded incompressible tori (meaning that $\pi_1(T)$ of each such a torus T is a subgroup of $\pi_1(N)$) such that each component of the 3-manifold removed by these tori is either "torus-irreducible" or "Seifert fibered." We refer to [1], [11] for more details on this and also for the precise meaning of "essential unique way by these geometric pieces" in the statement **3.3** above. Here we just give a simple example to explain this. If M is a 3-manifold which already has one of these 8 geometric structures, say $M = P^3$ (which has elliptic geometry) and if N is another 3-manifold also having one of these geometric structures, say $N = S^2 \times S^1$. Then $M \sharp N = P^3 \sharp (S^2 \times S^1)$ is the only decomposition of the manifold $L = M \sharp N$ in terms of the elliptic geometry on the piece P^3 and the $(S^2 \times S^1)$-geometry on the piece $S^2 \times S^1$.

If Thurston conjecture can be shown true then all 3-manifolds can be classified provided those of hyperbolic geometry can be classified (because the classifications of the remaining 7 geometries are known). Pertaining to Poincaré conjecture, Thurston conjecture implies the following that characterizes elliptic 3-manifolds.

Corollary 3.4. [to Thurston Conjecture] *A 3-manifold is an elliptic 3-manifold if and only if it has finite fundamental group.*

This immediately implies Poincaré conjecture since S^3 is the only elliptic 3-manifold with $\pi_1 = 0$.

From Riemannian geometry viewpoint, Thurston geometrization conjecture essentially asserts that for any 3-manifold M there exists a "best possible" metric to fit the picture of the geometry of M described in the conjecture. The space of metrics on M is $\mathbb{M} = C^\infty(M, \mathbb{R}^6)$ which is a huge space. How can one choose the "best possible" one? The Ricci flow introduced by Hamilton in 1982 (see [3] and also [4-7]) is an initial successful attempt to make a best choice for some of the 3-manifolds. Ricci flow is an evolution equation, that is, a differential equation, that involves Ricci curvatures parametrized in time variable t in the metric space $\mathbb{M}$. The reason to consider Ricci curvature is that the Riemannian curvature and the Ricci curvature are determined each other algebraically in dimension 3 and the latter is easier to work with, for example, it is a symmetric bilinear form and behaves nicely if it is positive when considering Ricci flow. Hamilton tried to apply his technique (Ricci flow) to more general 3-manifolds hoping to solve the geometrization conjecture. But there are difficulties, primarily from singularities that may arise. These singularities correspond to the passages from one geometry to another geometry in Thurston geometrization picture. Recent works ([12-14]) by Perelman are essentially an attempt to solve these singularities by introducing some functional analysis on the space of metrics $\mathbb{M}$ to interpret Ricci flow. Perelman's works are very deep. Much of his works has been validated by experts. So the classification of all 3-manifolds as suggested by Thurston seems to be possible in sight. We refer to [8] for a detailed survey on all of these developments from Thurston to Hamilton and then to Perelman.

It appears, from the development of the theory on 3-manifolds in the passing 100 years, that Poincaré conjecture, which is so simple in statement, has to be waited for its solution till the complete classification of all 3-manifolds is known. Can it be solved simply from the topological assumption $\pi_1 = 0$? More generally, can **Corollary 3.4** be proved without all the machineries that lead the complete classification of all 3-manifolds? With these questions we end this exposition of the Poincaré conjecture.

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