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CODES AND $L(2,1)$-LABELINGS IN SIERPIŃSKI GRAPHS

Sylvain Gravier, Sandi Klavžar and Michel Mollard

Abstract. The λ-number of a graph G is the minimum value λ such that G admits a labeling with labels from $\{0,1,\ldots,\lambda\}$ where vertices at distance two get different labels and adjacent vertices get labels that are at least two apart. Sierpiński graphs $S(n,k)$ generalize the Tower of Hanoi graphs—the graph $S(n,3)$ is isomorphic to the graph of the Tower of Hanoi with n disks. It is proved that for any $n \geq 2$ and any $k \geq 3$, $\lambda(S(n,k)) = 2k$. To obtain the result (perfect) codes in Sierpiński graphs are studied in detail. In particular a new proof of their (essential) uniqueness is obtained.

1. INTRODUCTION

An $L(2,1)$-labeling of a graph G is an assignment of labels from $\{0,1,\ldots,\lambda\}$ to the vertices of G such that vertices at distance two get different labels and adjacent vertices get labels that are at least two apart. The λ-number $\lambda(G)$ of G is the minimum value λ such that G admits an $L(2,1)$-labeling. The difference between the largest label and the smallest label assigned by an $L(2,1)$-labeling f is called the *span* of f.

These concepts arose from the problem of assigning frequencies to radio transmitters [9] and has been formulated as the $L(2,1)$-labeling problem by Griggs and Yeh [8]. The problem soon became an object of extensive research, of which [2, 4, 7, 11, 12, 16, 20, 22] is a sample of references. Note that many of these papers are very recent, so it seems that the interest for this topic is increasing. We also wish to point out that very recently Chang and Liaw [3] extended this concept to digraphs. Concerning the complexity issues of the problem we refer to [6] and references therein. One of the main messages of this extensive research is that it is usually very difficult to precisely determine the λ-number of a graph or of a family

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of graphs. For instance, to determine the λ-number of hypercubes seems to be an utmost difficult problem [23].

Consider an $L(2,1)$-labeling of a graph G and let C_i, $0 \le i \le \lambda$, be the set of vertices of G with label i. Then $C_0, \ldots, C_\lambda$ form a partition of $V(G)$ in which for each $i = 0, \ldots, \lambda$, distinct vertices in C_i have distance at least three. Such sets are called codes (in graphs). So an $L(2,1)$-labeling of a graph G is a partition of its vertex set into codes.

The study of (perfect) codes in (distance regular) graphs was initiated by Biggs [1]. Later Kratochvíl with his co-workers considered (perfect) codes in general graphs, see the monograph [17] and references therein. (For related complexity results we refer to [10].) Cull and Nelson [5] showed that the Tower of Hanoi graphs contain (essentially) unique 1-perfect codes, cf. also [18]. This result is in [14] extended to the so-called Sierpiński graphs that form a two parametric generalization of the Tower of Hanoi graphs.

The Sierpiński graphs $S(n,k)$ were introduced in [13]. The motivation for their introduction were topological studies in [19, 21] of Lipscomb's space, where it is shown that this space is a generalization of the Sierpiński triangular curve (Sierpiński gasket). For some recent results on the Sierpiński graphs from the area of topological graph theory see [15].

In this paper we determine the λ-numbers of the Sierpiński graphs. In the rest of this section we recall the concepts of the Sierpiński graphs and codes in graphs, and give a connection between the concepts. In Section 2 we closely analyze (perfect) codes in the Sierpiński graphs. In particular we prove that the perfect codes in these graphs are essentially unique, a result first proved in [14]. The present approach enables a shorter and also simpler proof of the theorem. In the last section we then prove that for any $n \ge 2$ and any $k \ge 3$, $\lambda(S(n,k)) = 2k$.

The graph $S(n,k)$ $(n, k \ge 1)$ is defined on the vertex set $\{1, 2, \ldots, k\}^n$, two different vertices $u = (i_1, i_2, \ldots, i_n)$ and $v = (j_1, j_2, \ldots, j_n)$ being adjacent if and only if $u \sim v$. The relation $\sim$ is defined as follows: $u \sim v$ if there exists an $h \in \{1, 2, \ldots, n\}$ such that

(i) $i_t = j_t$, for $t = 1, \ldots, h-1$;

(ii) $i_h \ne j_h$; and

(iii) $i_t = j_h$ and $j_t = i_h$ for $t = h+1, \ldots, n$.

In the rest of the paper we will write $\langle i_1 i_2 \ldots i_n \rangle$ as short for $(i_1, i_2, \ldots, i_n)$. The Sierpiński graphs $S(3,3)$ and $S(2,4)$, together with the corresponding vertex labelings are shown in Fig. 1.

A vertex of the form $\langle ii \ldots i \rangle$ of $S(n,k)$ is called an *extreme vertex*, the other vertices will be called *inner*. The extreme vertices of $S(n,k)$ are of degree $k-1$ while the degree of the inner vertices is k. Note also that in $S(n,k)$ there are k extreme vertices and that $|S(n,k)| = k^n$.

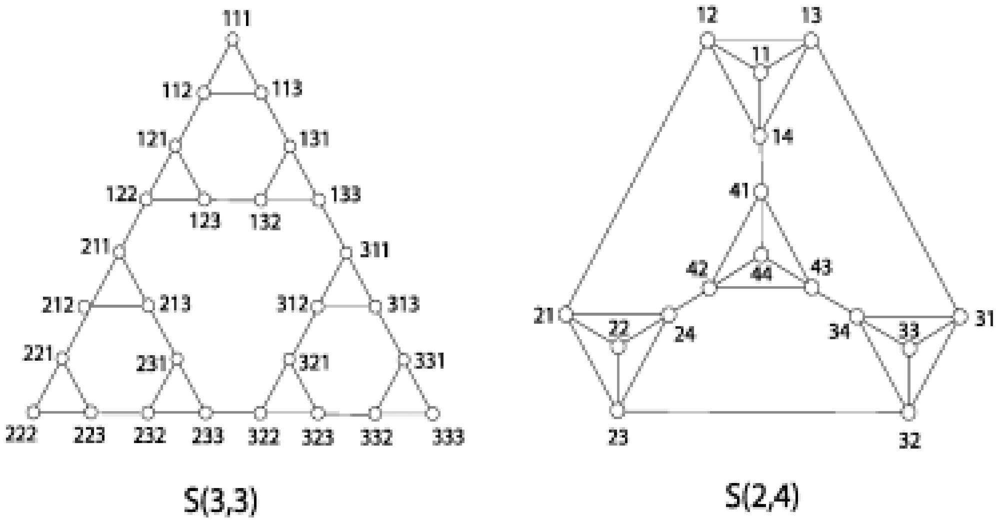

Fig. 1. Sierpiński graphs $S(3,3)$ and $S(2,4)$

A subset C of vertices of a graph G is a 1-*code* (or simply a *code*) if for any distinct vertices u and v in C, we have $d_G(u,v) \geq 3$, where the distance $d_G(u,v)$ (or, for short, $d(u,v)$) of vertices u and v is the number of edges on a shortest u,v-path. A vertex u is *dominated* by a set C if it has a neighbor in C or $u \in C$. A *perfect code* of $G = (V,E)$ is a code C for which every vertex of V is dominated by C. In other words, the closed neighborhoods of elements of C form a partition of V. We will call a code C of $S(n,k)$ an *almost perfect code* if all inner vertices are dominated by C. Clearly, a perfect code of $S(n,k)$ is also an almost perfect code.

We have already mentioned that an $L(2,1)$-labeling induces a partition of vertices into codes. Conversely, we can state the following easy observation that presents the starting point of this paper.

Proposition 1.1. *Let G be a graph and $\{C_0, C_1, \ldots, C_k\}$ a partition of $V(G)$, where C_i is a code for $0 \leq i \leq k$. Then $\lambda(G) \leq 2k$.*

Proof. Let $x \in V(G)$. Then set $\ell(x) = 2i$, where $x \in C_i$. ■

2. (Perfect) Codes in Sierpiński Graphs

In this section we will prove that $S(n,k)$ can be partitioned into perfect codes and almost perfect codes. This will allow us to construct (in the next section) an optimal $L(2,1)$-labeling. Along the way we will deduce the existence and uniqueness of perfect codes of the Sierpiński graphs.

The graph $S(n+1,k)$ can be constructed inductively from $S(n,k)$ as follows (cf. Fig. 1):

- Take k copies $G_1, G_2, \ldots, G_k$ of $S(n,k)$, where for $i = 1, 2, \ldots, k$ we have

$$V(G_i) = \{\langle i a_1 a_2 \ldots a_n \rangle;\ \langle a_1 a_2 \ldots a_n \rangle \in V(S(n,k))\}.$$

- For any i and any j with $i \neq j$, add an edge between the extreme vertex $\langle ijj \ldots j \rangle$ of G_i and the extreme vertex $\langle jii \ldots i \rangle$ of G_j.

Note that no edge incident to the extreme vertex $\langle ii \ldots i \rangle$ of G_i is added. Therefore these extreme vertices will be the only extreme vertices of $S(n+1,k)$.

Lemma 2.1. *Let C be a subset of vertices of $S(n+1,k)$. Then C is an almost perfect code of $S(n+1,k)$ if and only if for $i = 1, 2, \ldots, k$:*

(a) *$C \cap V(G_i)$ is an almost perfect code of G_i, and*

(b) *for every $j \neq i$, the extreme vertex $\langle ijj \ldots j \rangle$ of G_i belongs to C if and only if the extreme vertex $\langle jii \ldots i \rangle$ of G_j is not dominated by $C \cap V(G_j)$.*

Proof. Let C be an almost perfect code of $S(n+1,k)$. Clearly $C \cap V(G_i)$ is a code of G_i, and since the inner vertices of G_i are inner in $S(n+1,k)$, $C \cap V(G_i)$ is an almost perfect code of G_i, thus (a) holds. Assume that for some $i \neq j$, $\langle ijj \ldots j \rangle$ belongs to C. Since $\langle ijj \ldots j \rangle$ is adjacent to $\langle jii \ldots i \rangle$, the vertex $\langle jii \ldots i \rangle$ of G_j is not dominated by $C \cap V(G_j)$. The converse holds since C is an almost perfect code of $S(n+1,k)$ and since $\langle jii \ldots i \rangle$ is not extreme in $S(n+1,k)$. Hence (b) holds as well.

For the converse assume that we have a subset C which satisfies (a) and (b). First, we claim that C is a code in $S(n+1,k)$. Let $u, v \in C$, $u \in V(G_i)$ and $v \in V(G_j)$. By (a) we can assume that $j \neq i$. If u and v are inner vertices of G_i and G_j, respectively, then clearly $d(u,v) \geq 3$. Now suppose that $d(u,v) \leq 2$. Then a shortest u,v-path is unique and uses the edge $\langle ijj \ldots j \rangle \langle jii \ldots i \rangle$. We may without loss of generality assume that $u = \langle ijj \ldots j \rangle$. Thus $\langle jii \ldots i \rangle$ is dominated by $v \in C \cap V(G_j)$ which contradicts (b) and proves the claim.

To conclude the proof we must show that every inner vertex w of $S(n+1,k)$ is dominated. If w is an inner vertex of some G_i, this follows from (a). So let $w = \langle ijj \ldots j \rangle$ with $j \neq i$. If w is not dominated by $C \cap V(G_i)$ then by (b), it must be dominated by the vertex $\langle jii \ldots i \rangle$. ∎

Theorem 2.2. *Let $n \geq 1$, $k \geq 1$, and let C be an almost perfect code of $S(n,k)$.*

If n is an odd integer, then one of the following two possibilities occurs:

(1) *There is no extreme vertex in C. Then C is unique (denoted by S) and no extreme vertex is dominated by C.*

(2) *For some i, $\langle ii\dots i\rangle \in C$. Then C is unique (denoted by P_i) and for all $j \neq i$, $\langle jj\dots j\rangle \notin C$. Moreover, all extreme vertices are dominated by C.*

Furthermore the codes $S, P_1, \dots, P_k$ exist and partition the vertex set of $S(n,k)$.

If n is an even integer, then one of the following two possibilities occurs:

(3) *For some i, $\langle ii\dots i\rangle \in C$. Then C is unique (denoted by A) and for all j, $\langle jj\dots j\rangle \in C$.*

(4) *For some i, $\langle ii\dots i\rangle$ is not dominated by C. Then C is unique (denoted by B_i), and for all $j \neq i$, $\langle jj\dots j\rangle \notin C$ but is dominated by C.*

Furthermore the codes $A, B_1, \dots, B_k$ exist and partition the vertex set of $S(n,k)$.

The possibilities that occur in Theorem 2.2 are schematically presented in Fig. 2.

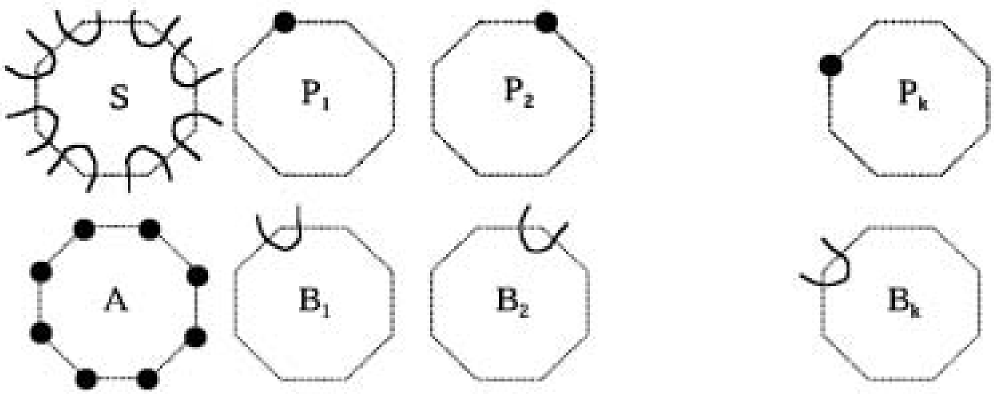

Fig. 2. Codes in $S(n,k)$ for odd n (above) and even n (below).

Before proving Theorem 2.2 we note that it immediately implies the following result from [14]:

Corollary 2.3. *If n is odd then the only perfect codes of $S(n,k)$ are $P_1, P_2, \dots, P_k$. If n is even then A is the unique perfect code of $S(n,k)$.* ■

Proof of Theorem 2.2. Our proof works by induction on n. Since $S(1,k)$ is the complete graph on k vertices the result is clear for $n = 1$. Assume now that the assertions hold for some $n \geq 1$ and let C be an almost perfect code of $S(n+1,k)$.

Suppose first that n is odd. By Lemma 2.1, for every i, $C \cap V(G_i)$ is an almost perfect code of $S(n,k)$. Moreover, by induction hypothesis, $C \cap V(G_i)$ must be equal to S or to P_j for some j.

Case 1. For some i, $\langle ii \dots i\rangle \in C$.
This implies that $C \cap V(G_i)$ is equal to P_i. By (2), for all $j \neq i$, the vertex $\langle ijj \dots j\rangle \notin C$ and is dominated by a vertex in $C \cap V(G_i)$. Thus, by Lemma 2.1, the vertex $\langle jii \dots i\rangle$ is dominated by some vertex in $C \cap V(G_j)$ and $\langle jii \dots i\rangle \notin C$. Since $\langle jii \dots i\rangle$ is dominated in G_j, by the induction hypothesis $C \cap V(G_j)$ cannot be S, hence $C \cap V(G_j) = P_j$. Thus, C is uniquely determined.

Now, consider the set A defined by $A \cap V(G_i) = P_i$ for all i. Then for every i and j, by the definition of P_i and P_j, A satisfies the hypothesis of Lemma 2.1, and so A is an almost perfect code satisfying (3).

Case 2. For every i, $\langle ii \dots i\rangle \notin C$ and $\langle ii \dots i\rangle$ is dominated by some vertex in C.
This implies that for every i, $C \cap V(G_i) \neq S$ and $C \cap V(G_i) \neq P_i$. Therefore, for a fixed i, there is some $j \neq i$ such that $C \cap V(G_i) = P_j$. Moreover, there is some $l \neq j$ such that $C \cap V(G_j) = P_l$. The vertices $\langle ijj \dots j\rangle$ and $\langle jii \dots i\rangle$ violate Condition (b) of Lemma 2.1. Therefore Case 2 never occurs.

Case 3. For every i, $\langle ii \dots i\rangle \notin C$ and there is some j with $\langle jj \dots j\rangle$ not dominated by C.
This implies that $C \cap V(G_j) = S$. By (1), none of the vertices $\langle jii \dots i\rangle$ is dominated by $C \cap V(G_j)$. Since C is an almost perfect code, Lemma 2.1 implies that $\langle jii \dots i\rangle$ must be dominated by $\langle ijj \dots j\rangle$, therefore for every $i \neq j$ we have $C \cap V(G_i) = P_j$. Thus, C is uniquely determined.

Now, consider the set B_j defined by $B_j \cap V(G_i) = P_j$ for all $i \neq j$ and $B_j \cap V(G_j) = S$. To verify that B_j is an almost perfect code, select a vertex $\langle abb \dots b\rangle$ and check condition (b) of Lemma 2.1 according to the cases:

- $a \neq j$ and $b \neq j$.
- $a = j$.

Furthermore, B_j satisfies (4).

To complete the case when n is odd, notice that for all j, we have $V(G_j) \cap A = P_j$, and for all $i \neq j$, $V(G_i) \cap B_j = P_j$, and $V(G_j) \cap B_j = S$. Moreover, by the induction hypothesis, $S, P_1, \dots, P_k$ partition $V(G_j) = S(n, k)$ which in turn implies that $A, B_1, \dots, B_k$ partition $S(n+1, k)$.

Now assume that n is even. By Lemma 2.1, for every i, $C \cap V(G_i)$ is an almost perfect code of $S(n, k)$. Moreover, by the induction hypothesis, $C \cap V(G_i)$ must be equal to A or to B_j (for some j).

Case 1. For every i, $\langle ii \dots i\rangle$ does not belong to C.
This implies that for a fixed i, we have $C \cap V(G_i) = B_j$. We claim that $j = i$. Indeed, in the opposite case the vertex $\langle ijj \dots j\rangle$ is neither dominated by $C \cap V(G_i)$,

nor by $\langle jii\dots i\rangle$ in G_j by property of the B_l's. Moreover $\langle ijj\dots j\rangle$ is not an extreme vertex of $S(n+1,k)$ which contradicts that C is an almost perfect code. Thus, C is uniquely determined with $C\cap V(G_i)=B_i$ for all i.

Now consider the set S defined by $S\cap V(G_i)=B_i$ for all i. By (4), S satisfies condition (b) of Lemma 2.1, so in each copy G_i, the vertex $\langle ii\dots i\rangle$ is the only vertex in S which is not dominated. Moreover, since $S\cap V(G_i)=B_i$ is an almost perfect code of G_i, we have that S is an almost perfect code of $S(n+1,k)$ satisfying (1).

Case 2. There is some i such that $\langle ii\dots i\rangle$ belongs to C.
This implies that $C\cap V(G_i)=A$. Choose $j\neq i$. Thus, the vertex $\langle ijj\dots j\rangle$ belongs to C, and so $\langle jii\dots i\rangle$ is not dominated by $C\cap V(G_j)$. Therefore, $C\cap V(G_j)=B_i$. Thus, C is uniquely determined with $C\cap V(G_i)=A$ and $C\cap V(G_j)=B_i$ for all $j\neq i$.

Now, consider the set P_i defined by $P_i\cap V(G_j)=B_i$ for all $j\neq i$ and $P_i\cap V(G_i)=A$. First, check property (b) of Lemma 2.1 for the edges $\langle ijj\dots j\rangle\langle jii\dots i\rangle$, and $\langle jll\dots l\rangle\langle ljj\dots j\rangle$ with j and $l\neq i$. Secondly, observe that $\langle jj\dots j\rangle\notin P_i$ and is dominated by $P_i\cap V(G_j)$ for all $j\neq i$. Finally, P_i satisfies (2).

To complete the proof of Theorem 2.2 it is enough to notice that for all j, we have $V(G_j)\cap S=B_j$, and for all $i\neq j$, $V(G_i)\cap P_j=B_j$, and $V(G_j)\cap P_j=A$. Moreover, by the induction hypothesis, $A,B_1,\dots,B_k$ partition $V(G_j)=S(n,k)$ which implies that $S,P_1,\dots,P_k$ partition $S(n+1,k)$. ■

3. $L(2,1)$-LABELINGS OF SIERPIŃSKI GRAPHS

In this section we give an optimal $L(2,1)$-labeling of the Sierpiński graphs. First we need the following lemma:

Lemma 3.1. *Let ℓ be an $L(2,1)$-labeling of the complete graph K_n such that the span of ℓ is at most $2n-1$. Then the image of ℓ is either $\{0,2,\dots,2n-2\}$, or $\{1,3,\dots,2n-1\}$, or there is an i such that the image of ℓ is $\{0,2,\dots,2i-2,2i+1,2i+3,\dots,2n-1\}$.*

Proof. Let I be the image of ℓ. If I contains no odd number then $I=\{0,2,\dots,2n-2\}$. Now, assume that $2i+1$ is the smallest odd number occurring in I. So, I contains at most i even numbers among $\{0,2,\dots,2i-2\}$. Thus I must contain at least $n-i-1$ numbers from $\{2i+3,2i+4,\dots,2n-1\}$; and the only possibility is to take all the odd numbers from this set. We conclude that $I=\{0,2,\dots,2i-2,2i+1,2i+3,\dots,2n-1\}$. ■

Theorem 3.2. *For any $n\geq 2$ and any $k\geq 3$, $\lambda(S(n,k))=2k$.*

Proof. By Theorem 2.2, the vertex set of $S(n,k)$ can be partitioned into $k+1$ codes $X_0, X_1, \ldots X_k$. Thus $\lambda(S(n,k)) \leq 2k$ by Proposition 1.1.

In the rest of the proof we need to show that there is no labeling of $S(n,k)$ with a smaller span. As $S(n,k)$ is an isometric subgraph of $S(n+1,k)$ (for any $n \geq 1$), it suffices to show that $\lambda(S(2,k)) \geq 2k$ for $k \geq 3$. The graph $S(2,k)$ consists of k complete subgraphs on k vertices induced by the vertex sets $L_i = \{\langle ij \rangle \mid j = 1,2,\ldots,k\}$. In addition, for $i \neq j$, the vertex $\langle ij \rangle \in L_i$ is adjacent to the vertex $\langle ji \rangle \in L_j$.

Let ℓ be an $L(2,1)$-labeling of $S(2,k)$ and define $\ell(L_i) = \{\ell(\langle ij \rangle) \mid j = 1,2,\ldots,k\}$. Clearly, the span of $\ell(L_1)$ is at least $2k-2$. If it is equal $2k-2$, then $\ell(L_1) = \{0,2,\ldots 2k-2\}$. Consider the vertex $\langle 12 \rangle$ of L_1 and let $\ell(\langle 12 \rangle) = 2r$, where $r \in \{0,1,\ldots,k-1\}$. The vertex $\langle 12 \rangle \in L_1$ is adjacent to the vertex $\langle 21 \rangle \in L_2$, hence the distance between $\langle 12 \rangle$ and a vertex of L_2 is at most 2. It follows that $\ell(\langle 2s \rangle) \neq 2r$ for $1 \leq s \leq k$, and consequently $\lambda(S(2,k)) \geq 2k-1$.

Suppose $\lambda(S(2,k)) = 2k-1$. Let

$$S_0 = \{1,3,5,\ldots,2k-1\}, \quad S_k = \{0,2,4,\ldots,2k-2\}\,,$$

and for $i = 1,2,\ldots,k-1$ set

$$S_i = \{0,2,\ldots,2i-2,2i+1,2i+3,\ldots,2k-1\}\,.$$

Since L_i induces a K_k and $\lambda(S(2,k)) = 2k-1$, then by Lemma 3.1, for any i there exists an $r \in \{0,1,\ldots,k\}$ such that $\ell(L_i) = S_r$. Moreover, we claim that $\ell(L_i) \neq \ell(L_j)$ whenever $i \neq j$. Indeed, consider the edge xy where $x \in L_i, y \in L_j$. Then $\ell(x) \notin \ell(L_j)$ which implies the claim. We now consider two cases.

Case 1. For any i, $\ell(L_i) \neq S_0$; or for any i, $\ell(L_i) \neq S_k$.
If for any i, $\ell(L_i) \neq S_0$, then the label 0 is used in each $\ell(L_i)$. We claim that $\ell(x) = 0$ if and only if x is an extreme vertex. Indeed, since a non extreme vertex of L_i is adjacent to some L_j with $j \neq i$, we have $\ell(x) \notin l(L_j)$. Therefore, all the extreme vertices of $S(2,k)$ are labeled 0. We may assume without loss of generality that $\ell(L_1) = S_k$. Let $\ell(\langle 1j \rangle) = 2$ and $\ell(\langle 1j' \rangle) = 4$. Then 2 and 4 are forbidden labels for L_j and $L_{j'}$, respectively. Hence $\ell(L_j) = S_1$. Moreover, since $\ell(L_{j'})$ is either S_1 or S_2, we must have $\ell(L_{j'}) = S_2$. Consider the edge xy between L_j and $L_{j'}$. Since x is on distance at most two from vertices of $L_{j'}$, $\ell(x) = 3$. Similarly it follows that $\ell(y) = 2$, but this is not possible as x is adjacent to y.

If for any i, $\ell(L_i) \neq S_k$, then we can argue analogously as above with the label $2k-1$ in place of the label 0 and by setting $\ell(L_1) = S_0$.

Case 2. For some i and j, $\ell(L_i) = S_0$ and $\ell(L_j) = S_k$.
We may assume that $\ell(L_1) = S_k$. Suppose first that $k \geq 4$. Because $\ell(L_1) = S_k$,

there are two vertices u and v of L_1 such that none of them is an extreme vertex and that $\ell(u) = 2t$ and $\ell(v) = 2(t+1)$ for some t. Let the neighbor of u outside L_1 lie in L_r and the neighbor of v outside L_1 belong to L_s. Then $\ell(L_r) = S_t$ and $\ell(L_s) = S_{t+1}$. Similar to Case 1, consider the edge xy between L_r and L_s. From the distances between x and L_s and between y and L_r we conclude that $\ell(x) = 2t+1$ and $\ell(y) = 2t$ which is not possible.

It remains to consider the case $k = 3$. We have $\ell(L_1) = \{0, 2, 4\}$ and we may assume in addition $\ell(L_2) = \{1, 3, 5\}$. Suppose $\ell(L_3) = \{0, 2, 5\}$. Then we immediately see that $\ell(\langle 13\rangle)$ cannot be 0 or 2, so $\ell(\langle 13\rangle) = 4$. Similarly, $\ell(\langle 31\rangle)$ must be 5, but this is impossible. The subcase $\ell(L_3) = \{0, 3, 5\}$ is treated analogously and the proof is complete. ■

We note that in the above proof Cases 1 and 2 could be merged into a single one for $k \geq 4$. However, this would make the analysis of the case $k = 3$ longer and pedestrian. We also wish to add that we have excluded the cases $k = 1$, $k = 2$, and $n = 1$ from the statement of the theorem from aesthetical reasons and since the corresponding λ-values are well known. Indeed, $S(n, 1)$ is isomorphic to K_1 for any n, and $S(n, 2)$ is the path on 2^n vertices, while $S(1, k)$ is K_k.

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Sylvain Gravier
CNRS, GeoD Research Group-Laboratoire Leibniz,
46 Avenue Félix Viallet,
38031 Grenoble Cédex,
France
E-mail: sylvain.gravier@imag.fr

Sandi Klavžar
Department of Mathematics PeF,
University of Maribor,
Koroška 160, 2000 Maribor,
Slovenia
E-mail: sandi.klavzar@uni-mb.si

Michel Mollard
CNRS, GeoD Research Group-Laboratoire Leibniz,
46 Avenue Félix Viallet,
38031 Grenoble Cédex,
France
E-mail: michel.mollard@imag.fr