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INHOMOGENEOUS CALDERÓN REPRODUCING FORMULAE ASSOCIATED TO PARA-ACCRETIVE FUNCTIONS ON METRIC MEASURE SPACES

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Abstract. Let $(X, \rho, \mu)_{d,\theta}$ be a space of homogeneous type which includes metric measure spaces and some fractals, namely, X is a set, ρ is a quasi-metric on X satisfying that there exist constants $C_0 > 0$ and $\theta \in (0, 1]$ such that for all $x,\ x',\ y \in X$,

$$|\rho(x, y) - \rho(x', y)| \leq C_0 \rho(x, x')^\theta [\rho(x, y) + \rho(x', y)]^{1-\theta},$$

and μ is a nonnegative Borel regular measure on X satisfying that for some $d > 0$, all $x \in X$ and all $0 < r < \operatorname{diam} X$,

$$\mu(\{y \in X: \ \rho(x, y) < r\}) \sim r^d.$$

In this paper, we first obtain the boundedness of Calderón-Zygmund operators on spaces of test functions; and using this, we then establish the continuous Calderón reproducing formulae associated with a given para-accretive function, which is a key tool for developing the theory of Besov and Triebel-Lizorkin spaces associated with para-accretive functions. By the Calderón reproducing formulae, we finally obtain a Littlewood-Paley theorem on the inhomogeneous g-function which gives a new characterization of Lebesgue spaces $L^p(X)$ for $p \in (1, \infty)$ and generalizes a corresponding result of David, Journé and Semmes.

1. INTRODUCTION

It is well-known that the remarkable $T1$ theorem of David and Journé provides a general criterion for the $L^2(\mathbb{R}^n)$-boundedness of generalized Calderón-Zygmund

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singular integral operators; see [3, 29]. The $T1$ theorem, however, cannot be directly applied to the Cauchy integral on Lipschitz curves. To this end, Meyer in [24] (see also [27]) observed that one needs to replace the function 1 in the $T1$ theorem by a bounded complex-valued function b satisfying $0 < \delta \le \mathrm{Re}\, b(x)$ almost everywhere. Replacing the function 1 by an accretive function b, McIntosh and Meyer in [24] proved the Tb theorem. David, Journé, and Semmes in [4] further introduced a more general class of $L^\infty(\mathbb{R}^n)$ functions b, namely, the so-called para-accretive functions and proved the so-called Tb theorem. Moreover, they verified that the para-accretivity is also necessary in the sense that the Tb theorem holds for a bounded function b, then b is para-accretive. Motivated by these results and the theory of the Hardy space $H^1_b(\mathbb{R}^n)$ of Meyer in [27], Han in [9], Han, Lee and Lin in [12] and Deng and the author in [5] further developed the theories of other spaces of functions including the homogeneous Besov and Triebel-Lizorkin spaces associated to para-accretive functions. The Tb theorems related to them were also established. A key tool for these theories is the homogeneous Calderón reproducing formulae.

The main purpose of this paper is to establish the inhomogeneous continuous Calderón reproducing formulae associated with a given para-accretive function b to pave a way for developing the theory of such type inhomogeneous spaces of functions, which will be considered in another paper; see [13-15, 17, 18]. When $b \equiv 1$, these formulae were obtained in [11]. We remark that due to the inhomogeneity, some new ideas and techniques different from the homogeneous case on $\mathbb{R}^n$ in [9, 12] are necessary. Moreover, we establish the inhomogeneous Calderón reproducing formulae on spaces of homogeneous type in the sense of Coifman and Weiss in [1, 2], which include metric measure spaces and some fractals.

We notice that the analysis on metric spaces has recently attracted an increasing interest; see [28, 19, 8, 21]. Especially, the theory of function spaces on metric spaces, or more generally, the spaces of homogeneous type has been well developed; see [22, 23, 16, 10, 13-15, 17, 18, 34, 36]. We point out that the spaces of homogeneous type considered in this paper include metric measure spaces, the Euclidean space, the C^∞-compact Riemannian manifolds, the boundaries of Lipschitz domains and, in particular, the Lipschitz manifolds introduced recently by Triebel in [33] and the isotropic and anisotropic d-sets in $\mathbb{R}^n$. It has been proved by Triebel in [31, 32] that the isotropic and anisotropic d-sets in $\mathbb{R}^n$ include various kinds of self-affine fractals, for example, the Cantor set, the generalized Sierpinski carpet and so forth. We particularly point out that the spaces of homogeneous type considered in this paper also include the post critically finite self-similar fractals studied by Kigami in [20] and by Strichartz in [28], and the metric spaces with heat kernel studied by Grigor'yan, Hu and Lau in [7]. More examples of spaces of homogeneous type can be found in [1, 2, 28].

To establish the inhomogeneous Calderón reproducing formulae on spaces of homogeneous type, we first need to establish the boundedness of Calderón-Zygmund operators on some spaces of test functions which itself generalizes a corresponding result of Han in [11] and is presented in Section 2. In Section 3, using our results in Section 2, we obtain the continuous Calderón reproducing formulae, where Coifman's idea (see [4]) plays a key tool. By these Calderón reproducing formulae, in Section 4, we obtain a Littlewood-Paley theorem for the inhomogeneous g-function, which gives a new characterization of Lebesgue spaces $L^p(X)$ for $p \in (1, \infty)$ and generalizes a corresponding result of David, Journé and Semmes in [4]. Such Littlewood-Paley theorem has also been proved to be useful in establishing the discrete Calderón reproducing formulae; see [15].

2. Boundedness of Calderón-zygmund Operators

The main purpose of this section is to establish the boundedness of Calderón-Zygmund operators on spaces of test functions associated with a given para-accretive function. We first recall some necessary definitions and notation of spaces of homogeneous type.

A quasi-metric ρ on a set X is a function $\rho: X \times X \to [0, \infty)$ satisfying that

(i) $\rho(x, y) = 0$ if and only if $x = y$;

(ii) $\rho(x, y) = \rho(y, x)$ for all $x,\ y \in X$;

(iii) There exists a constant $A \in [1, \infty)$ such that for all $x,\ y$ and $z \in X$,

$$\rho(x, y) \le A[\rho(x, z) + \rho(z, y)].$$

Any quasi-metric defines a topology, for which the balls

$$B(x, r) = \{y \in X : \ \rho(y, x) < r\}$$

for all $x \in X$ and all $r > 0$ form a basis.

In what follows, we set $\operatorname{diam} X = \sup\{\rho(x, y) : \ x,\ y \in X\}$. We also make the following conventions. We denote by $f \sim g$ that there is a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$. Throughout the paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. For any $q \in [1, \infty]$, we denote by q' its conjugate index, namely, $1/q + 1/q' = 1$. Let A be a set and we denote by χ_A the characteristic function of A.

Definition 2.1. ([17]) Let $d > 0$ and $\theta \in (0, 1]$. A space of homogeneous type, $(X, \rho, \mu)_{d,\theta}$, is a set X together with a quasi-metric ρ and a nonnegative

Borel regular measure μ on X, and there exists a constant $C_0 > 0$ such that for all $0 < r < \operatorname{diam} X$ and all $x,\ x',\ y \in X$,

$$\mu(B(x,r)) \sim r^d \tag{2.1}$$

and

$$|\rho(x,y) - \rho(x',y)| \le C_0 \rho(x,x')^\theta [\rho(x,y) + \rho(x',y)]^{1-\theta}. \tag{2.2}$$

In what follows, all the θ means the same θ as in (2.2).

The space of homogeneous type defined above is a variant of the space of homogeneous type introduced by Coifman and Weiss in [1]. In [22], Macias and Segovia have proved that one can replace the quasi-metric ρ of the space of homogeneous type in the sense of Coifman and Weiss by another quasi-metric $\bar{\rho}$ which yields the same topology on X as ρ such that $(X,\ \bar{\rho},\ \mu)$ is the space defined by Definition 2.1 with d = 1.

Let us now recall the definitions of the para-accretive function and the space of test functions.

Definition 2.2. A bounded complex-valued function b on X, a space of homogeneous type, is said to be para-accretive if there exist constants $C_1 > 0$ and $\kappa \in (0,1]$ such that for all balls $B \subset X$, there is a ball $B' \subset B$ with $\kappa\mu(B) \le \mu(B')$ satisfying

$$\frac{1}{\mu(B)} \left| \int_{B'} b(x)\, d\mu(x) \right| \ge C_1 > 0.$$

Definition 2.3. ([9]) Fix $\gamma > 0$ and $\theta \ge \beta > 0$. A function f defined on X is said to be a test function of type (x_0, r, β, γ) with $x_0 \in X$ and $r > 0$, if f satisfies the following conditions:

(i) $|f(x)| \le C \frac{r^\gamma}{(r+\rho(x,x_0))^{d+\gamma}}$;

(ii) $|f(x) - f(y)| \le C \left(\frac{\rho(x,y)}{r+\rho(x,x_0)}\right)^\beta \dfrac{r^\gamma}{(r+\rho(x,x_0))^{d+\gamma}}$ for $\rho(x,y) \le \frac{1}{2A}[r + \rho(x,x_0)]$.

If f is a test function of type (x_0, r, β, γ), we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{G}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} = \inf\{C:\ (i)\ and\ (ii)\ hold\}.$$

Now fix $x_0 \in X$ and let $\mathcal{G}(\beta,\gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to see that

$$\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$$

with an equivalent norm for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta, \gamma)$. Also, let the dual space $(\mathcal{G}(\beta, \gamma))'$ be all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta, \gamma)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in \mathcal{G}(\beta, \gamma)$,

$$|\mathcal{L}(f)| \leq C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

We denote by $\langle h, f\rangle$ the natural pairing of elements $h \in (\mathcal{G}(\beta, \gamma))'$ and $f \in \mathcal{G}(\beta, \gamma)$. Clearly, for all $h \in (\mathcal{G}(\beta, \gamma))'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$.

It is well-known that even when $X = \mathbb{R}^n$, $\mathcal{G}(\beta_1, \gamma)$ is not dense in $\mathcal{G}(\beta_2, \gamma)$ if $\beta_1 > \beta_2$, which bring us some inconvenience. To overcome this defect, in what follows, for a given $\epsilon \in (0, \theta]$, we let $\mathring{\mathcal{G}}(\beta, \gamma)$ be the completion of the space $\mathcal{G}(\epsilon, \epsilon)$ in $\mathcal{G}(\beta, \gamma)$ when $0 < \beta,\ \gamma < \epsilon$.

Let b be a para-accretive function. As usual, we write

$$b\mathcal{G}(\beta, \gamma) = \{f :\ f = bg \text{ for some } g \in \mathcal{G}(\beta, \gamma)\}.$$

If $f \in b\mathcal{G}(\beta, \gamma)$ and $f = bg$ for some $g \in \mathcal{G}(\beta, \gamma)$, then the norm of f is defined by

$$\|f\|_{b\mathcal{G}(\beta,\gamma)} = \|g\|_{\mathcal{G}(\beta,\gamma)}.$$

By this definition, it is easy to see that

$$f \in \left(b\mathring{\mathcal{G}}(\beta, \gamma)\right)' \quad \text{if and only if} \quad bf \in \left(\mathring{\mathcal{G}}(\beta, \gamma)\right)', \tag{2.3}$$

where we define $bf \in \left(\mathring{\mathcal{G}}(\beta, \gamma)\right)'$ by

$$\langle bf, g\rangle = \langle f, bg\rangle$$

for all $g \in \mathring{\mathcal{G}}(\beta, \gamma)$.

In what follows, we also let

$$\mathcal{G}_0^b(x_0, r, \beta, \gamma) = \left\{f \in \mathcal{G}(x_0, r, \beta, \gamma) :\ \int_X f(x)b(x)\, d\mu(x) = 0\right\};$$

for $\eta \in (0, \theta]$, we define $C_0^\eta(X)$ to be the set of all functions having compact support such that

$$\|f\|_{C_0^\eta(X)} = \sup_{x\neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\eta} < \infty.$$

Endow $C_0^\eta(X)$ with the natural topology and let $(C_0^\eta(X))'$ be its dual space.

Definition 2.4. Let $\epsilon \in (0, \theta]$. A continuous complex-valued function $K(x, y)$ on

$$\Omega = \{(x, y) \in X \times X : x \neq y\}$$

is called a Calderón-Zygmund kernel of type ϵ if there exist a constant $C_2 > 0$ such that

(i) $|K(x, y)| \leq C_2 \rho(x, y)^{-d}$,

(ii) $|K(x, y) - K(x', y)| \leq C_2 \rho(x, x')^{\epsilon} \rho(x, y)^{-d-\epsilon}$ for $\rho(x, x') \leq \frac{\rho(x,y)}{2A}$,

(iii) $|K(x, y) - K(x, y')| \leq C_2 \rho(y, y')^{\epsilon} \rho(x, y)^{-d-\epsilon}$ for $\rho(y, y') \leq \frac{\rho(x,y)}{2A}$.

A continuous linear operator $T : C_0^{\eta}(X) \to (C_0^{\eta}(X))'$ for all $\eta \in (0, \theta]$ is a Calderón-Zygmund singular integral operator of type ϵ if there is a Calderón-Zygmund kernel $K(x, y)$ of the type ϵ as above such that

$$\langle Tf, g\rangle = \int_X \int_X K(x, y) f(y) g(x) \, d\mu(x) \, d\mu(y)$$

for all $f,\ g \in C_0^{\eta}(X)$ with disjoint supports. In this case, we write $T \in CZO(\epsilon)$.

We also need the following notion of the strong weak boundedness property in [16, 11].

Definition 2.5. A Calderón-Zygmund singular integral operator T of the kernel K is said to have the strong weak boundedness property, if there exist $\eta \in (0, \theta]$ and constant $C_3 > 0$ such that

$$|\langle K, f\rangle| \leq C_3 r^d$$

for all $r > 0$ and all continuous f on $X \times X$ with $\operatorname{supp} f \subseteq B(x_1, r) \times B(y_1, r)$, where x_1 and $y_1 \in X$, $\|f\|_{L^\infty(X\times X)} \leq 1$, $\|f(\cdot, y)\|_{C_0^{\eta}(X)} \leq r^{-\eta}$ for all $y \in X$ and $\|f(x, \cdot)\|_{C_0^{\eta}(X)} \leq r^{-\eta}$ for all $x \in X$. We denote this by $T \in SWBP$.

The following theorem is the main theorem of this section, which when $b \equiv 1$ was obtained by Han in [11]. In what follows, we use $\mathcal{M}_b$ to denote the multiplication operator defined by b, namely, for suitable functions f, $\mathcal{M}_b(f) = bf$.

Theorem 2.1. *Let b be a para-accretive function as in Definition 2.2 and $\epsilon \in (0, \theta]$. Let T be a continuous linear operator from $C_0^{\eta}(X)$ to $(C_0^{\eta}(X))'$ for all $\eta \in (0, \theta]$ such that the kernels of T and $b^{-1}T^*\mathcal{M}_b$ respectively satisfy the conditions (i) and (ii) and only the condition (ii) of Definition 2.4 with the regularity exponent ϵ, $T(1) = 0$, and $T \in SWBP$. Furthermore, $K(x, y)$, the kernel of T, satisfies the following smoothness condition that for all $x,\ x',\ y,\ y' \in X$ such that $\rho(x, x'),\ \rho(y, y') \leq \frac{\rho(x,y)}{3A^2}$,*

$$
\begin{aligned}
(2.4) \qquad & \big|\big[K(x,y)b^{-1}(y) - K(x',y)b^{-1}(y)\big] \\
& \quad - \big[K(x,y')b^{-1}(y') - K(x',y')b^{-1}(y')\big]\big| \\
& \le C_4 \rho(x,x')^{\epsilon}\rho(y,y')^{\epsilon}\rho(x,y)^{-d-2\epsilon}.
\end{aligned}
$$

Then for any $x_0 \in X$*,* $r > 0$ *and* $0 < \beta,\ \gamma < \epsilon$*,* T *maps* $\mathcal{G}_0^b(x_0, r, \beta, \gamma)$ *into itself. Moreover, if we let* $\|T\| = \max\{C_2,\ C_3,\ C_4\}$*, then there exists a constant* $C > 0$ *such that*

$$\|Tf\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \le C\|T\|\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)}.$$

Proof. We prove Theorem 2.1 by following a procedure of the proof of Theorem 1 in [11] and we only give an outline to indicate their distinction.

Fix a function $\kappa \in C^1(\mathbb{R})$ with $\operatorname{supp}\kappa \subset \{x \in \mathbb{R} :\ |x| \le 2\}$, $0 \le \kappa(x) \le 1$ on $\mathbb{R}$ and $\kappa(x) = 1$ on $\{x \in \mathbb{R} :\ |x| \le 1\}$. Suppose that $f \in \mathcal{G}_0^b(x_0, r, \beta, \gamma)$ with $0 < \beta,\ \gamma < \epsilon$. We first verify that $T(f)(x)$ satisfies the size condition (i) of Definition 2.3. Using Lemma 2.1 in [11] (see also [25]), by the same argument as the proof of Theorem 1 in [11], we can verify that $T(f)(x)$ satisfies the size condition (i) of Definition 2.3 when $\rho(x,x_0) \le 5r$. We now consider the case $\rho(x,x_0) = R > 5r$. In this case, we set $1 = I(y) + J(y) + L(y)$, where $I(y) = \kappa\left(\frac{4A\rho(x,y)}{R}\right)$, $J(y) = \kappa\left(\frac{4A\rho(y,x_0)}{R}\right)$, and $f_1(y) = f(y)I(y)$, $f_2(y) = f(y)J(y)$, and $f_3(y) = f(y)L(y)$. Then, it is easy to check the following estimates:

$$|f_1(y)| \le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \frac{r^{\gamma}}{R^{d+\gamma}}; \tag{2.5}$$

$$|f_1(y) - f_1(y')| \le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \frac{\rho(y,y')^{\beta}}{R^{\beta}} \frac{r^{\gamma}}{R^{d+\gamma}} \quad \text{for all} \quad y,\ y' \in X; \tag{2.6}$$

$$|f_3(y)| \le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \frac{r^{\gamma}}{\rho(y,x_0)^{d+\gamma}} \chi_{\{\rho(y,x_0) > \frac{R}{4A}\}}(y); \tag{2.7}$$

$$\int_X |f_3(y)|\, d\mu(y) \le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \frac{r^{\gamma}}{R^{\gamma}}; \tag{2.8}$$

$$|f_2(y)| \le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \frac{r^{\gamma}}{\rho(y,x_0)^{d+\gamma}} \chi_{\{\rho(y,x_0) \le \frac{R}{2A}\}}(y); \tag{2.9}$$

and

$$\left|\int_X f_2(y)b(y)\, d\mu(y)\right| \le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \frac{r^{\gamma}}{R^{\gamma}}, \tag{2.10}$$

since $\int_X f(y)b(y)\,d\mu(y) = 0$ and b is bounded; see [11] for the details. From (2.6) and (2.5), and an argument similar to that in [11], it is easy to deduce that

$$|Tf_1(x)| \le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \frac{r^\gamma}{R^{d+\gamma}},$$

which is a desired estimate. For f_2, noting that $x \notin \operatorname{supp} f_2$, we write

$$\begin{aligned} T(f_2)(x) &= \int_X \left[b^{-1}(y)K(x,y) - b^{-1}(x_0)K(x,x_0)\right] b(y)f_2(y)\,d\mu(y) \\ &\quad + b^{-1}(x_0)K(x,x_0)\int_X f_2(y)b(y)\,d\mu(y) \\ &= \delta_1(x) + \delta_2(x). \end{aligned}$$

By the assumption that the kernel of $b^{-1}T^*M_b$ satisfies the condition (ii) of Definition 2.4, (2.9) and (2.10), we obtain

$$\begin{aligned} |\delta_1(x)| &\le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \int_{\rho(x_0,y)\le \frac{R}{2A}} \frac{\rho(x_0,y)^\epsilon}{R^{d+\epsilon}} \frac{r^\gamma}{\rho(y,x_0)^{d+\gamma}}\,d\mu(y) \\ &\le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \frac{r^\gamma}{R^{d+\gamma}} \end{aligned}$$

by $\gamma < \epsilon$, and

$$|\delta_2(x)| \le \frac{C}{R^d}\left|\int_X f_2(y)b(y)\,d\mu(y)\right| \le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \frac{r^\gamma}{R^{d+\gamma}},$$

since $b^{-1} \in L^\infty(X)$. Thus, we also obtain a desired estimate for $T(f_2)(x)$.

Similarly, since $x \notin \operatorname{supp} f_3$, the estimate (2.6) and the size condition of $K(x,y)$ then give us a desired estimate for $T(f_3)(x)$; see [11] for the details.

Thus, $Tf(x)$ also satisfies the size condition (i) of Definition 2.3 when $\rho(x,x_0) > 5r$.

Next, we verify that $T(f)(x)$ satisfies the smoothness condition (ii) of Definition 2.3. To this end, set $\rho(x,x_0) = R$ and $\rho(x,x') = \delta$. We only consider the case $R \ge 10r$ and $\delta \le \frac{1}{20A^2}(r+R)$. The other cases can be proved by a similar but easier argument. As in the above, set $1 = I(y) + J(y) + L(y)$, however, here $I(y) = \kappa\left(\frac{8A\rho(x,y)}{R}\right)$ and $J(y) = \kappa\left(\frac{8A\rho(x_0,y)}{R}\right)$. Moreover, we also set $f_1(y) = f(y)I(y)$, $f_2(y) = f(y)J(y)$ and $f_3(y) = f(y)L(y)$.

By Lemma 2.1 in [11], $T(1) = 0$, (2.6), the assumption on the kernel of T, we can obtain a desired estimate for $T(f_1)(x)$ and $T(f_3)(x)$ by the same argument as in [11].

Noting that for $\rho(x, x') = \delta \le \frac{1}{20A^2}(r+R)$ and $R \ge 10r$, $x,\ x' \notin \operatorname{supp}(f_2)$, by the assumption (2.4) and the estimates (2.9) and (2.10) for f_2, we obtain

$$\begin{aligned}
&|T(f_2)(x) - T(f_2)(x')| \\
&\quad = \left| \int_X \left[K(x,y) - K(x',y)\right] b^{-1}(y) f_2(y) b(y)\, d\mu(y) \right| \\
&\quad \le \int_X \big| \left[K(x,y) - K(x',y)\right] b^{-1}(y) \\
&\qquad - \left[K(x,x_0) - K(x',x_0)\right] b^{-1}(x_0) \big| \, |f_2(y) b(y)| \, d\mu(y) \\
&\qquad + \left| \left[K(x,x_0) - K(x',x_0)\right] b^{-1}(x_0) \right| \left| \int_X f_2(y) b(y)\, d\mu(y) \right| \\
&\quad \le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \left\{ \int_{\rho(x_0,y) \le \frac{R}{4A}} \frac{\rho(x,x')^\epsilon \rho(y,x_0)^\epsilon}{R^{d+2\epsilon}} \right. \\
&\qquad \left. \times \frac{r^\gamma}{(r+\rho(y,x_0))^{d+\gamma}}\, d\mu(y) + C\frac{\delta^\epsilon}{R^{d+\epsilon}} \frac{r^\gamma}{R^\gamma} \right\} \\
&\quad \le C\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \frac{\delta^\epsilon}{R^\epsilon} \frac{r^\gamma}{R^{d+\gamma}},
\end{aligned}$$

since $\gamma < \epsilon$, and $b,\ b^{-1} \in L^\infty(X)$, which is also a desired estimate. This finishes the proof of Theorem 2.1.

3. Calderón Reproducing Formulae

The main purpose of this section is to establish the continuous Calderón reproducing formulae associated to a given para-accretive function by using the results in Section 2. Another key tool for this is Coifman's idea; see [4]. We first recall the definition of approximations to the identity in [9].

Definition 3.1. Let b be a para-accretive function. A sequence $\{S_k\}^b_{k\in\mathbb{Z}_+}$ of linear operators is said to be an approximation to the identity of order $\epsilon \in (0, \theta]$ associated to b if there exists $C_5 > 0$ such that for all $k \in \mathbb{Z}_+$ and all $x,\ x',\ y$ and $y' \in X$, $S_k(x,y)$, the kernel of S_k is a function from $X \times X$ into $\mathbb{C}$ satisfying

(i) $|S_k(x,y)| \le C_5 \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$;

(ii) $|S_k(x,y) - S_k(x',y)| \le C_5 \left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$ for $\rho(x,x') \le \frac{1}{2A}(2^{-k}+\rho(x,y))$;

(iii) $|S_k(x,y)-S_k(x,y')|\le C_5\left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$ for $\rho(y,y')\le\frac{1}{2A}(2^{-k}+\rho(x,y))$;

(iv) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|\le C_5\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\times\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$ for $\rho(x,x')\le\frac{1}{2A}(2^{-k}+\rho(x,y))$ and $\rho(y,y')\le\frac{1}{2A}(2^{-k}+\rho(x,y))$;

(v) $\int_X S_k(x,y)b(y)\,d\mu(y)=1$;

(vi) $\int_X S_k(x,y)b(x)\,d\mu(x)=1$.

Remark 3.1. By Coifman's construction in [4], if b is a given para-accretive function, one can construct an approximation to the identity of order θ such that $S_k(x,y)$ has a compact support when one variable is fixed, namely, there is a constant $C_6>0$ such that for all $k\in\mathbb{Z}$, $S_k(x,y)=0$ if $\rho(x,y)\ge C_6 2^{-k}$.

Remark 3.2. We also remark that in the sequel, if the approximation to the identity as in Definition 3.1 exists, then all the results still hold when b and b^{-1} are bounded. It seems that we do not need to assume that b is a para-accretive function. However, in [4], it was proved that the existence of the approximation to the identity as in Definition 3.1 is equivalent to the para-accretivity of b.

In what follows, we let $a\wedge b=\min(a,b)$ for any $a,\ b\in\mathbb{R}$.

Lemma 3.1. *Let b be a given para-accretive function, $\epsilon\in(0,\theta]$, $\{S_k\}_{k\in\mathbb{Z}_+}$ and $\{E_k\}_{k\in\mathbb{Z}_+}$ be two approximations to the identity of order ϵ associated to b, $P_0=S_0$, $Q_0=E_0$, $P_k=S_k-S_{k-1}$ and $Q_k=E_k-E_{k-1}$ for $k\in\mathbb{N}$. Then for any $\epsilon'\in(0,\epsilon)$, there exist constants $C>0$, $\delta>0$ and $\sigma>0$ such that $P_l\mathcal{M}_bQ_k(x,y)$, the kernel of $P_l\mathcal{M}_bQ_k$, satisfies the following estimates:*

$$|P_l\mathcal{M}_bQ_k(x,y)|\le C2^{-|k-l|\epsilon'}\frac{2^{-(k\wedge l)\epsilon}}{(2^{-(k\wedge l)}+\rho(x,y))^{d+\epsilon}}\tag{3.1}$$

for all $x,\ y\in X$ and all $k,\ l\in\mathbb{Z}_+$;

$$|P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x,y')|\le C2^{-|k-l|\delta}\left(\frac{\rho(y,y')}{2^{-(l\wedge k)}+\rho(x,y)}\right)^{\epsilon'}\times\frac{2^{-(k\wedge l)\epsilon}}{(2^{-(k\wedge l)}+\rho(x,y))^{d+\epsilon}}\tag{3.2}$$

for $\rho(y,y')\le\frac{1}{4A^2}\rho(x,y)$ and all $k,\ l\in\mathbb{Z}_+$;

$$
\begin{aligned}
|P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x',y)| &\le C2^{-|k-l|\delta}\left(\frac{\rho(x,x')}{2^{-(l\wedge k)}+\rho(x,y)}\right)^{\epsilon'} \\
&\quad\times\frac{2^{-(k\wedge l)\epsilon}}{(2^{-(k\wedge l)}+\rho(x,y))^{d+\epsilon}}
\end{aligned}
\tag{3.3}
$$

for $\rho(x,x')\le\frac{1}{4A^2}\rho(x,y)$ *and all* $k,\ l\in\mathbb{Z}_+$*; and*

$$
\begin{aligned}
&|[P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x',y)]-[P_l\mathcal{M}_bQ_k(x,y')-P_l\mathcal{M}_bQ_k(x',y')]| \\
&\quad\le C2^{-|k-l|\delta}\rho(x,x')^{\epsilon'}\rho(y,y')^{\epsilon'}\frac{2^{-(k\wedge l)\sigma}}{(2^{-(k\wedge l)}+\rho(x,y))^{d+2\epsilon'+\sigma}}
\end{aligned}
\tag{3.4}
$$

for $\rho(x,x')\le\frac{1}{8A^3}\rho(x,y)$, $\rho(y,y')\le\frac{1}{8A^3}\rho(x,y)$ *and all* $k,\ l\in\mathbb{Z}_+$.

Proof. In what follows, by the symmetry, without loss of generality, we may assume that $k\ge l$.

Let us begin with proving (3.1). If $k\in\mathbb{N}$, by

$$
\int_X b(z)Q_k(z,y)\,d\mu(z)=0 \tag{3.5}
$$

and $b\in L^\infty(X)$, we write

$$
\begin{aligned}
|P_l\mathcal{M}_bQ_k(x,y)| &= \left|\int_X P_l(x,z)b(z)Q_k(z,y)\,d\mu(z)\right| \\
&= \left|\int_X [P_l(x,z)-P_l(x,y)]b(z)Q_k(z,y)\,d\mu(z)\right| \\
&\le C\int_X |P_l(x,z)-P_l(x,y)|\,|Q_k(z,y)|\,d\mu(z).
\end{aligned}
$$

Then the same argument as that for (3.9) in [11] leads a desired estimate.

If $k=0$, then by the assumption, we have also $l=0$. In this case, the size conditions of $S_0(x,z)$ and $E_0(z,y)$ imply that

$$
\begin{aligned}
|P_0\mathcal{M}_bQ_0(x,y)| &= \left|\int_X S_0(x,z)b(z)E_0(z,y)\,d\mu(z)\right| \\
&\le C\int_{\{z\in X:\ \rho(z,y)\le\frac{1}{2A}(1+\rho(x,y))\}}|S_0(x,z)E_0(z,y)|\,d\mu(z) \\
&\quad +C\int_{\{z\in X:\ \rho(z,y)>\frac{1}{2A}(1+\rho(x,y))\}}|S_0(x,z)E_0(z,y)|\,d\mu(z) \\
&\le C\frac{1}{(1+\rho(x,y))^{d+\epsilon}},
\end{aligned}
$$

which is also a desired estimate.

We now verify (3.2). It suffices to prove

$$(3.6)\quad |P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x,y')|\le C\left(\frac{\rho(y,y')}{2^{-l}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-l\epsilon}}{(2^{-l}+\rho(x,y))^{d+\epsilon}}$$

for $k\ge l$ and $\rho(y,y')\le\frac{1}{4A^2}\rho(x,y)$. To see this, noting that if $k\ge l$ and $\rho(y,y')\le\frac{1}{4A^2}\rho(x,y)$, then (3.1) implies

$$\begin{aligned}|P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x,y')| &\le |P_l\mathcal{M}_bQ_k(x,y)|+|P_l\mathcal{M}_bQ_k(x,y')|\\ &\le C2^{-(k-l)\epsilon'}\frac{2^{-l\epsilon}}{(2^{-l}+\rho(x,y))^{d+\epsilon}},\end{aligned}$$

this estimate together with (3.6) yields that for any $\sigma\in(0,1)$,

$$\begin{aligned}&|P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x,y')|\\ &\quad=|P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x,y')|^{1-\sigma}|P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x,y')|^{\sigma}\\ &\quad\le C2^{-(k-l)\epsilon'\sigma}\left(\frac{\rho(y,y')}{2^{-l}+\rho(x,y)}\right)^{(1-\sigma)\epsilon}\frac{2^{-l\epsilon}}{(2^{-l}+\rho(x,y))^{d+\epsilon}},\end{aligned}$$

which is (3.2). In what follows, we refer this method to the geometric mean. We now verify (3.6). From (3.5) when $k\in\mathbb{N}$ or

$$(3.7)\qquad \int_X Q_0(z,y)b(z)\,d\mu(z)=1=\int_X Q_0(z,y')b(z)\,d\mu(z)$$

when $k=0$ and $b\in L^\infty(X)$, it follows that

$$\begin{aligned}&|P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x,y')|\\ &\quad=\left|\int_X[P_l(x,z)-P_l(x,y)]b(z)[Q_k(z,y)-Q_k(z,y')]\,d\mu(z)\right|\\ &\quad\le C\int_X|P_l(x,z)-P_l(x,y)|\left|Q_k(z,y)-Q_k(z,y')\right|\,d\mu(z).\end{aligned}$$

Then, repeating the proof of (3.13) in [11] leads the estimate (3.6).

The proof of (3.3) is similar to that of (3.2) and we omit the details.

We now verifies (3.4). Similar to the proof of (3.2), we only need to prove

$$(3.8)\quad\begin{aligned}&|[P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x',y)]-[P_l\mathcal{M}_bQ_k(x,y')-P_l\mathcal{M}_bQ_k(x',y')]|\\ &\quad\le C\rho(x,x')^{\epsilon'}\rho(y,y')^{\epsilon'}\frac{2^{-(l\wedge k)\sigma}}{(2^{-(l\wedge k)}+\rho(x,y))^{d+2\epsilon'+\sigma}}\end{aligned}$$

for $\rho(x,x') \le \frac{1}{8A^3}\rho(x,y)$ and $\rho(y,y') \le \frac{1}{8A^3}\rho(x,y)$. To see this, if $\rho(x,x') \le \frac{1}{8A^3}\rho(x,y)$ and $\rho(y,y') \le \frac{1}{8A^3}\rho(x,y)$, (3.2) and (3.3) tell us that

$$\begin{aligned}&|[P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x',y)]-[P_l\mathcal{M}_bQ_k(x,y')-P_l\mathcal{M}_bQ_k(x',y')]|\\&\quad\le C2^{-|k-l|\delta}\left(\frac{\rho(x,x')}{2^{-(l\wedge k)}+\rho(x,y)}\right)^{\epsilon'}\frac{2^{-(l\wedge k)\epsilon}}{(2^{-(l\wedge k)}+\rho(x,y))^{d+\epsilon}}\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}&|[P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x',y)]-[P_l\mathcal{M}_bQ_k(x,y')-P_l\mathcal{M}_bQ_k(x',y')]|\\&\quad\le C2^{-|k-l|\delta}\left(\frac{\rho(y,y')}{2^{-(l\wedge k)}+\rho(x,y)}\right)^{\epsilon'}\frac{2^{-(l\wedge k)\epsilon}}{(2^{-(l\wedge k)}+\rho(x,y))^{d+\epsilon}}.\end{aligned}\tag{3.10}$$

Then similarly to the proof of (3.2), the geometric mean of the estimates (3.8), (3.9) and (3.10) gives (3.4).

Let us now verify (3.8). By (3.5) when $k \in \mathbb{N}$ or (3.7) when $k = 0$ and $b \in L^\infty(X)$, we can write

$$\begin{aligned}&|[P_l\mathcal{M}_bQ_k(x,y)-P_l\mathcal{M}_bQ_k(x',y)]-[P_l\mathcal{M}_bQ_k(x,y')-P_l\mathcal{M}_bQ_k(x',y')]|\\&=\left|\int_X\left\{[P_l(x,z)-P_l(x',z)]-[P_l(x,y)-P_l(x',y)]\right\}b(z)[Q_k(z,y)-Q_k(z,y')]\,d\mu(z)\right|\\&\le C\int_X\left|[P_l(x,z)-P_l(x',z)]-[P_l(x,y)-P_l(x',y)]\right||Q_k(z,y)-Q_k(z,y')|\,d\mu(z).\end{aligned}$$

Then, repeating the proof of (3.15) in [11] yields (3.7), and we have completed the proof of Lemma 3.1.

To establish the continuous Calderón reproducing formulae, we use the Coifman's idea; see [4]. In what follows, let $\{S_k\}_{k\in\mathbb{Z}_+}$ be an approximation to the identity of order $\epsilon \in (0,\theta]$ as in Definition 3.1. Set $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$, $D_0 = S_0$ and $D_k = 0$ for $k \in \mathbb{Z}\setminus\mathbb{Z}_+$. It is easy to see that

$$I=\sum_{k=0}^{\infty}D_k\mathcal{M}_b$$

in $L^2(X)$; see [9]. Let $N \in \mathbb{N}$. Coifman's idea is to rewrite the above identity into

$$\begin{aligned}I&=\left(\sum_{k=0}^{\infty}D_k\mathcal{M}_b\right)\left(\sum_{j=0}^{\infty}D_j\mathcal{M}_b\right)\\&=\sum_{|l|>N}\sum_{k=0}^{\infty}D_{k+l}\mathcal{M}_bD_k\mathcal{M}_b+\sum_{k=0}^{\infty}D_k^N\mathcal{M}_bD_k\mathcal{M}_b\\&=R_N+T_N,\end{aligned}\tag{3.11}$$

where

$$D_k^N = \sum_{|l|\le N} D_{k+l}.$$

To verify that T_N^{-1} exists and maps any space of test functions to itself, we first need to estimate the kernel, $R_N(x,y)$, of the operator R_N.

Lemma 3.2. *Let b be a para-accretive function, $\epsilon \in (0,\theta]$, R_N be as in (3.11) and $R_N(x,y)$ be its kernel. Then for any $\epsilon' \in (0,\epsilon)$, $R_N \in CZO(\epsilon') \cap SWBP$, $R_N(1) = 0$ and $(R_N)^*(b) = 0$. Moreover, $R_N(x,y)$ satisfies the following estimates: for any $\epsilon' \in (0,\epsilon)$, there exist constants $C > 0$ and $\delta > 0$, which are independent of N, such that*

$$|R_N(x,y)| \le C2^{-N\delta}\rho(x,y)^{-d}, \tag{3.12}$$

$$|R_N(x,y)b^{-1}(y) - R_N(x,y')b^{-1}(y')| \le C2^{-N\delta}\rho(y,y')^{\epsilon'}\rho(x,y)^{-(d+\epsilon')} \tag{3.13}$$

for $\rho(y,y') \le \frac{1}{2A}\rho(x,y)$,

$$|R_N(x,y) - R_N(x',y)| \le C2^{-N\delta}\rho(x,x')^{\epsilon'}\rho(x,y)^{-(d+\epsilon')} \tag{3.14}$$

for $\rho(x,x') \le \frac{1}{2A}\rho(x,y)$,

$$\begin{aligned} &|[R_N(x,y) - R_N(x',y)]b^{-1}(y) - [R_N(x,y') - R_N(x',y')]b^{-1}(y')| \\ &\quad\le C2^{-N\delta}\rho(x,x')^{\epsilon'}\rho(y,y')^{\epsilon'}\rho(x,y)^{-(d+2\epsilon')} \end{aligned} \tag{3.15}$$

for $\rho(x,x'),\ \rho(y,y') \le \frac{1}{3A^2}\rho(x,y)$, and

$$|\langle R_N, f\rangle| \le C2^{-N\delta}r^d \tag{3.16}$$

for all $r > 0$ and all continuous functions f on $X \times X$ with $\operatorname{supp} f \subseteq B(x_0,r) \times B(y_0,r)$, where $x_0,\ y_0 \in X$, $\|f\|_{L^\infty(X\times X)} \le 1$, $\|f(\cdot,y)\|_{C_0^\eta(X)} \le r^{-\eta}$ for all $y \in X$ and $\|f(x,\cdot)\|_{C_0^\eta(X)} \le r^{-\eta}$ for all $x \in X$.

Proof. To verify (3.12), (3.13), (3.14) and (3.15), we write

$$\begin{aligned} R_N(x,y) &= \sum_{l=N+1}^{\infty}\sum_{k=0}^{\infty}(D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b)(x,y) + \sum_{l=-\infty}^{-N-1}\sum_{k=0}^{\infty}(D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b)(x,y) \\ &= R_N^1(x,y) + R_N^2(x,y). \end{aligned}$$

We only verify $R_N^1(x,y)$ satisfying (3.12), (3.13), (3.14) and (3.15). The proof for $R_N^2(x,y)$ is similar. In what follows, let $[t]$ be the maximum integer no more than t for any $t \in \mathbb{R}$.

If $\rho(x,y) \geq 1$, by (3.1) and $b \in L^\infty(X)$, we have

$$\begin{aligned}|R_N^1(x,y)| &\leq C \sum_{l=N+1}^{\infty} \sum_{k=0}^{\infty} 2^{-l\epsilon'} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}} \\ &\leq \frac{C}{\rho(x,y)^{d+\epsilon}} \sum_{l=N+1}^{\infty} 2^{-l\epsilon'} \sum_{k=0}^{\infty} 2^{-k\epsilon} \\ &\leq C2^{-N\epsilon'}\rho(x,y)^{-d},\end{aligned}$$

which is a desired estimate. If $\rho(x,y) < 1$, let

$$k_0 = [-\log_2 \rho(x,y)] \tag{3.17}$$

be the maximal integer no more than $-\log_2 \rho(x,y)$. Then $k_0 \in \mathbb{Z}_+$. We then have

$$\begin{aligned}|R_N^1(x,y)| &\leq C \sum_{l=N+1}^{\infty} 2^{-l\epsilon'} \left[\sum_{k=0}^{k_0} 2^{kd} + \frac{1}{\rho(x,y))^{d+\epsilon}} \sum_{k=k_0+1}^{\infty} 2^{-k\epsilon}\right] \\ &\leq C2^{-N\epsilon'}\rho(x,y)^{-d}.\end{aligned}$$

Thus, (3.12) holds.

Let us now verify (3.13) with $R_N^1(x,y)$. We only prove this when $\rho(y,y') \leq \frac{1}{4A^2}\rho(x,y)$. The case $\frac{1}{4A^2}\rho(x,y) \leq \rho(y,y') \leq \frac{1}{2A}\rho(x,y)$ follows from (3.12). We also consider two cases. If $\rho(x,y) \geq 1$, then (3.2) tells us that

$$\begin{aligned}&|R_N^1(x,y)b^{-1}(y) - R_N^1(x,y')b^{-1}(y')| \\ &\leq C \sum_{l=N+1}^{\infty} \sum_{k=0}^{\infty} 2^{-l\delta} \left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon'} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}} \\ &\leq C\rho(y,y')^{\epsilon'} \frac{1}{\rho(x,y)^{d+\epsilon+\epsilon'}} \sum_{l=N+1}^{\infty} 2^{-l\delta} \left(\sum_{k=0}^{\infty} 2^{-k\epsilon}\right) \\ &\leq C2^{-N\delta}\rho(y,y')^{\epsilon'}\rho(x,y)^{-(d+\epsilon')},\end{aligned}$$

which is a desired estimate. If $\rho(x,y) < 1$, letting k_0 be as in (3.17), we then obtain

$$\begin{aligned}&|R_N^1(x,y)b^{-1}(y) - R_N^1(x,y')b^{-1}(y')| \\ &\leq C\rho(y,y')^{\epsilon'} \sum_{l=N+1}^{\infty} 2^{-l\delta} \left[\sum_{k=0}^{k_0} 2^{k(\epsilon'+d)} + \frac{1}{\rho(x,y)^{d+\epsilon+\epsilon'}} \sum_{k=k_0+1}^{\infty} 2^{-k\epsilon}\right] \\ &\leq C2^{-N\delta}\rho(y,y')^{\epsilon'}\rho(x,y)^{-(d+\epsilon')},\end{aligned}$$

which verifies (3.13). The proof of (3.14) is similar to that of (3.13).

For the proof of (3.15), we only need to verify this under the assumptions $\rho(x,x') \le \frac{1}{8A^3}\rho(x,y)$ and $\rho(y,y') \le \frac{1}{8A^3}\rho(x,y)$. The other cases can be easily deduced from (3.13) and (3.14). In this case, we also consider two cases. If $\rho(x,y) \ge 1$, (3.4) yields

$$\begin{aligned}
&|[R_N^1(x,y) - R_N^1(x',y)]b^{-1}(y) - [R_N^1(x,y') - R_N^1(x',y')]b^{-1}(y')|\\
&\quad \le C\rho(x,x')^{\epsilon'}\rho(y,y')^{\epsilon'} \sum_{l=N+1}^{\infty} 2^{-l\delta} \sum_{k=0}^{\infty} \frac{2^{-k\sigma}}{(2^{-k}+\rho(x,y))^{d+2\epsilon'+\sigma}}\\
&\quad \le C\rho(x,x')^{\epsilon'}\rho(y,y')^{\epsilon'}\rho(x,y)^{-(d+2\epsilon'+\sigma)}2^{-l\delta}\sum_{k=0}^{\infty}2^{-k\sigma}\\
&\quad \le C2^{-l\delta}\rho(x,x')^{\epsilon'}\rho(y,y')^{\epsilon'}\rho(x,y)^{-(d+2\epsilon')},
\end{aligned}$$

which is what we want. If $\rho(x,y) < 1$, letting k_0 be as in (3.17), by (3.4), we have

$$\begin{aligned}
&|[R_N^1(x,y) - R_N^1(x',y)] - [R_N^1(x,y') - R_N^1(x',y')]|\\
&\quad \le C\rho(x,x')^{\epsilon'}\rho(y,y')^{\epsilon'} \sum_{l=N+1}^{\infty} 2^{-l\delta} \left[\sum_{k=0}^{k_0} 2^{k(d+2\epsilon')} + \frac{1}{\rho(x,y)^{d+2\epsilon'+\sigma}} \sum_{k=k_0+1}^{\infty} 2^{-k\sigma}\right]\\
&\quad \le C2^{-l\delta}\rho(x,x')^{\epsilon'}\rho(y,y')^{\epsilon'}\rho(x,y)^{-(d+2\epsilon')},
\end{aligned}$$

which proves (3.15).

It remains to verify (3.16). Suppose that $f \in C_0^\eta(X\times X)$ with

$$\operatorname{supp} f \subseteq B(x_0,r)\times B(y_0,r),$$

where x_0 and $y_0 \in X$, $\|f\|_{L^\infty(X\times X)} \le 1$, $\|f(\cdot,y)\|_{C_0^\eta(X)} \le r^{-\eta}$ for all $y \in X$ and

$$\|f(x,\cdot)\|_{C_0^\eta(X)} \le r^{-\eta}$$

for all $x \in X$. By (3.1) and $b \in L^\infty(X)$, we have

$$\begin{aligned}
&|\langle D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b, f\rangle|\\
&\quad = \left|\int_X\int_X (D_{k+l}\mathcal{M}_b D_k)(x,y)b(y)f(x,y)\,d\mu(y)\,d\mu(x)\right|\\
&\quad \le C2^{-|l|\epsilon'}\|f\|_{L^\infty(X\times X)}\int_{B(x_0,r)}\left[\int_X \frac{2^{-(k\wedge l)\epsilon}}{(2^{-(k\wedge l)}+\rho(x,y))^{d+\epsilon}}\,d\mu(y)\right]\,d\mu(x)\\
&\quad \le C2^{-|l|\epsilon'}\mu(B(x_0,r))\\
&\quad \le C2^{-|l|\epsilon'}r^d.
\end{aligned}\tag{3.18}$$

On the other hand, for $k \in \mathbb{N}$ and $\eta < \epsilon$, (3.1), $b \in L^\infty(X)$ and the fact that

$$\int_X D_k(z,y)b(y)\,d\mu(y) = 0$$

tell us that

$$\begin{aligned}
&|\langle D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b, f\rangle| \\
&= \left|\int_X\int_X\int_X D_{k+l}(x,z)b(z)D_k(z,y)b(y)f(x,y)\,d\mu(z)\,d\mu(y)\,d\mu(x)\right| \\
&= \left|\int_X\int_X\int_X D_{k+l}(x,z)b(z)D_k(z,y)b(y)[f(x,y)-f(x,z)]\,d\mu(z)\,d\mu(y)\,d\mu(x)\right| \\
&\le Cr^{-\eta}\int_{B(x_0,r)}\left\{\int_X |D_{k+l}(x,z)|\right. \\
&\quad \times \left.\left[\int_X |D_k(z,y)|\rho(z,y)^\eta\,d\mu(y)\right]\,d\mu(z)\right\}\,d\mu(x) \\
&\le C2^{-k\eta}r^{-\eta}r^d.
\end{aligned} \tag{3.19}$$

We have also, by Definition 3.1 and $b \in L^\infty(X)$, that

$$\begin{aligned}
&|\langle D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b, f\rangle| \\
&= \left|\int_X\int_X\int_X D_{k+l}(x,z)b(z)D_k(z,y)b(y)f(x,y)\,d\mu(z)\,d\mu(y)\,d\mu(x)\right| \\
&\le C\|f\|_{L^\infty(X)}\|D_k\|_{L^\infty(X\times X)} \\
&\quad \times \int_{B(x_0,r)}\int_{B(y_0,r)}\left[\int_X |D_{k+l}(x,z)|\,d\mu(z)\right]\,d\mu(y)\,d\mu(x) \\
&\le C2^{kd}r^{2d}.
\end{aligned} \tag{3.20}$$

The geometric mean of (3.18) and (3.19) yields

$$|\langle D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b, f\rangle| \le C2^{-|l|\delta}2^{-k\eta'}r^{-\eta'}r^d \tag{3.21}$$

for all $k \in \mathbb{N}$ and some positive δ and η', and the geometric mean of (3.18) and (3.20) yields

$$|\langle D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b, f\rangle| \le C2^{-|l|\delta}2^{-k\eta''}r^{-\eta''}r^d \tag{3.22}$$

for all $k \in \mathbb{N}$ and some positive δ and η''. Thus, (3.18), (3.21) and (3.22) tell us that

$$|\langle R_N, f\rangle| = \left|\left\langle \sum_{|l|>N}\sum_{k=0}^{\infty} D_{k+l}\mathcal{M}_b D_k\mathcal{M}_b, f\right\rangle\right|$$

$$
\begin{aligned}
&\le \left|\left\langle \sum_{l>N} D_l\mathcal{M}_b D_0\mathcal{M}_b, f\right\rangle\right| + \left|\left\langle \sum_{|l|>N}\sum_{2^{-k}>r} D_{k+l}\mathcal{M}_b D_k\mathcal{M}_b, f\right\rangle\right| \\
&\quad + \left|\left\langle \sum_{|l|>N}\sum_{2^{-k}\le r} D_{k+l}\mathcal{M}_b D_k\mathcal{M}_b, f\right\rangle\right| \\
&\le C2^{-N\delta}r^d,
\end{aligned}
$$

which verifies (3.16), and, hence, Lemma 3.2.

As a consequence of Lemma 3.2 and Theorem 2.1, we obtain the following theorem.

Theorem 3.1. *Let b be a given para-accretive function. Suppose that $\{S_k\}_{k=0}^\infty$ is an approximation to the identity of order ϵ as in Definition 3.1. For $N\in\mathbb{N}$, let T_N be as in (3.11). If N is large enough, then T_N^{-1} exists and maps any space of test functions to itself. More precisely, if N is sufficiently large, then there exists a constant $C>0$ such that for all $f\in\mathcal{G}_0^b(x_1,r,\beta,\gamma)$ with $x_1\in X$, $r>0$ and $0<\beta,\ \gamma<\epsilon$,*

$$\|T_N^{-1}(f)\|_{\mathcal{G}(x_1,r,\beta,\gamma)} \le C\|f\|_{\mathcal{G}(x_1,r,\beta,\gamma)}.$$

Proof. By Theorem 2.1 and Lemma 3.2, there exist constants $C_7>0$ and $\delta>0$, which are independent of N, such that for all $f\in\mathcal{G}_0^b(x_1,r,\beta,\gamma)$ with $x_1\in X$, $r>0$ and $0<\beta,\ \gamma<\epsilon$,

$$\|R_N(f)\|_{\mathcal{G}(x_1,r,\beta,\gamma)} \le C_7 2^{-N\delta}\|f\|_{\mathcal{G}(x_1,r,\beta,\gamma)}.$$

If we choose $N\in\mathbb{N}$ such that

$$C_7 2^{-N\delta}<1, \tag{3.23}$$

then we have that for all $f\in\mathcal{G}_0^b(x_1,r,\beta,\gamma)$,

$$
\begin{aligned}
\left\|T_N^{-1}(f)\right\|_{\mathcal{G}_0^b(x_1,r,\beta,\gamma)} &= \left\|(I-R_N)^{-1}(f)\right\|_{\mathcal{G}_0^b(x_1,r,\beta,\gamma)} \\
&= \left\|\sum_{l=0}^\infty (R_N)^l(f)\right\|_{\mathcal{G}_0^b(x_1,r,\beta,\gamma)} \\
&\le \sum_{l=0}^\infty \left(C_7 2^{-N\delta}\right)^l \|f\|_{\mathcal{G}(x_1,r,\beta,\gamma)} \\
&\le C\|f\|_{\mathcal{G}(x_1,r,\beta,\gamma)},
\end{aligned}
$$

which completes the proof of Theorem 3.1.

We can now prove the inhomogeneous continuous Calderón reproducing formulae on spaces of homogenous type.

Theorem 3.2. *Let b be a para-accretive function, $\epsilon \in (0,\theta]$, $\{S_k\}_{k=0}^{\infty}$ be an approximation to the identity of order ϵ associated to b. Set $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. Then there exist a family of linear operators $\widetilde{D}_k$ for $k \in \mathbb{Z}_+$ and a fixed large integer $N \in \mathbb{N}$ such that for all $f \in \mathcal{G}(\beta,\gamma)$ with $0 < \beta,\ \gamma < \epsilon$,*

$$f = \sum_{k=0}^{\infty} \widetilde{D}_k \mathcal{M}_b D_k \mathcal{M}_b(f), \tag{3.24}$$

where the series converge in the norm of $\mathcal{G}(\beta',\gamma')$ for $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$. Moreover, (3.24) also converge in the norm of $L^p(X)$ for $p \in (1,\infty)$, and the kernels of the operators $\widetilde{D}_k$ satisfy the conditions (i) and (ii) of Definition 3.1 with ϵ replaced by ϵ' for $0 < \epsilon' < \epsilon$, and

$$\int_X \widetilde{D}_k(x,y)b(y)\,d\mu(y) = \int_X \widetilde{D}_k(x,y)b(y)\,d\mu(x) = \begin{cases} 1, & k = 0,1,\cdots,N; \\ 0, & k \ge N+1. \end{cases}$$

Proof. Fix a large integer N such that (3.23) and, therefore, Theorem 3.1 holds. It is easy to check that $D_j(\cdot,y) \in \mathcal{G}_0^b(y,2^{-j},\epsilon,\epsilon)$ for all $j \in \mathbb{N}$ and $D_k^N(\cdot,y) \in \mathcal{G}_0^b(y,2^{-j},\epsilon,\epsilon)$ for $k > N$. Set $\widetilde{D}_k(x,y) = T_N^{-1}(D_k^N(\cdot,y))(x)$ for $k \in \mathbb{Z}_+$, where T_N^{-1} is defined as Theorem 3.1. Notice that

$$D_k^N = \sum_{|j|\le N} D_{k+j} = \sum_{j=0}^{k+N} D_j$$

for $k = 0,1,\cdots,N$, and

$$D_k^N = \sum_{|j|\le N} D_{k+j} = \sum_{j=k-N}^{k+N} D_j$$

for $k = N+1,\cdots$. By Theorem 3.1, when $k = N+1,\cdots$, $\widetilde{D}_k(\cdot,y) \in \mathcal{G}(y,2^{-k},\epsilon',\epsilon')$ with $\epsilon' \in (0,\epsilon)$ (In fact, in this case, this is also true for $\epsilon' = \epsilon$) and this proves that $\widetilde{D}_k(x,y)$, the kernel of $\widetilde{D}_k$, satisfies the conditions (i) and (ii) of Definition 3.1 with ϵ replaced by ϵ', and it is easy to see that

$$\int_X \widetilde{D}_k(x,y)b(x)\,d\mu(x) = 0,$$

since $(R_N)^*(b) = 0$. Obviously we also have

$$\int_X \widetilde{D}_k(x,y)b(y)\,d\mu(y) = 0$$

for $k = N+1, \cdots$. If $k = 0,\ 1,\ \cdots,\ N$, since

$$\begin{aligned}\widetilde{D}_k(x,y) &= T_N^{-1}\left(D_k^N(\cdot,y)\right)(x)\\ &= T_N^{-1}\left(\sum_{j=0}^{k+N} D_j(\cdot,y)\right)\\ &= \sum_{j=0}^{k+N} T_N^{-1}\left(D_j(\cdot,y)\right)(x)\\ &= T_N^{-1}(S_0(\cdot,y))(x) + \sum_{j=1}^{k+N} T_N^{-1}\left(D_j(\cdot,y)\right)(x),\end{aligned}$$

to verify that $\widetilde{D}_k(x,y)$ satisfies the conditions (i) and (ii) of Definition 3.1 with ϵ replaced by ϵ' and

$$\int_X \widetilde{D}_k(x,y)b(y)\,d\mu(y) = \int_X \widetilde{D}_k(x,y)b(x)\,d\mu(x) = 1,$$

it suffices to prove that $T_N^{-1}(S_0(\cdot,y))(x)$ satisfies the conditions (i) and (ii) of Definition 3.1 with ϵ replaced by ϵ' and

$$\int_X T_N^{-1}(S_0(\cdot,y))(x)b(y)\,d\mu(y) = \int_X T_N^{-1}(S_0(\cdot,y))(x)b(x)\,d\mu(x) = 1, \tag{3.25}$$

since, by Theorem 3.1, $T_N^{-1}(D_j(\cdot,y)) \in \mathcal{G}(y, 2^{-j}, \epsilon', \epsilon')$ for any $\epsilon' \in (0,\epsilon)$ (In fact, in this case, this is also true for $\epsilon' = \epsilon$) and $j = 1, \cdots, k+N$. To estimate $T_N^{-1}(S_0(\cdot,y))(x)$, Theorem 3.1 can not be directly applied since $S_0(\cdot,y) \notin \mathcal{G}_0^b(y, 2^{-j}, \epsilon', \epsilon')$. Let us first verify that for any $\epsilon' \in (0,\epsilon)$, there are constants $C > 0$ and $\delta > 0$ such that

$$|(R_N S_0)(x,y)| \le C 2^{-N\delta} \frac{1}{(1+\rho(x,y))^{d+\epsilon}}, \tag{3.26}$$

and

$$|(R_N S_0)(x,y) - (R_N S_0)(x',y)| \le C 2^{-N\delta} \left(\frac{\rho(x,x')}{1+\rho(x,y)}\right)^{\epsilon'} \frac{1}{(1+\rho(x,y))^{d+\epsilon}} \tag{3.27}$$

for $\rho(x,x') \le \frac{1}{4A^2}(1+\rho(x,y))$. The estimates (3.26), (3.27) and $(R_N)^*(b) = 0$ imply that

$$(R_N S_0)(\cdot,y) \in \mathcal{G}_0^b(y, 1, \epsilon', \epsilon').$$

Then Theorem 3.1, together with the fact that

$$T_N^{-1}(S_0(\cdot, y))(x) = \sum_{l=0}^{\infty}(R_N)^l S_0(x, y),$$

yields that if N is sufficiently large, then $T_N^{-1}(S_0(\cdot, y))(x)$ satisfies the conditions (i) of Definition 3.1 and (ii) of Definition 3.1 for the case $\rho(x, x') \le \frac{1}{4A^2}(1 + \rho(x, y))$, where ϵ is replaced by ϵ'. However, (ii) of Definition 3.1 for the case $\frac{1}{4A^2}(1 + \rho(x, y)) \le \rho(x, x') \le \frac{1}{2A}(1 + \rho(x, y))$ follows from (i) of Definition 3.1. The fact (3.25) then follows from the facts that

$$\int_X S_0(x, y)b(y)\, d\mu(y) = \int_X S_0(x, y)b(x)\, d\mu(x) = 1$$

and $R_N(1) = (R_N)^*(b) = 0$.

To verify (3.26) and (3.27), it suffices to verify that for $k \in \mathbb{Z}_+$ and $k + l \in \mathbb{Z}_+$,

$$|(D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b S_0)(x, y)| \le C2^{-k\epsilon'} \frac{1}{(1 + \rho(x, y))^{d+\epsilon}}, \tag{3.28}$$

$$|(D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b S_0)(x, y)| \le C2^{-|l|\epsilon'} \frac{1}{(1 + \rho(x, y))^{d+\epsilon}}, \tag{3.29}$$

and

$$\begin{aligned} &|(D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b S_0)(x, y) - (D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b S_0)(x', y)| \\ &\quad \le C \left(\frac{\rho(x, x')}{1 + \rho(x, y)}\right)^{\epsilon} \frac{1}{(1 + \rho(x, y))^{d+\epsilon}} \end{aligned} \tag{3.30}$$

for $\rho(x, x') \le \frac{1}{4A^2}(1 + \rho(x, y))$.

Assuming these estimates for the moment, by (3.28), (3.29) and the geometric mean, we obtain

$$|(D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b S_0)(x, y)| \le C2^{-\frac{1}{2}(|l|+k)\epsilon'} \frac{1}{(1 + \rho(x, y))^{d+\epsilon}}, \tag{3.31}$$

and

$$\begin{aligned} &|(D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b S_0)(x, y) - (D_{k+l}\mathcal{M}_b D_k \mathcal{M}_b S_0)(x', y)| \\ &\quad \le C2^{-\frac{1}{2}(|l|+k)\epsilon'} \frac{1}{(1 + \rho(x, y))^{d+\epsilon}} \end{aligned} \tag{3.32}$$

for $\rho(x, x') \le \frac{1}{4A^2}(1 + \rho(x, y))$. The geometric mean of (3.30) and (3.32) yields

$$(3.33)\quad \begin{aligned}&|(D_{k+l}\mathcal{M}_bD_k\mathcal{M}_bS_0)(x,y)-(D_{k+l}\mathcal{M}_bD_k\mathcal{M}_bS_0)(x',y)|\\&\le C2^{-(|l|+k)\delta}\left(\frac{\rho(x,x')}{1+\rho(x,y)}\right)^{\epsilon'}\frac{1}{(1+\rho(x,y))^{d+\epsilon}}\end{aligned}$$

for $\rho(x,x')\le\frac{1}{4A^2}(1+\rho(x,y))$, any $\epsilon'\in(0,\epsilon)$ and some $\delta>0$. Then, the estimates (3.31) and (3.33), together with

$$(R_NS_0)(x,y)=\sum_{|l|>N}\sum_{k=0}^{\infty}(D_{k+l}\mathcal{M}_bD_k\mathcal{M}_bS_0)(x,y),$$

tell us (3.26) and (3.27). Thus, we still need to verify (3.28), (3.29) and (3.30). The estimate (3.1) in Lemma 3.1 and the estimate (3.6) in the proof of Lemma 3.1 tell us that $(D_k\mathcal{M}_bS_0)(x,y)$, the kernel of $D_k\mathcal{M}_bS_0$, satisfies the following estimates that for $k\in\mathbb{Z}_+$,

$$(3.34)\quad |(D_k\mathcal{M}_bS_0)(x,y)|\le C2^{-k\epsilon'}\frac{1}{(1+\rho(x,y))^{d+\epsilon}},$$

and

$$(3.35)\quad |(D_k\mathcal{M}_bS_0)(x,y)-(D_k\mathcal{M}_bS_0)(x',y)|\le C\left(\frac{\rho(x,x')}{1+\rho(x,y)}\right)^{\epsilon}\frac{1}{(1+\rho(x,y))^{d+\epsilon}}$$

for $\rho(x,x')\le\frac{1}{4A^2}(1+\rho(x,y))$. Then, (3.34) and (3.35), together with (3.1) and (3.6) again, tell us (3.28) and (3.30) for $k\in\mathbb{Z}_+$ and $k+l\in\mathbb{Z}_+$. Similarly, by first estimating the kernel of operator $D_{k+l}\mathcal{M}_bD_k$ and then the kernel of operator $D_{k+l}\mathcal{M}_bD_k\mathcal{M}_bS_0$, we can verify (3.29).

Now it remains to prove that the series in (3.24) converges in the norm of $\mathcal{G}(\beta',\gamma')$ for $0<\beta'<\beta$ and $0<\gamma'<\gamma$. Suppose first that $f\in\mathcal{G}(\beta,\gamma)$. Then, for $M>N+1$, we write

$$\begin{aligned}\sum_{k=0}^{M}\widetilde{D}_k\mathcal{M}_bD_k\mathcal{M}_b(f)&=T_N^{-1}\left(\sum_{k=0}^{M}D_k^N\mathcal{M}_bD_k\mathcal{M}_b\right)(f)\\&=T_N^{-1}\left(T_N-\sum_{k=M+1}^{\infty}D_k^N\mathcal{M}_bD_k\mathcal{M}_b\right)(f)\\&=T_N^{-1}T_N(f)-T_N^{-1}\left(\sum_{k=M+1}^{\infty}D_k^N\mathcal{M}_bD_k\mathcal{M}_b\right)(f)\\&=f-\lim_{j\to\infty}(R_N)^j(f)-T_N^{-1}\left(\sum_{k=M+1}^{\infty}D_k^N\mathcal{M}_bD_k\mathcal{M}_b\right)(f).\end{aligned}$$

Thus,

$$
\begin{aligned}
&\left\| \sum_{k=0}^{M} \widetilde{D}_k \mathcal{M}_b D_k \mathcal{M}_b(f) - f \right\|_{\mathcal{G}(\beta',\gamma')} \\
&\quad \le \lim_{j\to\infty} \|(R_N)^j(f)\|_{\mathcal{G}(\beta',\gamma')} + \left\| T_N^{-1} \left(\sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b \right)(f) \right\|_{\mathcal{G}(\beta',\gamma')}.
\end{aligned}
\tag{3.36}
$$

Similarly to (3.26) and (3.27), we can verify that

$$
|R_N(f)(x)| \le C 2^{-N\delta} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}}, \tag{3.37}
$$

and

$$
|R_N(f)(x) - R_N(f)(x')| \le C 2^{-N\delta} \left(\frac{\rho(x,x')}{1+\rho(x,x_0)} \right)^{\beta'} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}} \tag{3.38}
$$

for $\rho(x,x') \le \frac{1}{4A^2}(1+\rho(x,x_0))$. Moreover, (3.37) and (3.38) imply that

$$
|R_N(f)(x) - R_N(f)(x')| \le C 2^{-N\delta} \left(\frac{\rho(x,x')}{1+\rho(x,x_0)} \right)^{\beta'} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}} \tag{3.39}
$$

for $\rho(x,x') \le \frac{1}{2A}(1+\rho(x,x_0))$. Thus, (3.37), (3.39) and $(R_N)^*(b) = 0$ indicate that $R_N(f) \in \mathcal{G}_0^b(\beta',\gamma')$. Furthermore, Lemma 3.2 and Theorem 2.1 yield

$$
\|(R_N)^j(f)\|_{\mathcal{G}(\beta',\gamma')} \le (C_7 2^{-N\delta})^j \|f\|_{\mathcal{G}(\beta',\gamma')},
$$

which indicates the first term in the right-hand side of (3.36) equals to 0, if we choose $N \in \mathbb{N}$ such that (3.23) holds.

To prove the second term in the right-hand side of (3.36) tends to 0 as M tends to infinity, by Theorem 3.1, it suffices to establish the following estimate

$$
\left\| \sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f) \right\|_{\mathcal{G}(\beta',\gamma')} \le C 2^{-\sigma M} \|f\|_{\mathcal{G}(\beta,\gamma)} \tag{3.40}
$$

for all $0 < \beta' < \beta$, $0 < \gamma' < \gamma$ and some $\sigma > 0$, where $C > 0$ is a constant independent of f and M.

In fact, we will verify that for $0 < \beta' < \beta$ and some $\sigma > 0$, there exists a constant $C > 0$ which is independent of f and M such that

$$
\left| \sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f)(x) \right| \le C 2^{-\beta M} \|f\|_{\mathcal{G}(\beta,\gamma)} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}}, \tag{3.41}
$$

and

$$\left|\sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f)(x) - \sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f)(x')\right| \le C\|f\|_{\mathcal{G}(\beta,\gamma)} \left(\frac{\rho(x,x')}{1+\rho(x,x_0)}\right)^{\beta'} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}} \tag{3.42}$$

for $\rho(x,x') \le \frac{1}{4A^2}(1+\rho(x,x_0))$. Let us first see now how (3.41) and (3.42) imply (3.40). To see this, note that if $\rho(x,x') \le \frac{1}{4A^2}(1+\rho(x,x_0))$, then, by (3.41), we have

$$\left|\sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f)(x) - \sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f)(x')\right| \le C2^{-\beta M}\|f\|_{\mathcal{G}(\beta,\gamma)} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}}. \tag{3.43}$$

The geometric mean of (3.42) and (3.43) tells us that

$$\left|\sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f)(x) - \sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f)(x')\right| \le C2^{-\sigma M}\|f\|_{\mathcal{G}(\beta,\gamma)} \left(\frac{\rho(x,x')}{1+\rho(x,x_0)}\right)^{\beta'} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}}$$

for $\rho(x,x') \le \frac{1}{4A^2}(1+\rho(x,x_0))$, which together with (3.41) indicates (3.40).

We now verify (3.41). Denote $D_k^N \mathcal{M}_b D_k$ by E_k. By Lemma 3.1, it is easy to see that $E_k(x,y)$, the kernel of E_k, satisfies the conditions (i), (ii) and (iii) of Definition 3.1 with ϵ replaced by $\epsilon' \in (\max(\beta,\gamma),\epsilon)$, and $E_k(b) = 0$ for $k \in \mathbb{N}$. Then

$$\begin{aligned}
&\left|\sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f)(x)\right| \\
&\quad = \left|\sum_{k=M+1}^{\infty} E_k \mathcal{M}_b(f)(x)\right| \\
&\quad = \left|\sum_{k=M+1}^{\infty} \int_X E_k(x,y) b(y)[f(y)-f(x)]\, d\mu(y)\right| \\
&\quad \le C \sum_{k=M+1}^{\infty} \int_{\{y\in X:\, \rho(x,y)\le \frac{1}{2A}(1+\rho(x,x_0))\}} |E_k(x,y)[f(y)-f(x)]|\, d\mu(y) \\
&\qquad + C \sum_{k=M+1}^{\infty} \int_{\{y\in X:\, \rho(x,y)> \frac{1}{2A}(1+\rho(x,x_0))\}} |E_k(x,y)[f(y)-f(x)]|\, d\mu(y)
\end{aligned}$$

$$\leq C\|f\|_{\mathcal{G}(\beta,\gamma)}\sum_{k=M+1}^{\infty}\left\{\frac{1}{(1+\rho(x,x_0))^{d+\gamma}}\int_{\{y\in X:\,\rho(x,y)\leq\frac{1}{2A}(1+\rho(x,x_0))\}}|E_k(x,y)|\right.$$

$$\times\left(\frac{\rho(x,y)}{1+\rho(x,x_0)}\right)^{\beta}d\mu(y)+\int_{\{y\in X:\,\rho(x,y)>\frac{1}{2A}(1+\rho(x,x_0))\}}|E_k(x,y)|$$

$$\left.\times\left[\frac{1}{(1+\rho(x,y))^{d+\gamma}}+\frac{1}{(1+\rho(x,x_0))^{d+\gamma}}\right]d\mu(y)\right\}$$

$$\leq C\|f\|_{\mathcal{G}(\beta,\gamma)}\frac{1}{(1+\rho(x,x_0))^{d+\gamma}}$$

$$\times\sum_{k=M+1}^{\infty}\left\{\int_X\frac{2^{-k\epsilon'}\rho(x,y)^{\beta}}{(2^{-k}+\rho(x,y))^{d+\epsilon'}}\,d\mu(y)+\int_X\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(x,y))^{d+\epsilon'-\beta}}\,d\mu(y)\right\}$$

$$\leq C2^{-\beta M}\|f\|_{\mathcal{G}(\beta,\gamma)}\frac{1}{(1+\rho(x,x_0))^{d+\gamma}},$$

which indicates (3.41).

To verify (3.42), we write

$$\left|\sum_{k=M+1}^{\infty}D_k^N\mathcal{M}_bD_k\mathcal{M}_b(f)(x)-\sum_{k=M+1}^{\infty}D_k^N\mathcal{M}_bD_k\mathcal{M}_b(f)(x')\right|$$

$$=\left|\sum_{k=M+1}^{\infty}\int_X[E_k(x,y)-E_k(x',y)]b(y)f(y)\,d\mu(y)\right|$$

$$=\left|\sum_{k=M+1}^{\infty}\int_X[E_k(x,y)-E_k(x',y)]b(y)[f(y)-f(x)]\,d\mu(y)\right|$$

$$\leq C\sum_{k=M+1}^{\infty}\int_{V_1}|[E_k(x,y)-E_k(x',y)][f(y)-f(x)]|\,d\mu(y)$$

$$+C\sum_{k=M+1}^{\infty}\int_{V_2}\cdots+C\sum_{k=M+1}^{\infty}\int_{V_3}\cdots$$

$$=N_1+N_2+N_3,$$

where

$$V_1=\left\{y\in X:\ \rho(x,x')\leq\frac{1}{4A^2}(1+\rho(x,x_0))\leq\frac{1}{2A}(2^{-k}+\rho(x,y))\right\},$$

$$V_2=\left\{y\in X:\ \rho(x,x')\leq\frac{1}{2A}(2^{-k}+\rho(x,y))\leq\frac{1}{4A^2}(1+\rho(x,x_0))\right\},$$

and

$$V_3 = \left\{ y \in X : \ \rho(x, x') > \frac{1}{2A}(2^{-k} + \rho(x, y)) \right\}.$$

From the smoothness condition of E_k and the size condition of f, it follows that

$$\begin{aligned}
N_1 &\le C \sum_{k=M+1}^{\infty} \int_{V_1} \left(\frac{\rho(x, x')}{2^{-k} + \rho(x, y)} \right)^{\epsilon'} \frac{2^{-k\epsilon'}}{(2^{-k} + \rho(x, y))^{d+\epsilon'}} \\
&\quad \times \{|f(y)| + |f(x)|\} \, d\mu(y) \\
&\le C \left(\frac{\rho(x, x')}{1+\rho(x, x_0)} \right)^{\epsilon'} \sum_{k=M+1}^{\infty} \left\{ \frac{2^{-k\epsilon'}}{(1+\rho(x, x_0))^{d+\epsilon'}} \int_{V_1} |f(y)| \, d\mu(y) \right. \\
&\quad \left. + |f(x)| \int_{V_1} \frac{2^{-k\epsilon'}}{(2^{-k} + \rho(x, y))^{d+\sigma}} \, d\mu(y) \right\} \\
&\le C \left(\frac{\rho(x, x')}{1 + \rho(x, x_0)} \right)^{\epsilon'} \frac{1}{(1 + \rho(x, x_0))^{d+\gamma}} \|f\|_{\mathcal{G}(\beta,\gamma)} \\
&\quad \times \sum_{k=M+1}^{\infty} \left[2^{-k\epsilon'} + 2^{-k(\epsilon'-\sigma)} \right] \\
&\le C \left(\frac{\rho(x, x')}{1 + \rho(x, x_0)} \right)^{\epsilon'} \frac{1}{(1 + \rho(x, x_0))^{d+\gamma}} \|f\|_{\mathcal{G}(\beta,\gamma)},
\end{aligned} \tag{3.44}$$

where $\sigma \in (0, \epsilon')$.

Using the smoothness conditions on E_k and f, we obtain

$$\begin{aligned}
N_2 &\le C\|f\|_{\mathcal{G}(\beta,\gamma)} \sum_{k=M+1}^{\infty} \int_{V_2} \left(\frac{\rho(x, x')}{2^{-k} + \rho(x, y)} \right)^{\epsilon'} \frac{2^{-k\epsilon'}}{(2^{-k} + \rho(x, y))^{d+\epsilon'}} \\
&\quad \times \left(\frac{\rho(x, y)}{1 + \rho(x, x_0)} \right)^{\beta} \frac{1}{(1 + \rho(x, x_0))^{d+\gamma}} \, d\mu(y) \\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)} \left(\frac{\rho(x, x')}{1 + \rho(x, x_0)} \right)^{\beta'} \frac{1}{(1 + \rho(x, x_0))^{d+\gamma}} \\
&\quad \times \sum_{k=M+1}^{\infty} \int_X \frac{2^{-k\epsilon'}}{(2^{-k} + \rho(x, y))^{d+\epsilon'}} \rho(x, y)^{\beta-\beta'} \, d\mu(y) \\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)} \left(\frac{\rho(x, x')}{1+\rho(x, x_0)} \right)^{\beta'} \frac{1}{(1+\rho(x, x_0))^{d+\gamma}} \sum_{k=M+1}^{\infty} 2^{-k(\beta-\beta')} \\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)} \left(\frac{\rho(x, x')}{1 + \rho(x, x_0)} \right)^{\beta'} \frac{1}{(1 + \rho(x, x_0))^{d+\gamma}}.
\end{aligned} \tag{3.45}$$

The size condition of E_k and the smoothness condition of f tell us that

$$\begin{aligned}
N_3 &\le C\|f\|_{\mathcal{G}(\beta,\gamma)} \sum_{k=M+1}^{\infty} \int_{V_3} \left\{|E_k(x,y)| + |E_k(x',y)|\right\} \\
&\quad \times \left(\frac{\rho(x,y)}{1+\rho(x,x_0)}\right)^{\beta} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}}\, d\mu(y) \\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)} \left(\frac{\rho(x,x')}{1+\rho(x,x_0)}\right)^{\beta'} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}} \\
&\quad \times \sum_{k=M+1}^{\infty} \int_{X} \left\{|E_k(x,y)| + |E_k(x',y)|\right\} \rho(x,y)^{\beta-\beta'}\, d\mu(y) \\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)} \left(\frac{\rho(x,x')}{1+\rho(x,x_0)}\right)^{\beta'} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}} \sum_{k=M+1}^{\infty} 2^{-k(\beta-\beta')} \\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)} \left(\frac{\rho(x,x')}{1+\rho(x,x_0)}\right)^{\beta'} \frac{1}{(1+\rho(x,x_0))^{d+\gamma}},
\end{aligned}$$

which, together with (3.44) and (3.45), verifies (3.42). This proves that the series in (3.24) converge in $\mathcal{G}(\beta',\gamma')$ for $0<\beta'<\beta$ and $0<\gamma'<\gamma$.

Finally, let us verify the series in (3.24) also converge in $L^p(X)$ for $p\in(1,\infty)$. To this end, by the above proof, we only need to verify the last two terms in (3.36) tend to 0 as $M\to\infty$ if the norm of $\mathcal{G}(\beta',\gamma')$ is replaced by the norm of $L^p(X)$ for $p\in(1,\infty)$. The estimates in Lemma 3.2 and the Tb theorem in [4] tell us that $R_N b^{-1}$ is a Calderón-Zygmund operator with the operator norm at most $C_7 2^{-N\delta}$ and, hence, R_N is bounded on $L^p(X)$ for $p\in(1,\infty)$ with the operator norm at most $C_7 2^{-N\delta}$. This yields that

$$\left\|(R_N)^j(f)\right\|_{L^p(X)} \le \left(C_7 2^{-N\delta}\right)^j \|f\|_{L^p(X)}. \tag{3.46}$$

Thus, $\lim_{j\to\infty}\left\|(R_N)^j(f)\right\|_{L^p(X)}=0$ if we choose $N\in\mathbb{N}$ such that (3.23) holds. The estimate (3.46) also implies that T_N^{-1} is bounded on $L^p(X)$ for $p\in(1,\infty)$ if N satisfies (3.23). Thus, it suffices to prove that

$$\lim_{M\to\infty}\left\|\sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f)\right\|_{L^p(X)} = 0$$

for $p\in(1,\infty)$. To see this, letting $1/p+1/p'=1$, by duality and the result in [4], we have

$$\begin{aligned}
&\left\|\sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f)\right\|_{L^p(X)} \\
&\quad = \sup_{\|g\|_{L^{p'}(X)}\le 1} \left|\left\langle \sum_{k=M+1}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b(f), g\right\rangle\right| \\
&\quad \le \sup_{\|g\|_{L^{p'}(X)}\le 1} \left\|\left\{\sum_{k=M+1}^{\infty} |D_k\mathcal{M}_b(f)|^2\right\}^{\frac12}\right\|_{L^p(X)} \left\|\left\{\sum_{k=M+1}^{\infty} |\mathcal{M}_b(D_k^N)^*(g)|^2\right\}^{\frac12}\right\|_{L^{p'}(X)} \\
&\quad \le C \sup_{\|g\|_{L^{p'}(X)}\le 1} \left\|\left\{\sum_{k=M+1}^{\infty} |D_k\mathcal{M}_b(f)|^2\right\}^{\frac12}\right\|_{L^p(X)} \|g\|_{L^{p'}(X)} \\
&\quad \le C \left\|\left\{\sum_{k=M+1}^{\infty} |D_k\mathcal{M}_b(f)|^2\right\}^{\frac12}\right\|_{L^p(X)} \to 0,
\end{aligned}$$

as $M\to\infty$. This finishes the proof of Theorem 3.2.

By an argument of duality, from Theorem 3.2, we can deduce the following continuous Calderón-type reproducing formulae in $\left(\mathring{\mathcal{G}}(\beta,\gamma)\right)'$ with $\beta,\ \gamma\in(0,\epsilon)$ and $\epsilon\in(0,\theta]$; see also [9, 11].

Theorem 3.3. *Let b be a para-accretive function, $\epsilon\in(0,\theta]$, $\{S_k\}_{k=0}^{\infty}$ be an approximation to the identity of order ϵ associated to b. Set $D_k=S_k-S_{k-1}$ for $k\in\mathbb{N}$ and $D_0=S_0$. Then there exist a family of linear operators $\widetilde{E}_k$ for $k\in\mathbb{Z}_+$ and a fixed large integer $N\in\mathbb{N}$ such that for all $f\in\left(\mathring{\mathcal{G}}(\beta,\gamma)\right)'$ with $0<\beta,\ \gamma<\epsilon$,*

$$f=\sum_{k=0}^{\infty}\mathcal{M}_b D_k \mathcal{M}_b \widetilde{E}_k(f), \tag{3.47}$$

where the series converge in $\left(\mathring{\mathcal{G}}(\beta',\gamma')\right)'$ with $\beta<\beta'<\epsilon$ and $\gamma<\gamma'<\epsilon$. Moreover, the kernels of the operators $\widetilde{E}_k$ satisfy the conditions (i) and (iii) of Definition 3.1 with ϵ replaced by ϵ' for $0<\epsilon'<\epsilon$, and

$$\int_X \widetilde{E}_k(x,y)b(y)\,d\mu(y)=\int_X \widetilde{E}_k(x,y)b(x)\,d\mu(x)=\begin{cases}1, & k=0,1,\cdots,N;\\ 0, & k\ge N+1.\end{cases}$$

The following theorems can be proved by some arguments similar to those of Theorem 3.2 and Theorem 3.3. We leave the details to the reader.

Theorem 3.4. *With all the notation as in Theorem 3.3, then for all $f\in\mathcal{G}(\beta,\gamma)$,*

$$f=\sum_{k=0}^{\infty} D_k\mathcal{M}_b\widetilde{E}_k\mathcal{M}_b(f)$$

holds in both the norm of $\mathcal{G}(\beta', \gamma')$ *for* $0 < \beta' < \beta$ *and* $0 < \gamma' < \gamma$*, and the norm of* $L^p(X)$ *with* $p \in (1, \infty)$.

Theorem 3.5. *With all the notation as in Theorem 3.2 and Theorem 3.3, then for all* $f \in b\mathcal{G}(\beta, \gamma)$,

$$f = \sum_{k=0}^{\infty} \mathcal{M}_b \widetilde{D}_k \mathcal{M}_b D_k(f) = \sum_{k=0}^{\infty} \mathcal{M}_b D_k \mathcal{M}_b \widetilde{E}_k(f)$$

holds in both the norm of $b\mathcal{G}(\beta', \gamma')$ *for* $0 < \beta' < \beta$ *and* $0 < \gamma' < \gamma$*, and the norm of* $L^p(X)$ *with* $p \in (1, \infty)$.

Theorem 3.6. *With all the notation as in Theorem 3.2, then for all* $f \in \left(\mathring{\mathcal{G}}(\beta, \gamma)\right)'$ *with* $0 < \beta,\ \gamma < \epsilon$,

$$f = \sum_{k=0}^{\infty} \mathcal{M}_b \widetilde{D}_k \mathcal{M}_b D_k(f)$$

holds in $\left(\mathring{\mathcal{G}}(\beta', \gamma')\right)'$ *with* $\beta < \beta' < \epsilon$ *and* $\gamma < \gamma' < \epsilon$.

Theorem 3.7. *With all the notation as in Theorem 3.2 and Theorem 3.3, then for all* $f \in \left(b\mathring{\mathcal{G}}(\beta, \gamma)\right)'$ *with* $0 < \beta,\ \gamma < \epsilon$,

$$f = \sum_{k=0}^{\infty} \widetilde{D}_k \mathcal{M}_b D_k \mathcal{M}_b(f) = \sum_{k=0}^{\infty} D_k \mathcal{M}_b \widetilde{E}_k \mathcal{M}_b(f)$$

holds in $\left(b\mathring{\mathcal{G}}(\beta', \gamma')\right)'$ *with* $\beta < \beta' < \epsilon$ *and* $\gamma < \gamma' < \epsilon$.

Remark 3.3. By rearranging the orders of D_k, $\widetilde{D}_k$ and $\widetilde{E}_k$, we can actually take $N = 0$ in Theorems 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7, which is convenient in applications of these formulae.

4. One Application

In this section, we use the Calderón reproducing formulae in Section 3 to establish a Littlewood-Paley theorem for the following g-function defined by

$$g(f)(x) = \left[\sum_{k=0}^{\infty} |D_k(f)(x)|^2\right]^{1/2}, \tag{4.1}$$

where $\{D_k\}_{k\in\mathbb{Z}_+}$ is as in Theorem 3.2. This theorem generalizes a corresponding result of David, Journé and Semmes in [4] and will be used in establishing the discrete Calderón reproducing formulae associated to para-accretive functions which will be discussed in another paper; see also [15]. We establish our theorem based on the Tb Theorem in [4]. To state it, we first need to recall the definition of the weak boundedness property.

Definition 4.1. A Calderón-Zygmund singular integral operator T of the kernel K is said to have the weak boundedness property, if there exist $\eta \in (0,\theta]$ and constant $C_8 > 0$ such that

$$|\langle Tf, g\rangle| \le C_8 r^{d+2\eta} \|f\|_{C_0^\eta(X)} \|g\|_{C_0^\eta(X)} \tag{4.2}$$

for all $r > 0$ and all $f,\ g \in C_0^\eta(X)$ supporting in some ball of radius r. We denote this by $T \in WBP$.

Remark 4.1. It was proved in [4] that if $T \in WBP$, then (4.2) is true for all $\eta \in (0,\theta]$.

Lemma 4.1. *Let $b_1,\ b_2$ be two para-accretive functions, $\epsilon \in (0,\theta]$ and $T \in CZO(\epsilon)$, which here means that T is a continuous linear operator from $b_1C_0^\eta(X)$ into $(b_1C_0^\eta(X))'$ for all $\eta \in (0,\theta]$ and there is a kernel $K(x,y)$ satisfying the conditions (i), (ii) and (iii) of Definition 2.4 such that for all $f,\ g \in C_0^\eta(X)$ with disjoint supports,*

$$\langle Tf, g\rangle = \int_X \int_X b_2(x)K(x,y)b_1(y)f(y)g(x)\, d\mu(x)\, d\mu(y).$$

Then T is bounded on $L^2(X)$ if and only if (i) $Tb_1 \in BMO(X)$, (ii) $T^*b_2 \in BMO(X)$, and (iii) $\mathcal{M}_{b_2}T\mathcal{M}_{b_1} \in WBP$.

We also need the following Fefferman-Stein vector-valued maximal function inequality in [6].

Lemma 4.2. *Let $1 < p < \infty$, $1 < q \le \infty$ and M be the Hardy-Littlewood maximal operator on X. Let $\{f_k\}_{k=0}^\infty \subset L^p(X)$ be a sequence of measurable functions on X. Then*

$$\left\| \left\{ \sum_{k=0}^\infty |M(f_k)|^q \right\}^{1/q} \right\|_{L^p(X)} \le C \left\| \left\{ \sum_{k=0}^\infty |f_k|^q \right\}^{1/q} \right\|_{L^p(X)},$$

where C is independent of $f \in L^p(X)$.

We can now establish the following Littlewood-Paley theorem on spaces of homogeneous type.

Theorem 4.1. *Let* $\{S_k\}_{k=0}^{\infty}$ *be an approximation to the identity of order* $\epsilon \in (0, \theta]$ *as in Definition 3.1. Let* $\{D_k\}_{k\in\mathbb{Z}_+}$ *be as in Theorem 3.2 and* $g(f)$ *be defined as in (4.1). Then for any* $p \in (1, \infty)$, *there exist two constants* A_p *and* B_p *depending on* p *such that*

$$A_p\|f\|_{L^p(X)} \le \|g(f)\|_{L^p(X)} \le B_p\|f\|_{L^p(X)} \tag{4.3}$$

for all $f \in L^p(X)$.

Proof. To verify the inequality on the right hand side of (4.3), by the Khintchine inequality (see [26]), we first have that for any fixed $N \in \mathbb{N}$,

$$\begin{aligned} \left\|\left\{\sum_{k=0}^{N}|D_k(f)|^2\right\}^{1/2}\right\|_{L^p(X)} &\le C_p\left\|\frac{1}{2^N}\sum_{\sigma_0}\sum_{\sigma_1}\cdots\sum_{\sigma_N}\sum_{k=0}^{N}\sigma_k D_k(f)\right\|_{L^p(X)} \\ &\le \frac{C_p}{2^N}\sum_{\sigma_0}\sum_{\sigma_1}\cdots\sum_{\sigma_N}\left\|\sum_{k=0}^{N}\sigma_k D_k(f)\right\|_{L^p(X)}, \end{aligned} \tag{4.4}$$

where $\sigma_k = 1$ or -1 for $k \in \{0, 1, \cdots, N\}$ and C_p is independent of f and N. For any fixed $\sigma = \{\sigma_k\}_{k=0}^{N}$, we set $T_N^{\sigma}f = \sum_{k=0}^{N}\sigma_k D_k(f)$ and denote its kernel by $K_N^{\sigma}(x, y)$. We first verify that K_N^{σ} is a standard Calderón-Zygmund kernel with the constant independent of N and σ. In fact, we have

$$\begin{aligned} |K_N^{\sigma}(x,y)| &\le \sum_{k=0}^{N}|D_k(x,y)| \le C\sum_{k=0}^{N}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}} \\ &\le C\rho(x,y)^{-(d+\epsilon)}\sum_{\{k\in\mathbb{Z}:\ 2^{-k}\le\rho(x,y)\}}2^{-k\epsilon} + C\sum_{\{k\in\mathbb{Z}:\ 2^{-k}>\rho(x,y)\}}2^{kd} \\ &\le C\rho(x,y)^{-d}. \end{aligned} \tag{4.5}$$

For $\rho(x, x') \le \rho(x, y)/(2A)$, we have

$$\begin{aligned} |K_N^{\sigma}(x,y) - K_N^{\sigma}(x',y)| &\le \sum_{k=0}^{N}|S_k(x,y) - S_k(x',y)| \\ &\le C\sum_{k=0}^{\infty}\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}} \\ &\le C\rho(x,x')^{\epsilon}\rho(x,y)^{-(d+2\epsilon)}\sum_{\{k\in\mathbb{Z}:\ 2^{-k}\le\rho(x,y)\}}2^{-k\epsilon} \\ &\quad + C\rho(x,x')^{\epsilon}\sum_{\{k\in\mathbb{Z}:\ 2^{-k}>\rho(x,y)\}}2^{k(d+\epsilon)} \\ &\le C\,\rho(x,x')^{\epsilon}\rho(x,y)^{-(d+\epsilon)}. \end{aligned} \tag{4.6}$$

Similarly, we can prove that for $\rho(y,y') \le \rho(x,y)/(2A)$, we have

$$|K_N^\sigma(x,y) - K_N^\sigma(x,y')| \le C\rho(y,y')^\epsilon \rho(x,y)^{-(d+\epsilon)}. \tag{4.7}$$

By combining (4.5), (4.6) with (4.7), we know that K_N^σ is a standard Calderón-Zygmund kernel with the constant independent of N and σ. Now, we claim that T_N^σ is bounded on $L^2(X)$ with the operator norm independent of N and σ. If this is true, then T_N^σ is a standard Calderón-Zygmund operator bounded on $L^p(X)$ for any $p \in (1,\infty)$; see [1]. Thus,

$$\left\|\sum_{k=0}^N \sigma_k D_k(f)\right\|_{L^p(X)} \le C\|f\|_{L^p(X)},$$

where C is independent of σ and N. By this and (4.4), we obtain

$$\left\|\left\{\sum_{k=0}^N |D_k(f)|^2\right\}^{1/2}\right\|_{L^p(X)} \le C\|f\|_{L^p(X)},$$

where C_p is independent of N. Then by the Fatou lemma, we have

$$\|g(f)\|_{L^p(X)} \le C_p\|f\|_{L^p(X)}. \tag{4.8}$$

This is just the right hand side inequality of (4.3).

Now, we still need to prove T_N^σ is bounded on $L^2(X)$. We do this by applying Lemma 4.1 with $b_1 = b_2 = b$. It is easy to see that $T_N^\sigma b = (T_N^\sigma)^* b = \sigma_0 \in BMO(X)$ with the norm 0. It suffices to check that $\mathcal{M}_b T_N^\sigma \mathcal{M}_b \in WBP$. To this end, by Definition 4.1, let $f,\ g \in C_0^\eta(X)$ support in the ball $B(x_0,r)$ for some $x_0 \in X$ and some $r > 0$. We then write

$$\begin{aligned}
|\langle \mathcal{M}_b T_N^\sigma \mathcal{M}_b f, g\rangle| &= \left|\sum_{k=0}^N \sigma_k \int_X\int_X b(x)D_k(x,y)b(y)f(y)g(x)\,d\mu(y)\,d\mu(x)\right| \\
&\le \int_X\int_X |b(x)D_0(x,y)b(y)f(y)g(x)|\,d\mu(y)\,d\mu(x) \\
&\quad + \sum_{k=1}^N \left|\int_X\int_X b(x)D_k(x,y)b(y)f(y)g(x)\,d\mu(y)\,d\mu(x)\right| \\
&= O_1 + O_2.
\end{aligned}$$

For O_1, from the trivial estimates that

$$|f(x)| \le Cr^\eta\|f\|_{C_0^\eta(X)} \quad \text{and} \quad |g(x)| \le Cr^\eta\|g\|_{C_0^\eta(X)} \tag{4.9}$$

and the size condition of D_0, it follows that

$$\begin{aligned} O_1 &\le Cr^{\eta}\|f\|_{C_0^\eta(X)} \int_{B(x_0,r)} \left[\int_X |D_0(x,y)|\, d\mu(y)\right] |g(x)|\, d\mu(x) \\ &\le Cr^{d+2\eta}\|f\|_{C_0^\eta(X)}\|g\|_{C_0^\eta(X)}, \end{aligned}$$

which is a desired estimate.

We now estimate O_2. We consider two cases. *Case 1.* $r \ge 1$. In this case, by (4.9) and $b \in L^\infty(X)$, we have

$$\begin{aligned} O_2 &= \sum_{k=1}^N \left|\int_X\int_X b(x)D_k(x,y)b(y)f(y)g(x)\,d\mu(y)\,d\mu(x)\right| \\ &= \sum_{k=1}^N \left|\int_X\int_X b(x)D_k(x,y)b(y)[f(y)-f(x)]g(x)\,d\mu(y)\,d\mu(x)\right| \\ &\le C\sum_{k=1}^N \int_X\int_X |D_k(x,y)|\,|f(y)-f(x)|^\kappa\,[|f(y)| \\ &\quad + |f(x)|]^{1-\kappa}\,|g(x)|\,d\mu(y)\,d\mu(x) \\ &\le C\sum_{k=1}^N r^{(1-\kappa)\eta}\|f\|_{C_0^\eta(X)}\int_{B(x_0,r)}|g(x)| \\ &\quad \times \left[\int_X \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}\rho(x,y)^{\kappa\eta}\,d\mu(y)\right]\,d\mu(x) \\ &\le Cr^{d+2\eta}\|f\|_{C_0^\eta(X)}\|g\|_{C_0^\eta(X)}\sum_{k=1}^N 2^{-k\eta\kappa} \\ &\le Cr^{d+2\eta}\|f\|_{C_0^\eta(X)}\|g\|_{C_0^\eta(X)}, \end{aligned} \tag{4.10}$$

where $\kappa \in (0,1)$ such that $\kappa\eta < \epsilon$, which is also a desired estimate. *Case 2.* $r \sim 2^{k_0}$ for some $k_0 \in \mathbb{N}$. In this case, we write

$$\begin{aligned} O_2 &= \sum_{k=1}^N \left|\int_X\int_X b(x)D_k(x,y)b(y)f(y)g(x)\,d\mu(y)\,d\mu(x)\right| \\ &= \sum_{k=1}^{k_0} \left|\int_X\int_X b(x)D_k(x,y)b(y)f(y)g(x)\,d\mu(y)\,d\mu(x)\right| + \sum_{k=k_0+1}^N \cdots \\ &= O_2^1 + O_2^2. \end{aligned}$$

For O_2^1, the estimate (4.9), $b \in L^\infty(X)$ and the size condition of D_k yield that

$$\begin{aligned} O_2^1 &\le Cr^{d+2\eta}\|f\|_{C_0^\eta(X)}\|g\|_{C_0^\eta(X)}\sum_{k=1}^{k_0}2^{kd}r^d \\ &\le Cr^{d+2\eta}\|f\|_{C_0^\eta(X)}\|g\|_{C_0^\eta(X)}\sum_{k=1}^{k_0}2^{(k-k_0)d} \\ &\le Cr^{d+2\eta}\|f\|_{C_0^\eta(X)}\|g\|_{C_0^\eta(X)}, \end{aligned}$$

which is a desired estimate.

For O_2^2, similarly to (4.10), we have

$$\begin{aligned} O_2 &= \sum_{k=k_0+1}^{N}\left|\int_X\int_X b(x)D_k(x,y)b(y)f(y)g(x)\,d\mu(y)\,d\mu(x)\right| \\ &= \sum_{k=k_0+1}^{N}\left|\int_X\int_X b(x)D_k(x,y)b(y)[f(y)-f(x)]g(x)\,d\mu(y)\,d\mu(x)\right| \\ &\le C\sum_{k=k_0+1}^{N}\int_X\int_X |D_k(x,y)|\,|f(y)-f(x)|^\kappa\,[|f(y)| \\ &\quad +|f(x)|]^{1-\kappa}\,|g(x)|\,d\mu(y)\,d\mu(x) \\ &\le Cr^{(1-\kappa)\eta}\|f\|_{C_0^\eta(X)}\sum_{k=k_0+1}^{N}\int_{B(x_0,r)}|g(x)| \\ &\quad\times\left[\int_X\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}\rho(x,y)^{\kappa\eta}\,d\mu(y)\right]\,d\mu(x) \\ &\le Cr^{d+(1-\kappa)\eta+\eta}\|f\|_{C_0^\eta(X)}\|g\|_{C_0^\eta(X)}\sum_{k=k_0+1}^{N}2^{-k\eta\kappa} \\ &\le Cr^{d+2\eta}\|f\|_{C_0^\eta(X)}\|g\|_{C_0^\eta(X)}, \end{aligned}$$

where $\kappa\in(0,1)$ such that $\kappa\eta<\epsilon$, which is also a desired estimate. Thus, $\mathcal{M}_bT_N^\sigma\mathcal{M}_b\in WBP$, and we have proved the second inequality in (4.3).

To prove the first inequality in (4.3), we claim that if $\{E_k\}_{k\in\mathbb{Z}_+}$ is a family of linear operators and $E_k(x,y)$, the kernel of E_k, satisfies the conditions of (i) and (iii) of Definition 3.1, and

$$\int_X E_k(x,y)\,d\mu(y)=\begin{cases}0, & k\in\mathbb{N}\\ 1, & k=0,\end{cases}$$

then there exists a constant $C>0$ such that

$$\left\|\left\{\sum_{k=0}^{\infty}|E_k(f)|^2\right\}^{1/2}\right\|_{L^p(X)}\le C\|f\|_{L^p(X)} \tag{4.11}$$

for $p \in (1, \infty)$. We verify (4.11) by using Theorem 3.5. From Theorem 3.5, it follows that

$$|E_k(f)(x)| = \left| E_k \left(\sum_{j=0}^{\infty} \mathcal{M}_b \widetilde{D}_j \mathcal{M}_b D_j(f) \right)(x) \right| \leq \sum_{j=0}^{\infty} \left| \left(E_k \mathcal{M}_b \widetilde{D}_j \mathcal{M}_b D_j \right)(f)(x) \right|.$$

By Lemma 3.1 and $b \in L^\infty(X)$, we further obtain

$$\sum_{j=0}^{\infty} \left| \left(E_k \mathcal{M}_b \widetilde{D}_j \mathcal{M}_b D_j \right)(f)(x) \right| \leq C \sum_{j=0}^{\infty} 2^{-|k-j|\epsilon'} M\left(D_j(f)\right)(x).$$

Thus, by Lemma 4.2, the Hölder inequality and (4.8), we have

$$\begin{aligned}
\left\| \left\{ \sum_{k=0}^{\infty} |E_k(f)|^2 \right\}^{1/2} \right\|_{L^p(X)} &\leq C \left\| \left\{ \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} 2^{-|k-j|\epsilon'} M\left(D_j(f)\right) \right]^2 \right\}^{1/2} \right\|_{L^p(X)} \\
&\leq C \left\| \left\{ \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} 2^{-|k-j|\epsilon'} \left[M\left(D_j(f)\right)\right]^2 \right) \right\}^{1/2} \right\|_{L^p(X)} \\
&\leq C \left\| \left\{ \sum_{j=0}^{\infty} \left[M\left(D_j(f)\right)\right]^2 \right\}^{1/2} \right\|_{L^p(X)} \\
&\leq C \left\| \left\{ \sum_{j=0}^{\infty} |D_j(f)|^2 \right\}^{1/2} \right\|_{L^p(X)} \\
&\leq C \|f\|_{L^p(X)}.
\end{aligned}$$

Thus, our claim is true.

Now, using Theorem 3.5, the Hölder inequality, $b \in L^\infty(X)$, the claim (4.11) with $E_k = (\widetilde{D}_k)^*$ and (4.8), we have

$$\begin{aligned}
\|f\|_{L^p(X)} &= \sup_{\|h\|_{L^{p'}(X)} \leq 1} |\langle f, h \rangle| \\
&= \sup_{\|h\|_{L^{p'}(X)} \leq 1} \left| \left\langle \sum_{k=0}^{\infty} \mathcal{M}_b \widetilde{D}_k \mathcal{M}_b D_k(f), h \right\rangle \right| \\
&\leq \sup_{\|h\|_{L^{p'}(X)} \leq 1} \left\{ \sum_{k=0}^{\infty} |\langle \mathcal{M}_b D_k(f), (\widetilde{D}_k)^* \mathcal{M}_b(h) \rangle| \right\} \\
&\leq C \sup_{\|h\|_{L^{p'}(X)} \leq 1} \|g(f)\|_{L^p(X)} \left\| \left\{ \sum_{k=0}^{\infty} |(\widetilde{D}_k)^*(bh)|^2 \right\}^{1/2} \right\|_{L^{p'}(X)}
\end{aligned}$$

$$
\begin{aligned}
&\le C \sup_{\|h\|_{L^{p'}(X)}\le 1} \|g(f)\|_{L^p(X)} \left\| \left\{ \sum_{k=0}^{\infty} |D_k(bh)|^2 \right\}^{1/2} \right\|_{L^{p'}(X)} \\
&\le C\|g(f)\|_{L^p(X)} \sup_{\|h\|_{L^{p'}(X)}\le 1} \|h\|_{L^{p'}(X)} \\
&\le C\|g(f)\|_{L^p(X)}.
\end{aligned}
$$

This verifies the first inequality in (4.3) and we have finished the proof of Theorem 4.1.

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