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SPIKE SOLUTIONS OF A NONLINEAR ELECTRIC CIRCUIT WITH A PERIODIC INPUT

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Abstract. We consider spike solutions of a second order differential equation with a forcing modeling a nonlinear circuit used in converting analog signals to digital ones. It is shown that the number of spikes which correspond to bits in digital signals can be provided by asymptotic expansions. Numerical results are also presented.

1. Introduction

An electronic circuit is an interconnection of components which can be modelled by a system of ordinary differential equations by using Kirchhoff voltage and current Laws. The type of circuits we are interested in for this article is illustrated in Fig. 1.

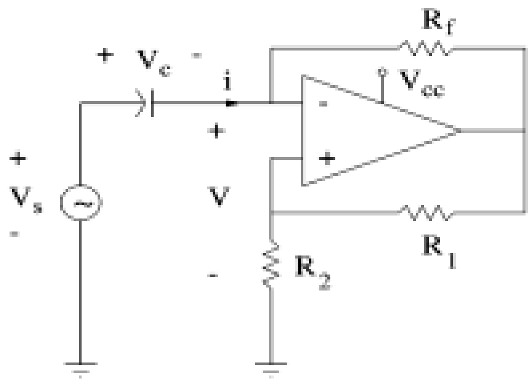

Fig. 1. A nonlinear electronic circuit.

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This circuit can be modelled by the following system of nonlinear differential equations

$$(1.1)\qquad \begin{cases} \dfrac{dV}{ds} = \dfrac{dV_s}{ds} - \dfrac{i}{C}, \\ \mu\dfrac{di}{ds} = V - \psi(i), \end{cases}$$

where C is the capacitance, μ is related to inductance, V_s is the input voltage, V is the output voltage, $\psi(i)$ is the *current-voltage* characteristic of the circuit. Note that $\psi(i)$ is piece-wise linear and is of the following form:

$$\psi(i) = \begin{cases} K_1 i, & \text{if } i > 0; \\ K_2 i, & \text{if } i_0 < i \le 0; \\ K_2 i_0 + K_1(i - i_0), & \text{if } i \le i_0, \end{cases}$$

where $K_1 = R_f$, $K_2 = -(R_2/R_1)R_f$ and $i_0 = -(R_1/R_2)V_1/R_f$. The graph of the function is of (single) S-shape (cf. Figure 2).

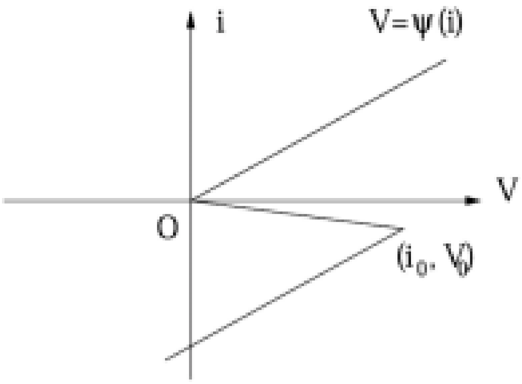

Fig. 2. Function $V = \psi(i)$

For the circuit we are interested, C and μ are usually very small. The problem is thus a singularly perturbed problem with two small parameters. To ease the numerical computation and also the theoretical analysis we make a time variable transformation

$$s = Ct.$$

The differential Equations (1.1) then become

$$(1.2)\qquad \begin{cases} \dfrac{dV}{dt} = \dfrac{dV_s}{dt} - i, \\ \varepsilon\dfrac{di}{dt} = V - \psi(i), \end{cases}$$

where $\varepsilon = \mu/C$ which we assume ε is relatively small.

There are many practical applications of such nonlinear electric circuits in engineering (cf. [9]). Our interest in this particular circuit is in its application in the ultra wide band technology in wire or wireless communications. We refer our readers to the website ”www.cellonics.com” for its connection to information technology. In [1], Chow and Huang gave a detailed study for the existence and stability of spiking solutions of (1.2). A precise definition of spike solutions and the number of spikes associated with theses solutions were given. In this paper, we are interested in the precise number of spikes under a periodic input and finding their asymptotic formulae.

We are particularly interested in studying the relation between the amplitude and frequency of a periodic input voltage V_s and the number of spikes of its output. If there is no input ($V_s = 0$) and ε is small, then system (1.2) has a stable limit cycle and its orbit in the phase plane will rotate around a limit cycle which is unique. For each rotation in the phase plane, the time series of the corresponding solution gives a spike or pulse as ε is small. We call such solution a *spike solution*. More precise definition of the spike solution will be given in the next section (see also [1]). We will study the formation of spike solutions and compute the number of spikes in terms of input parameters. The work has been of interest to researchers in wireless communication. We believe that our work would be of interest to researchers in demodulation scheme in communication but also be useful to other nonlinear circuits.

In this paper we consider a single S-shaped characteristic function first. The formation of a spike solution is intuitively described with a phase plane analysis in §2. Then in §3 the time interval of one spike (one-spike time) is calculated based on asymptotic analysis. In §4 formulae are derived for computing the number of spikes associated with piecewise linear periodic and sinusoidal inputs. Numerical experiments are given to demonstrate our computation. Bifurcation diagrams are drawn to show how input frequency-amplitude regions are associated with the number of spikes. In §5, a characteristic function which combines two single S-shaped ones is studied so that much richer and more interesting output spike wave patterns are obtained.

We note that existence and uniqueness of a limit cycle for a periodically forced van der Pol equation have been proved in [3] and asymptotic solutions were given in [5] and [10] (see also, [6]). In [4], a geormetric approach to relaxation oscillation is presented. However, all these work are related to asymptotic behavior of solutions as time approached infinity. Whereas we are only interested in the number of spikes of an orbit in one period which is the period of the periodic forcing.

2. Formation of a Spike Solution for Single S-Shaped $\psi(i)$

The system (1.2) is not autonomous because of the input signal term

$$\frac{dV_s}{dt} = f(t).$$

In practice $f(t)$ is usually a periodic function. We first consider orbits in phase plan when $f(t)$ is a constant. We then let time to flow to obtain properties of these solutions. We are able to do this because of the small parameter ε. For a rigorous proof for such solutions, we refer to [1].

Let $\Gamma : V = \psi(i)$ be the characteristic curve of the system. Assume that $f(t)$ is a constant. Thus, the intersection point of the curve Γ and the horizontal line $i = f(t)$ is a fixed point.

Consider a solution that starts at a point $P = (V(0), i(0))$. If $f(t)$ is between zero and i_0, then the fixed point is unstable. If $f(t)$ is larger than zero or smaller than i_0, then the fixed point is stable and every solution approaches to the fixed point.

For any fixed $f(t)$ we note that $\dfrac{di}{dt} = 0$ on the characteristic curve Γ for all ε, but at all other points $\dfrac{di}{dt}$ is very large as ε is close to zero. In other words, the directional field would be nearly vertical at all points except those very close to the characteristic curve Γ. With this in mind it is not very difficult to argue formally how solutions of Equation (1.2) behave in the i-V phase plane.

First consider the case where $i_0 < f(t) < 0$. Consider an orbit of (1.2) which starts at P. The orbit will be nearly a vertical straight line up to P_1, where it reaches Γ. Since the direction field at all points other than those near Γ is nearly vertical, the solution curve will tend to follow Γ, staying above it, until it gets to a vicinity of P_2. At this point the curve turns almost vertically downwards until Γ is reached once more at P_3. The curve then follows Γ, staying below it, until P_4 is reached, where it turns vertically upwards again to intersect Γ at P_5. Then it tends to follow the path from P_5 to P_2 (cf. Figure 3). Therefore the limit of the solution as $\varepsilon \to 0$

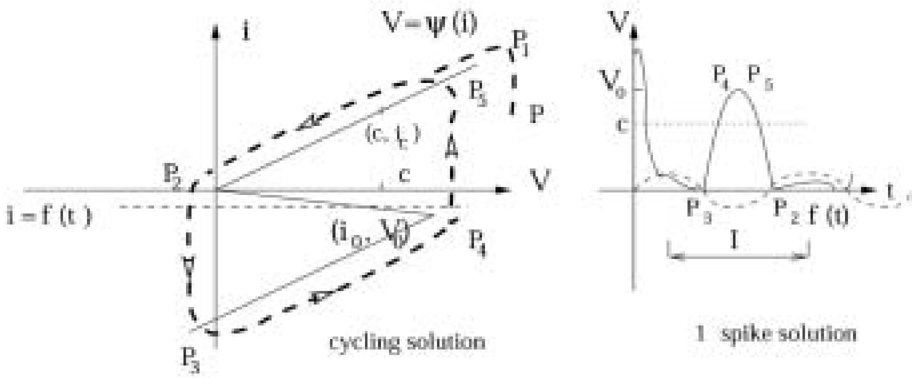

Fig. 3. Spike solution and spike solution when $0 > f(t) > i_0$

consists of the two segments P_5P_2 and P_3P_4 of Γ, and the two vertical lines P_4P_5 and P_2P_3(for a rigorous proof of this statement, see [1] or [4]). Note that the limit solution satisfies $V = \psi(i)$ except at certain points (i.e., P_2 and P_4) where i has jump discontinuities. These discontinuities cause difficulty in constructing the asymptotic formulae for the time it takes to go through a whole cycle. For convenience of description we will call such phase-plane solution a *spike solution*. The corresponding time-series for $V(t)$ goes from near zero to near V_0 and then from near V_0 back to near zero.

The cases $f(t) \geq 0$ and $f(t) \leq i_0$ can be described similarly and are illustrated in Figures 4-5, respectively. In these two cases we would not have a spike solution since the solution approaches a stable fixed point.

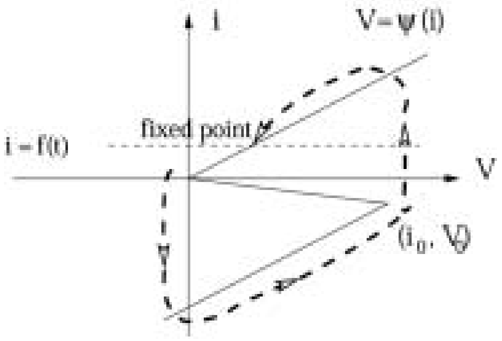

Fig. 4. Phase portrait when $f(t) \geq 0$

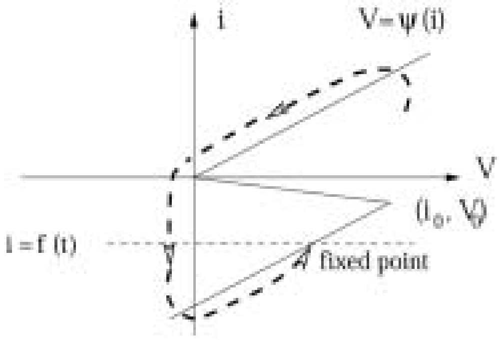

Fig. 5. Phase portrait when $f(t) \leq i_0$

Now let $f(t)$ be varying in time t. We assume that $f(t)$ is oscillatory around the axis $i = 0$ and its amplitude is less than $i_c = c/K_1$. Suppose $f(t)$ starts from some point, say its maximum [1], and moves down. After some time, say t_+, $f(t)$ moves down to zero. During this time segment $f(t)$ is above $i = 0$. The solution

The maximum should be positive since $f(t)$ is oscillatory around $i = 0$

of the system will be near the fixed point – a crossing point (moving with t) of the curve Γ and the horizontal line $i = f(t)$. After $t = t_+$, $f(t)$ moves down to the region $0 > i > i_0$. If $f(t)$ stays long enough (i.e. longer than the time the solution travels one cycle in the phase plane) in the region $(i_0, 0)$ then the solution will turn around the $P_3P_4P_5P_2$ cycle a few times. For each cycle the output voltage solution V moves from near zero to near V_0 and then turns back to near zero. The number of cycles the solution travels (or the number of spikes the output voltage produces) will depend on how long $f(t)$ stays in the region $(i_0, 0)$ and how long the solution needs to travel one cycle (*one-cycle time*). We will consider this in details in §3 and §4.

For convenience we only consider the voltage V above. A corresponding result for the current $i(t)$ can be similarly obtained.

If the minimum of $f(t)$ is larger than i_0 then after some time, say t_-, $f(t)$ will move up to positive side and the cycling behavior will stop until it turns back to below zero again. A typical such spike pattern is shown in Figure 6.

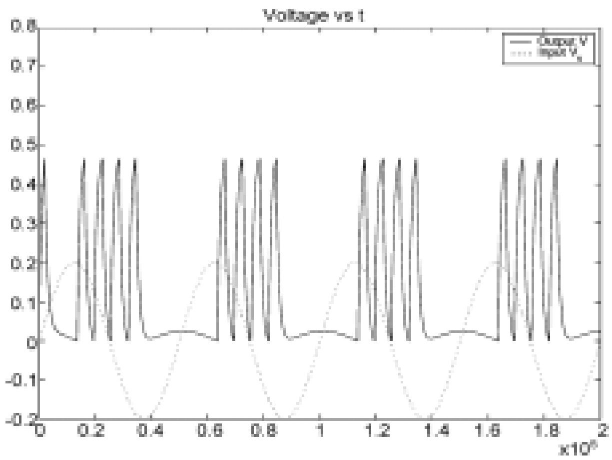

Fig. 6. Input and output voltage when the minimum of $f(t)$ does not drop below i_0.

The transformed system (1.2) is a singularly perturbed problem with one small parameter ϵ. When $\epsilon = 0$ it is an index-1 differential equation. In order to see the behavier of the solution we would like to solve it numerically. We have to use stiff ODE solver because of above mentioned properties. We adopt a variable order stiff solver which is a quasi-constant step size implementation in terms of backward differences of the Klopfenstein-Shampine family of numerical differentiation formulas of orders one to five (details may be found in [11]). The method works very well for the system (1.2). Note that in the computational results we take $f(t) = \frac{dV_s}{dt}$ and V_s is a sinusoidal input. From the figure we see that the negative part of $f(t)$ corresponds to the spikes in the output V.

If the minimum of $f(t)$ is smaller than i_0 then after $f(t)$ drops below i_0 the cycling behavior will also stop until it reaches the minimum and turns back to i_0. A typical spike pattern in this case is shown in Figure 7. From the figure we again see the negative part of $f(t) = dV_s/dt$ corresponds to the spike solution in the output V. When $f(t)$ drops below i_0 we see a flat part of V in the middle of spikes, which may be counted as a spike as well. But this flat spike solution can be avoided if we control the amplitude of the input $f(t)$ to be smaller than $|i_0|$.

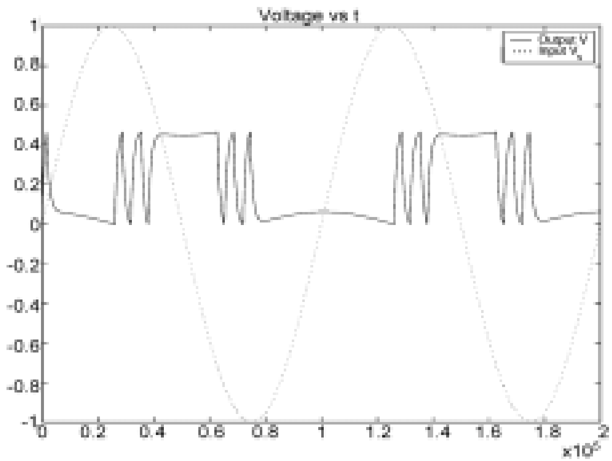

Fig. 7. Input and output voltage when the minimum of $f(t)$ drops below i_0.

To make the notation more conventional we denote

$$x = V, \quad y = i$$

in the rest of this paper. And the system (1.2) becomes

$$\begin{cases} \dfrac{dx}{dt} &= f(t) - y, \\ \varepsilon\dfrac{dy}{dt} &= x - \psi(y). \end{cases} \tag{2.1}$$

Definition 1. Let $f(t)$ be periodic function and $(x(t), y(t))$ be solution of system (2.1). If $(x(t), y(t))$ rotates n $P_2P_3P_4P_5$-cycles in one period of $f(t)$, then $(x(t, y(t))$ is called a n-spike solution.

3. Asymptotic Expansion of the One-cycle Time of the Solution

The computation of the one-cycle time t_o involves a construction of asymptotic expansion of the spike solution. Zeroth order approximation of t_o with respect to the small parameter ε is not very difficult to construct. It is just the time travelling from P_5 to P_2 along the characteristic curve Γ plus the time travelling from P_3 to P_4 along Γ (cf. Figure 3) and can be calculated by using the solution of the reduced system. In practice ε is not always very small. Higher order approximation of t_o is generally needed. In this section we provide the asymptotic expansions for general oscillatory function $f(t)$. We are only interested in the time segment where

$$f(t) \in (i_0, 0) \tag{3.1}$$

since we study the case that the system has a spike solution.

We divide the solution cycle into four parts according to the piecewise expression of the function $\psi(i)$ as illustrated in Figure 8.

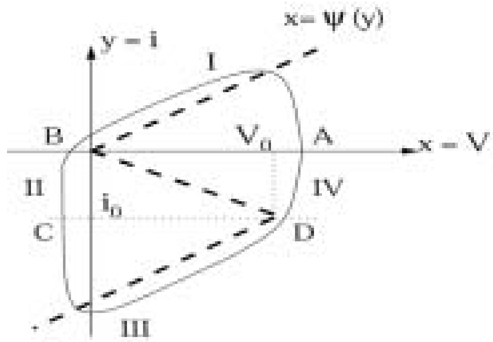

Fig. 8. Illustration of the construction procedure.

We will construct asymptotic solution of the differential equations and then the time it travels in each part. The construction is based on the matched asymptotic expansion combined with some specific techniques used in [7, 8, 12]. Motivated from a simpler example in [12] we should include $\varepsilon \ln \varepsilon$ in the expansions.

Region I where $\psi(y) = K_1 y$.

Let the solution start at a point A near $(V_0, 0)$ (cf. Figure 8), that is, initial conditions are

$$x(0) = x_A, \quad y(0) = y_A = 0. \tag{3.2}$$

Let x_A take the following expansion:

$$x(0) = V_0 + \beta_0 \varepsilon \ln \varepsilon + \gamma_0 \varepsilon + \cdots, \tag{3.3}$$

where β_0 and γ_0 are parameters to be determined later. Note that in this region both slow and fast modes are involved. We first construct the outer asymptotic expansion (slow mode solution) $x^o(t)$ and $y^o(t)$. Suppose [2]

$$x^o(t) = x_0(t) + x_1(t)\varepsilon\ln\varepsilon + x_2(t)\varepsilon + \cdots,$$
$$y^o(t) = y_0(t) + y_1(t)\varepsilon\ln\varepsilon + y_2(t)\varepsilon + \cdots. \tag{3.4}$$

Substituting (3.4) into (2.1) and equating like powers of $\varepsilon\ln\varepsilon$ and ε we have

$$\varepsilon^0 \quad \begin{cases} \dfrac{dx_0}{dt} &= f(t) - y_0, \\ 0 &= x_0 - K_1 y_0, \end{cases} \tag{3.5}$$

$$\varepsilon\ln\varepsilon \quad \begin{cases} \dfrac{dx_1}{dt} &= -y_1, \\ 0 &= x_1 - K_1 y_1, \end{cases} \tag{3.6}$$

$$\varepsilon \quad \begin{cases} \dfrac{dx_2}{dt} &= -y_2, \\ \dfrac{dy_0}{dt} &= x_2 - K_1 y_2, \end{cases} \tag{3.7}$$

$$\vdots \qquad\qquad \vdots$$

From (3.4) and (3.3), x_i, $i = 0, 1, 2$, should satisfy the following initial conditions

$$x_0(0) = V_0, \quad x_1(0) = \beta_0, \quad x_2(0) = \gamma_0. \tag{3.8}$$

Solving equations (3.5)-(3.7) with initial conditions (3.8), respectively, we obtain

$$\begin{cases} x_0(t) = V_0 e^{-\frac{t}{K_1}} + e^{-\frac{t}{K_1}} \displaystyle\int_0^t e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi, \\ y_0(t) = \dfrac{V_0}{K_1} e^{-\frac{t}{K_1}} + \dfrac{1}{K_1} e^{-\frac{t}{K_1}} \displaystyle\int_0^t e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi, \end{cases}$$

$$\begin{cases} x_1(t) = \beta_0 e^{-\frac{t}{K_1}}, \\ y_1(t) = \dfrac{\beta_0}{K_1} e^{-\frac{t}{K_1}}, \end{cases}$$

$$\begin{cases} x_2(t) = \gamma_0 e^{-\frac{t}{K_1}} + \dfrac{1}{K_1^3}(K_1 f(0) - V_0) t e^{-\frac{t}{K_1}} + \dfrac{1}{K_1^2} e^{-\frac{t}{K_1}} \displaystyle\int_0^t \int_0^\xi e^{\frac{\eta}{K_1}} f'(\eta)\mathrm{d}\eta\mathrm{d}\xi, \\ y_2(t) = \left(\dfrac{\gamma_0}{K_1} + \dfrac{V_0}{K_1^3}\right) e^{-\frac{t}{K_1}} + \dfrac{1}{K_1^4}(K_1 f(0) - V_0) t e^{-\frac{t}{K_1}} - \dfrac{1}{K_1^2} f(t) \\ \qquad + \dfrac{1}{K_1^3} e^{-\frac{t}{K_1}} \displaystyle\int_0^t \int_0^\xi e^{\frac{\eta}{K_1}} f'(\eta)\mathrm{d}\eta\mathrm{d}\xi + \dfrac{1}{K_1^3} e^{-\frac{t}{K_1}} \displaystyle\int_0^t e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi. \end{cases}$$

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[2] The term $\varepsilon\ln\varepsilon$ is expected without surprising. Recall that in [2] the period of periodic solution of a forced Van der Pol equation has the form $\varepsilon^{\frac{1}{3}} + \varepsilon\ln\varepsilon + \cdots$. Because our system is piece-wise linear, the term $\varepsilon^{\frac{1}{3}}$ does not appear.

Hence,

$$\begin{cases} x^o(t) = & V_0 e^{-\frac{t}{K_1}} + e^{-\frac{t}{K_1}} \int_0^t e^{\frac{\xi}{K_1}} f(\xi) \mathrm{d}\xi + \beta_0 e^{-\frac{t}{K_1}} \varepsilon \ln \varepsilon + \Big[\gamma_0 e^{-\frac{t}{K_1}} \\ & + \frac{1}{K_1^3}(K_1 f(0) - V_0) t e^{-\frac{t}{K_1}} + \frac{1}{K_1^2} e^{-\frac{t}{K_1}} \int_0^t \int_0^\xi e^{\frac{\eta}{K_1}} f'(\eta) \mathrm{d}\eta \mathrm{d}\xi \Big] \varepsilon + \cdots, \\ y^o(t) = & \frac{V_0}{K_1} e^{-\frac{t}{K_1}} + \frac{1}{K_1} e^{-\frac{t}{K_1}} \int_0^t e^{\frac{\xi}{K_1}} f(\xi) \mathrm{d}\xi + \frac{\beta_0}{K_1} e^{-\frac{t}{K_1}} \varepsilon \ln \varepsilon \\ & + \Big[\Big(\frac{\gamma_0}{K_1} + \frac{V_0}{K_1^3} \Big) e^{-\frac{t}{K_1}} + \frac{1}{K_1^4}(K_1 f(0) - V_0) t e^{-\frac{t}{K_1}} - \frac{1}{K_1^2} f(t) \\ & + \frac{1}{K_1^3} e^{-\frac{t}{K_1}} \int_0^t \int_0^\xi e^{\frac{\eta}{K_1}} f'(\eta) \mathrm{d}\eta \mathrm{d}\xi + \frac{1}{K_1^3} e^{-\frac{t}{K_1}} \int_0^t e^{\frac{\xi}{K_1}} f(\xi) \mathrm{d}\xi \Big] \varepsilon + \cdots. \end{cases}$$

We then construct the inner asymptotic expansion (fast mode solution). Making the stretched time transformation $\tau = \dfrac{t}{\varepsilon}$ in the system (2.1), we have

$$\begin{cases} \frac{dx}{d\tau} = \varepsilon(f(\varepsilon\tau) - y), \\ \frac{dy}{d\tau} = x - K_1 y \end{cases} \tag{3.9}$$

satisfying initial conditions (3.2) where $x(0)$ has the expansion (3.3). Let $x^i(\tau)$ and $y^i(\tau)$ stand for inner expansion and assume

$$\begin{cases} x^i(\tau) = \bar{x}_0(\tau) + \bar{x}_1(\tau)\varepsilon \ln \varepsilon + \bar{x}_2(\tau)\varepsilon + \cdots, \\ y^i(\tau) = \bar{y}_0(\tau) + \bar{y}_1(\tau)\varepsilon \ln \varepsilon + \bar{y}_2(\tau)\varepsilon + \cdots. \end{cases} \tag{3.10}$$

Substituting (3.10) into (3.9) and equating like powers of $\varepsilon \ln \varepsilon$ and ε, we have

$$\varepsilon^0 \quad \begin{cases} \frac{d\bar{x}_0}{d\tau} = 0, \\ \frac{d\bar{y}_0}{d\tau} = \bar{x}_0 - K_1 \bar{y}_0, \end{cases} \tag{3.11}$$

$$\varepsilon \ln \varepsilon \quad \begin{cases} \frac{d\bar{x}_1}{d\tau} = 0, \\ \frac{d\bar{y}_1}{d\tau} = \bar{x}_1 - K_1 \bar{y}_1, \end{cases} \tag{3.12}$$

$$\varepsilon \quad \begin{cases} \frac{d\bar{x}_2}{d\tau} = f(0) - \bar{y}_0, \\ \frac{d\bar{y}_2}{d\tau} = \bar{x}_2 - K_1 \bar{y}_2, \end{cases} \tag{3.13}$$

$$\vdots \qquad\qquad \vdots$$

From (3.2) $\bar{y}_i$, $i = 0, 1, 2$, should satisfy the following initial conditions

$$\bar{y}_0(0) = \bar{y}_1(0) = \bar{y}_2(0) = 0. \tag{3.14}$$

Solving equations (3.11)-(3.12) with initial conditions (3.14), respectively, we obtain

$$\begin{cases} \bar{x}_0(\tau) = c_0, \\ \bar{y}_0(\tau) = -\dfrac{c_0}{K_1}e^{-K_1\tau} + \dfrac{c_0}{K_1}, \end{cases}$$

$$\begin{cases} \bar{x}_1(\tau) = c_1, \\ \bar{y}_1(\tau) = -\dfrac{c_1}{K_1}e^{-K_1\tau} + \dfrac{c_1}{K_1}, \end{cases}$$

$$\begin{cases} \bar{x}_2(\tau) = -\dfrac{c_0}{K_1^2}e^{-K_1\tau} + (f(0) - \dfrac{c_0}{K_1})\tau + c_2, \\ \bar{y}_2(\tau) = \left[\dfrac{1}{K_1^2}(f(0) - \dfrac{c_0}{K_1}) - \dfrac{c_2}{K_1}\right]e^{-K_1\tau} - \dfrac{c_0}{K_1^2}\tau e^{-K_1\tau} \\ \qquad\qquad + \dfrac{1}{K_1^2}(f(0) - \dfrac{c_0}{K_1})(K_1\tau - 1) + \dfrac{c_2}{K_1}. \end{cases}$$

Hence,

$$\begin{cases} x^i(\tau) = \; c_0 + c_1\varepsilon\ln\varepsilon + \left[-\dfrac{c_0}{K_1^2}e^{-K_1\tau} + (f(0) - \dfrac{c_0}{K_1})\tau + c_2\right]\varepsilon + \cdots, \\ y^i(\tau) = \; -\dfrac{c_0}{K_1}e^{-K_1\tau} + \dfrac{c_0}{K_1} + \dfrac{c_1}{K_1}(1 - e^{-K_1\tau})\varepsilon\ln\varepsilon \\ \qquad\quad + \left[\left(\dfrac{1}{K_1^2}(f(0) - \dfrac{c_0}{K_1}) - \dfrac{c_2}{K_1}\right)e^{-K_1\tau}\right. \\ \qquad\quad \left. -\dfrac{c_0}{K_1^2}\tau e^{-K_1\tau} + \dfrac{1}{K_1^2}(f(0) - \dfrac{c_0}{K_1})(K_1\tau - 1) + \dfrac{c_2}{K_1}\right]\varepsilon + \cdots, \end{cases}$$

Here c_0, c_1, c_2 are constants which will be determined by matching with the outer solution. Using Van Dyke's matching principle (cf. [7], Section 4.1) we have

$$c_0 = V_0,\; c_1 = \beta_0 \;\text{ and }\; c_2 = \gamma_0$$

and the composite asymptotic solution

(3.15)

$$\begin{cases} x^I(t) = V_0 e^{-\frac{t}{K_1}} + e^{-\frac{t}{K_1}} \int_0^t e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi + \beta_0 e^{-\frac{t}{K_1}} \varepsilon \ln \varepsilon + \Big[\gamma_0 e^{-\frac{t}{K_1}} - \frac{V_0}{K_1^2} e^{-K_1 \frac{t}{\varepsilon}} \\ \qquad + \frac{1}{K_1^3}(K_1 f(0) - V_0) t e^{-\frac{t}{K_1}} + \frac{1}{K_1^2} e^{-\frac{t}{K_1}} \int_0^t \int_0^\xi e^{\frac{\eta}{K_1}} f'(\eta)\mathrm{d}\eta \mathrm{d}\xi \Big] \varepsilon + \cdots, \\ y^I(t) = \frac{V_0}{K_1} e^{-\frac{t}{K_1}} - \frac{V_0}{K_1} e^{-K_1 \frac{t}{\varepsilon}} + \frac{1}{K_1} e^{-\frac{t}{K_1}} \int_0^t e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi - \frac{V_0}{K_1^2} t e^{-K_1 \frac{t}{\varepsilon}} \\ \qquad + \Big[\frac{\beta_0}{K_1} e^{-\frac{t}{K_1}} - \frac{\beta_0}{K_1} e^{-K_1 \frac{t}{\varepsilon}} \Big] \varepsilon \ln \varepsilon + \Big[\Big(\frac{\gamma_0}{K_1} + \frac{V_0}{K_1^3} \Big) e^{-\frac{t}{K_1}} - \frac{1}{K_1^2} f(t) \\ \qquad + \frac{1}{K_1^4}(K_1 f(0) - V_0) t e^{-\frac{t}{K_1}} + \frac{1}{K_1^3} e^{-\frac{t}{K_1}} \int_0^t \int_0^\xi e^{\frac{\eta}{K_1}} f'(\eta)\mathrm{d}\eta \mathrm{d}\xi \\ \qquad + \frac{1}{K_1^3} e^{-\frac{t}{K_1}} \int_0^t e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi + \Big(\frac{1}{K_1^2}(f(0) - \frac{V_0}{K_1}) - \frac{\gamma_0}{K_1} \Big) e^{-K_1 \frac{t}{\varepsilon}} \Big] \varepsilon + \cdots. \end{cases}$$

Thus the time t_1 needed for the solution of (2.1) to travel from A to B is determined by

(3.16) $$y^I(t) = 0.$$

Assume

(3.17) $$t_1 = p_1 + q_1 \varepsilon \ln \varepsilon + r_1 \varepsilon + \cdots.$$

Then by (3.15) and (3.16), p_1, q_1 and r_1 satisfy the following equations, respectively

(3.18) $$\int_0^{p_1} e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi = -V_0,$$

(3.19) $$q_1 = -\frac{\beta_0}{f(p_1)} e^{-\frac{p_1}{K_1}},$$

(3.20) $$r_1 = \frac{1}{K_1} - \frac{K_1}{f(p_1)} L_1(p_1) e^{-\frac{p_1}{K_1}},$$

where

$$L_1(p_1) = \frac{\gamma_0}{K_1} + \frac{1}{K_1^4}(K_1 f(0) - V_0) p_1 + \frac{1}{K_1^3} \int_0^{p_1} \int_0^\xi e^{\frac{\eta}{K_1}} f'(\eta)\mathrm{d}\eta \mathrm{d}\xi.$$

We then have

$$x^I(t_1) = \alpha_1 + \beta_1 \varepsilon \ln \varepsilon + \gamma_1 \varepsilon + \cdots,$$

where

(3.21) $$\alpha_1 = \beta_1 = 0,\ \gamma_1 = \frac{f(p_1)}{K_1}.$$

Region II where $\psi(y) = K_2 y$.

Only fast mode is involved in this region. Introducing the stretched time $\tau = \frac{t - t_1}{\varepsilon}$ we have

(3.22) $$\begin{cases} \dfrac{dx}{d\tau} = \varepsilon(f(\varepsilon\tau + t_1) - y), \\ \dfrac{dy}{d\tau} = x - K_2 y \end{cases}$$

with initial conditions

(3.23) $$\begin{cases} x(t_1) & = \alpha_1 + \beta_1 \varepsilon \ln \varepsilon + \gamma_1 \varepsilon + \cdots, \\ y(t_1) & = 0. \end{cases}$$

Suppose

(3.24) $$\begin{cases} x^{II}(\tau) = \bar{x}_0(\tau) + \bar{x}_1(\tau)\varepsilon \ln \varepsilon + \bar{x}_2(\tau)\varepsilon + \cdots, \\ y^{II}(\tau) = \bar{y}_0(\tau) + \bar{y}_1(\tau)\varepsilon \ln \varepsilon + \bar{y}_2(\tau)\varepsilon + \cdots. \end{cases}$$

Substituting (3.23) into (3.22) and equating like powers of $\varepsilon \ln \varepsilon$ and ε, we have

(3.25) $$\varepsilon^0 \quad \begin{cases} \dfrac{d\bar{x}_0}{d\tau} & = 0, \\ \dfrac{d\bar{y}_0}{d\tau} & = \bar{x}_0 - K_2 \bar{y}_0, \end{cases}$$

(3.26) $$\varepsilon \ln \varepsilon \quad \begin{cases} \dfrac{d\bar{x}_1}{d\tau} & = 0, \\ \dfrac{d\bar{y}_1}{d\tau} & = \bar{x}_1 - K_2 \bar{y}_1, \end{cases}$$

(3.27) $$\varepsilon \quad \begin{cases} \dfrac{d\bar{x}_2}{d\tau} & = f(t_1) - \bar{y}_0, \\ \dfrac{d\bar{y}_2}{d\tau} & = \bar{x}_2 - K_2 \bar{y}_2, \end{cases}$$
$$\vdots \qquad\qquad \vdots$$

From (3.23), $\bar{x}_i$ and $\bar{y}_i$, $i = 0, 1, 2$, should satisfy the following initial conditions

(3.28) $\bar{x}_0(0) = \alpha_1, \quad \bar{x}_1(0) = \beta_1, \quad \bar{x}_2(0) = \gamma_1, \quad \bar{y}_0(0) = \bar{y}_1(0) = \bar{y}_2(0) = 0.$

Noting $\alpha_1 = \beta_1 = 0$ and solving equation (3.25)-(3.27) with initial conditions (3.28), respectively, we obtain

$$\begin{cases} \bar{x}_0(\tau) \equiv 0, \\ \bar{y}_0(\tau) \equiv 0, \end{cases}$$

$$\begin{cases} \bar{x}_1(\tau) \equiv 0, \\ \bar{y}_1(\tau) \equiv 0, \end{cases}$$

$$\begin{cases} \bar{x}_2(\tau) = f(t_1)\tau + \gamma_1, \\ \bar{y}_2(\tau) = \dfrac{1}{K_2^2}(f(t_1) - K_2\gamma_1)e^{-K_2\tau} + \dfrac{f(t_1)}{K_2}\tau + \dfrac{1}{K_2^2}(K_2\gamma_1 - f(t_1)). \end{cases}$$

Hence, changing the time variable back to t, yields

$$\begin{cases} x^{II}(t) = f(t_1)(t-t_1) + \gamma_1\varepsilon + \cdots, \\ y^{II}(t) = \dfrac{f(t_1)}{K_2}(t-t_1) + \left[\dfrac{1}{K_2^2}(f(t_1) - K_2\gamma_1)e^{-K_2\frac{t}{\varepsilon}} + \dfrac{1}{K_2^2}(K_2\gamma_1 - f(t_1))\right]\varepsilon + \cdots. \end{cases}$$

Solving

$$y^{II}(t) = i_0,$$

we get

$$t_2 = t_1 + \frac{1}{K_2}\varepsilon \ln \varepsilon + \frac{1}{K_2}\varepsilon \ln \frac{f(t_1) - K_2\gamma_1}{K_2 V_0} + \cdots,$$

which is the time needed for the solution to reach C in Figure 8. We can also compute

$$x^{II}(t_2) = \alpha_2 + \beta_2\varepsilon \ln \varepsilon + \gamma_2\varepsilon + \cdots,$$

where

(3.29) $$\alpha_2 = 0, \ \beta_2 = \frac{f(t_1)}{K_2}, \ \gamma_2 = \frac{f(t_1)}{K_2} \ln \frac{f(t_1) - K_2\gamma_1}{K_2 V_0} + \gamma_1.$$

Region III where $\psi(y) = K_1 y + (K_2 - K_1)i_0$.

The construction is similar to that for Region I. Let

$$a = (K_1 - K_2)i_0.$$

We can obtain the following composite expansion

$$\begin{cases} x^{III}(t) = & ae^{-\frac{t-t_2}{K_1}} - a + e^{-\frac{t-t_2}{K_1}}\displaystyle\int_{t_2}^{t} e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi + \beta_2 e^{-\frac{t-t_2}{K_1}}\varepsilon\ln\varepsilon \\ & +\Big[\dfrac{1}{K_1}\Big(\dfrac{V_0}{K_2} - \dfrac{a}{K_1}\Big)e^{-K_1\frac{t-t_2}{\varepsilon}} + \gamma_2 e^{-\frac{t-t_2}{K_1}} + \dfrac{1}{K_1^3}(K_1 f(t_2) - a)(t-t_2)e^{-\frac{t-t_2}{K_1}} \\ & +\dfrac{1}{K_1^2}e^{-\frac{t-t_2}{K_1}}\displaystyle\int_{t_2}^{t}\int_{t_2}^{\xi} e^{\frac{\eta}{K_1}} f'(\eta)\mathrm{d}\eta\mathrm{d}\xi\Big]\varepsilon + \cdots, \\ y^{III}(t) = & \dfrac{a}{K_1}e^{-\frac{t-t_2}{K_1}} + \dfrac{1}{K_1}e^{-\frac{t-t_2}{K_1}}\displaystyle\int_{t_2}^{t} e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi + \Big(\dfrac{V_0}{K_2} - \dfrac{a}{K_1}\Big)e^{-K_1\frac{t-t_2}{\varepsilon}} \\ & +\dfrac{1}{K_1}\Big(\dfrac{V_0}{K_2} - \dfrac{a}{K_1}\Big)(t-t_2)e^{-K_1\frac{t-t_2}{\varepsilon}} + \Big(\dfrac{\beta_2}{K_1}e^{-\frac{t-t_2}{K_1}} - \dfrac{\beta_2}{K_1}e^{-K_1\frac{t-t_2}{\varepsilon}}\Big)\varepsilon\ln\varepsilon \\ & +\Big\{\Big(\dfrac{\gamma_2}{K_1} + \dfrac{a}{K_1^3}\Big)e^{-\frac{t-t_2}{K_1}} + \dfrac{1}{K_1^4}(K_1 f(t_2) - a)(t-t_2)e^{-\frac{t-t_2}{K_1}} \\ & -\dfrac{1}{K_1^2}f(t) + \dfrac{1}{K_1^3}e^{-\frac{t-t_2}{K_1}}\displaystyle\int_{t_2}^{t}\int_{t_2}^{\xi} e^{\frac{\eta}{K_1}} f'(\eta)\mathrm{d}\eta\mathrm{d}\xi + \dfrac{1}{K_1^3}e^{-\frac{t-t_2}{K_1}}\displaystyle\int_{t_2}^{t} e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi \\ & +\Big(\dfrac{1}{K_1^2}(f(t_2) - \dfrac{a}{K_1}) - \dfrac{\gamma_2}{K_1}\Big)e^{-K_1\frac{t-t_2}{\varepsilon}}\Big\}\varepsilon + \cdots. \end{cases}$$

The time for the solution to reach D is obtained from

$$y^{III}(t_3) = i_0,$$

that is,

$$t_3 = t_2 + p_3 + q_3\varepsilon\ln\varepsilon + r_3\varepsilon + \cdots,$$

where p_3, q_3 and r_3 are determined by the following equations

$$\int_0^{p_3} e^{\frac{\xi}{K_1}} f(t_2 + \xi)\mathrm{d}\xi = \frac{K_1}{K_2}V_0 e^{\frac{p_3}{K_1}} - a, \tag{3.30}$$

$$q_3 = \frac{K_2\beta_2}{V_0 - K_2 f(t_2 + p_3)}\ e^{-\frac{p_3}{K_1}}, \tag{3.31}$$

$$r_3 = \frac{K_2 f(t_2 + p_3)}{K_1(K_2 f(t_2 + p_3) - V_0)} - \frac{K_1 K_2 L_3(p_3)}{K_2 f(t_2 + p_3) - V_0}\ e^{-\frac{p_3}{K_1}}, \tag{3.32}$$

and in (3.32)

$$L_3(p_3) = \frac{\gamma_2}{K_1} + \frac{V_0}{K_1^2 K_2} e^{\frac{p_3}{K_1}} + \frac{1}{K_1^4}(K_1 f(t_2) - a)p_3 + \frac{1}{K_1^3}\int_0^{p_3}\int_0^{\xi} e^{\frac{\eta}{K_1}} f'(t_2+\eta)\mathrm{d}\eta\mathrm{d}\xi.$$

We can then obtain

$$x^{III}(t_3) = \alpha_3 + \beta_3 \varepsilon \ln\varepsilon + \gamma_3 \varepsilon + \cdots,$$

where

$$\alpha_3 = V_0,\ \beta_3 = 0,\ \gamma_3 = \frac{1}{K_1^2} f(t_2 + p_3) - \frac{V_0}{K_1 K_2}. \tag{3.33}$$

Region IV where $\psi(y) = K_2 y$ again.

The construction is similar to that in Region II. The composite asymptotic expansion in this region is

$$\begin{cases} x^{IV}(t) = V_0 + (f(t_3) - \frac{V_0}{K_2})(t - t_3) + \gamma_3\varepsilon + \cdots, \\ y^{IV}(t) = \frac{V_0}{K_2} + \frac{1}{K_2}\left(f(t_3) - \frac{V_0}{K_2}\right)(t - t_3) \\ \quad + \left[\left(\frac{1}{K_2^2}\left(f(t_3) - \frac{V_0}{K_2}\right) - \frac{\gamma_3}{K_2}\right) e^{-K_2\frac{t-t_3}{\varepsilon}} \right. \\ \quad \left. + \frac{\gamma_3}{K_2} - \frac{1}{K_2^2}\left(f(t_3) - \frac{V_0}{K_2}\right)\right]\varepsilon + \cdots. \end{cases}$$

The time needed for the solution to reach E is determined from $y^{IV}(t_4) = 0$

$$t_4 = t_3 + \frac{1}{K_2}\varepsilon\ln\varepsilon + \frac{1}{K_2}\ln\left(\frac{\gamma_3}{V_0} - \frac{f(t_3)}{K_2 V_0} + \frac{1}{K_2^2}\right)\varepsilon + \cdots.$$

We can calculate

$$\begin{aligned} x^{IV}(t_4) &= V_0 + \frac{1}{K_2}\left(f(t_3) - \frac{V_0}{K_2}\right)\varepsilon\ln\varepsilon \\ &\quad + \left[\frac{1}{K_2}\left(f(t_3) - \frac{V_0}{K_2}\right)\ln\left(\frac{\gamma_3}{V_0} - \frac{f(t_3)}{K_2 V_0} + \frac{1}{K_2^2}\right) + \gamma_3\right]\varepsilon + \cdots, \end{aligned}$$

To obtain a spike solution E must coincide with A. We thus have the following formula for calculating β_0 and γ_0:

$$\beta_0 = \frac{1}{K_2}(f(t_3) - \frac{V_0}{K_2}), \tag{3.34}$$

(3.35) $$\gamma_0 = \frac{1}{K_2}\left(f(t_3) - \frac{V_0}{K_2}\right)\ln\left(\frac{\gamma_3}{V_0} - \frac{f(t_3)}{K_2V_0} + \frac{1}{K_2^2}\right) + \gamma_3.$$

Now we can summarize above result into a theorem.

Theorem 1. *Assume that for periodic input $f(t)$, system (2.1) has at least one-spike solution. Then the one-cycle time of the spike solution is given by*

(3.36) $$t_o = p + q\varepsilon\ln\varepsilon + r\varepsilon,$$

where

$$p = p_1 + p_3,\ q = q_1 + q_3 + \frac{2}{K_2},$$

$$r = r_1 + r_3 + \frac{1}{K_2}\ln\frac{K_1f(t_1) - K_2f(p_1)}{K_1K_2V_0}$$

$$+\frac{1}{K_2}\ln\left(\frac{K_1f(t_2+p_3) - K_2f(t_3)}{K_2^2V_0} + \frac{K_1 - K_2}{K_1K_2^2}\right)$$

and $p_1, p_3, q_1, q_3, r_1, r_3$ satisfy the following equations, respectively

$$\int_0^{p_1} e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi = -V_0,$$

$$q_1 = -\frac{\beta_0}{f(p_1)}e^{-\frac{p_1}{K_1}},$$

$$r_1 = \frac{1}{K_1} - \frac{K_1}{f(p_1)}L_1(p_1)e^{-\frac{p_1}{K_1}},$$

$$\int_0^{p_3} e^{\frac{\xi}{K_1}} f(t_2+\xi)\mathrm{d}\xi = \frac{K_1}{K_2}V_0e^{\frac{p_3}{K_1}} - a,$$

$$q_3 = \frac{K_2\beta_2}{V_0 - K_2f(t_2+p_3)}\ e^{-\frac{p_3}{K_1}},$$

$$r_3 = \frac{K_2f(t_2+p_3)}{K_1(K_2f(t_2+p_3) - V_0)} - \frac{K_1K_2L_3(p_3)}{K_2f(t_2+p_3) - V_0}\ e^{-\frac{t_2+p_3}{K_1}},$$

here

$$L_1(p_1) = \frac{\gamma_0}{K_1} + \frac{1}{K_1^4}(K_1f(0) - V_0)p_1 + \frac{1}{K_1^3}\int_0^{p_1}\int_0^{\xi} e^{\frac{\eta}{K_1}} f'(\eta)\mathrm{d}\eta\mathrm{d}\xi.$$

$$L_3(p_3) = \frac{\gamma_2}{K_1} + \frac{V_0}{K_1^2K_2}e^{\frac{p_3}{K_1}} + \frac{1}{K_1^4}(K_1f(t_2) - a)p_3$$

$$+\frac{1}{K_1^3}\int_0^{p_3}\int_0^{\xi} e^{\frac{\eta}{K_1}} f'(t_2+\eta)\mathrm{d}\eta\mathrm{d}\xi.$$

and other parameters can be determined by (3.21), (3.29), (3.33), (3.34) *and* (3.35).

Remark 1. In order for the system to have a spike solution, we need to assume that

$$t_o < t_-,$$

where t_- is the time $f(t)$ spends in the region $(i_0, 0)$ in one period of $f(t)$.

4. Formulas for Computing the Number of Spikes and Numerical Demonstration

As we analyzed above when $f(t)$ locates and stays long enough in the region $(i_0, 0)$ (e.g. $t_o < t_-$), the phase plane solution will produce cycles and the output voltage will produce spikes. Generally, from the phase plane analysis the number of spikes in one period of $f(t)$ produced in the output voltage can be determined roughly as $[\frac{t_-}{t_o}] + 1$, where $[\cdot]$ denotes the integral part of the number. Since $f(t)$ is given it is not difficult to obtain t_-. The formula for computing t_o has been found in the previous section after a construction of uniform asymptotic expansion of the cycling solution. From the formula we can see that the number of spikes depends mainly on the slopes of the characteristic curve Γ, i_0 (or V_0), and the function $f(t)$. Next we are going to consider a couple of special cases where the formula may be simpler. Then we run some numerical experiments to demonstrate the correctness of these formulas. In numerical simulation, taking numerical errors into account, it is better to choose parameters so that $\dfrac{t_-}{t_o}$ locates near the middle of the interval $([\dfrac{t_-}{t_o}], [\dfrac{t_-}{t_o}] + 1)$ to ensure that the expected number of spikes is produced.

Case I. (Periodic piecewise linear inputs)

$$V_\tau(t) = \begin{cases} -kt + A, & t \in [2nt_-, (2n+1)t_-], \\ kt - 3A, & t \in [(2n+1)t_-, (2n+2)t_-]. \end{cases}$$

That is,

$$f(t) = \begin{cases} -k, & t \in [2nt_-, (2n+1)t_-], \\ k, & t \in [(2n+1)t_-, (2n+2)t_-], \end{cases}$$

where $0 < k < -i_0, t_- = \dfrac{2A}{k}$, $A > 0$.

In this case the formula (3.36) can be obtained explicitly:

$$\begin{aligned} t_o &= p_1 + p_3 + (q_1 + q_3 + \frac{2}{K_2})\varepsilon \ln \varepsilon + \Bigg[r_1 + r_3 \\ &+ \frac{1}{K_2} \ln \frac{(K_2 - K_1)k}{K_1 K_2 V_0} + \frac{1}{K_2} \ln \frac{(kK_1 - V_0)(K_2 - K_1)}{K_1 K_2^2 V_0} \Bigg] \varepsilon + \cdots , \end{aligned}$$

where

$$\begin{aligned} p_1 &= K_1 \ln(1 + \frac{V_0}{K_1 k}), \\ q_1 &= -\frac{\beta_0}{f(p_1)} e^{-\frac{p_1}{K_1}} = -\frac{K_1(V_0 + K_2 k)}{K_2^2(V_0 + K_1 k)}, \\ r_1 &= \frac{1}{K_1} - \frac{K_1}{f(p_1)} L_1(p_1) e^{-\frac{p_1}{K_1}} = \frac{1}{K_1} + \frac{K_1^2}{K_1 k + V_0} L_1(p_1), \\ p_3 &= K_1 \ln \left(1 - \frac{K_2 V_0}{K_1(K_2 k + V_0)} \right), \\ q_3 &= \frac{K_2 \beta_2}{V_0 - K_2 f(p_3)} e^{-\frac{p_3}{K_1}} = -\frac{K_1 k}{K_1 K_2 k + (K_1 - K_2)V_0}, \\ r_3 &= \frac{K_2 k}{K_1(V_0 + K_2 k)} + \frac{K_1^2 K_2}{K_1 K_2 k + (K_1 - K_2)V_0} L_3(p_3), \end{aligned}$$

with

$$\begin{aligned} L_1(p_1) &= -\frac{V_0 + K_2 k}{K_1 K_2^2} \ln \left(-\frac{K_1^2 k + K_2 V_0}{K_1 K_2^2 V_0} + \frac{V_0 + K_2 k}{V_0 K_2^2} \right) \\ &\quad -\frac{K_1^2 k + K_2 V_0}{K_1^2 K_2^2} - \frac{V_0 + K_1 k}{K_1^3} \ln \left(1 + \frac{V_0}{K_1 k} \right), \\ L_3(p_3) &= -\frac{k}{K_1^2} - \frac{k}{K_1 K_2} \ln \frac{(K_1 + K_2)k}{-K_1 K_2 V_0} + \frac{V_0(K_1 K_2 k + K_1 V_0 - K_2 V_0)}{K_1^2 K_2 (K_1 K_2 k + K_1 V_0)} \\ &\quad -\frac{K_1 k + a}{K_1^3} \ln \left(1 - \frac{K_2 V_0}{K_1 K_2 k + K_1 V_0} \right). \end{aligned}$$

To demonstrate above formulas we take the following data from a real electronic circuit which is modeled by (1.2): $K_1 = 10^3$, $K_2 = -10^4$, $i_0 = -10^{-5}$. In order to have spike solution we take a pretty large $t_- = 5 \times 10^4$ so the condition (3.1) is satisfied. Let n be the number of spikes in each period of the input. Then from above formula we can roughly obtain a relationship between the amplitude k of $f(t)$ and the number of spikes n:

$$k = \frac{1}{2} \left[1 + \sqrt{1 - \frac{440}{e^{\frac{50}{n}} - 1}} \right] \times 10^{-5}.$$

The following table shows some data obtained by the above relationship.

k	.99999e-5	.99959e-5	.9498e-5	.97281e-5	.90370e-5	.69297e-5
n	3	4	5	6	7	8

Now we use again the variable order stiff ODE solver to solve the system (1.2) with k given in the table. Figure 9 shows numerical solutions where the number of spikes exactly matches those given in the table.

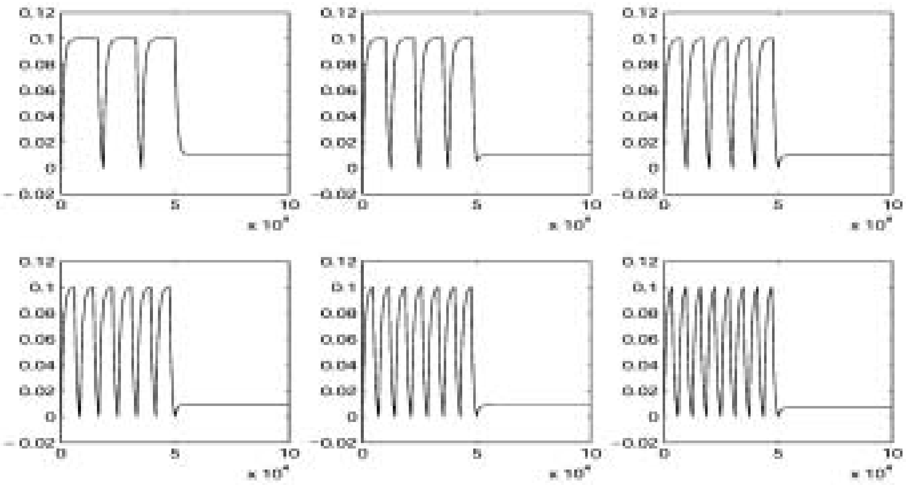

Fig. 9. V vs t for various choices of k

Furthermore, it is interesting to examine for which values of amplitude k and frequency $1/2t_{-}$ we have certain number of spikes. For above given input and $i_0 = -10^{-5}$ we draw bifurcation diagrams about the number of spikes for various choices of slopes of the piecewise linear characteristic function. From the computational results shown in Figures 10-13 we observe that when the slope ratio $|K_2/K_1|$ is larger the region to have a certain number of spikes is longer, and when the slope ratio is smaller the region is wider.

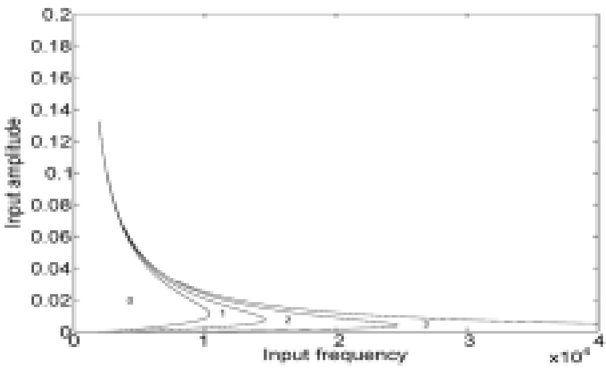

Fig. 10. Bifurcation diagram for $K_1 = 1000$ and $K_2/K_1 = -0.5$.

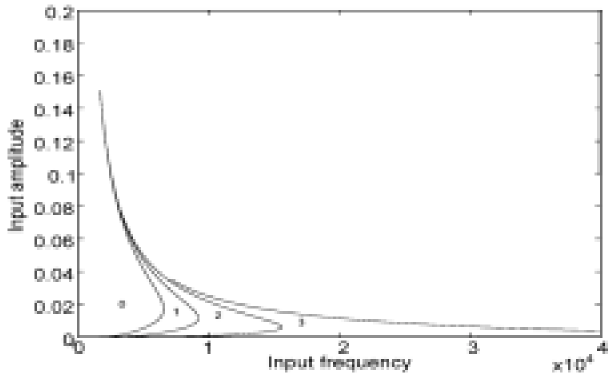

Fig. 11. Bifurcation diagram for $K_1 = 1000$ and $K_2/K_1 = -1$.

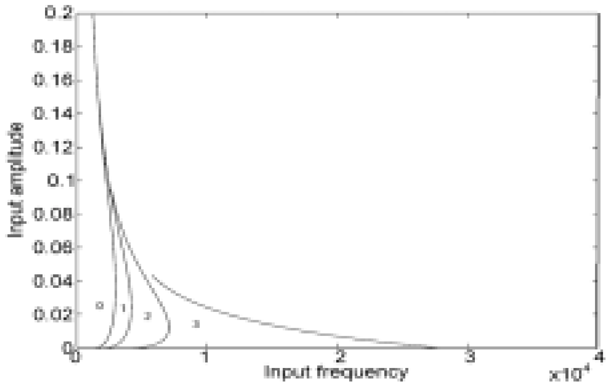

Fig. 12. Bifurcation diagram for $K_1 = 1000$ and $K_2/K_1 = -5$.

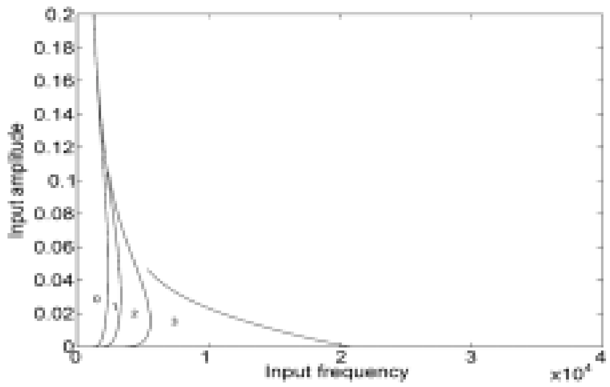

Fig. 13. Bifurcation diagram for $K_1 = 1000$ and $K_2/K_1 = -10$.

Case 2. (Sinusoidal inputs)

$$V_\tau = \frac{k}{\omega}\cos\omega t \text{ or } f(t) = -k\sin\omega t,$$

where $0 < k < -i_0$. Unlike case 1, there is no explicit expression for the parameters in the formula (3.36). However, if we ignore the complicated $O(\varepsilon\ln\varepsilon)$ terms we can have a relatively simple expression for the one-cycle time:

$$t_o = p_1 + p_3 + O(\varepsilon\ln\varepsilon),$$

where p_1 and p_3 can be obtained from the following equations

$$\begin{aligned} e^{\frac{p_1}{K_1}}(\sin\omega p_1 - K_1\omega\cos\omega p_1) - \frac{1+K_1^2\omega^2}{kK_1}V_0 + K_1\omega &= 0, \\ e^{\frac{p_3}{K_1}}\left(\sin\omega(p_1+p_3) - K_1\omega\cos\omega(p_1+p_3) + \frac{1+K_1^2\omega^2}{kK_2}V_0\right) & \\ -\frac{1+K_1^2\omega^2}{kK_1}(K_1-K_2)i_0 + K_1\omega\cos\omega p_1 + \sin\omega p_1 &= 0. \end{aligned}$$

We take the same data (K_1, K_2 and i_0) as in case 1 from a real electronic circuit. In this case we are going to fix the amplitude of V_τ, for example, $k/\omega = 0.05$ and making the frequency ω change. We can then obtain a set of data showing relationship between the frequency ω and the number of spikes n in the following table, where $fr = \omega/2\pi$.

$fr = \omega/2\pi$	3.20e-5	2.40e-5	1.70e-5	1.35e-5	1.10e-5	0.92e-5
n	3	4	5	6	7	8

We use the variable order stiff ODE solver again to solve the system (1.2) with ω given in the table. Figure 14 shows numerical solutions where the number of spikes matches those given in the table.

5. Spike Solution for a Double S-Shaped Characteristic Function

In this section we consider again the system (2.1) with a double S-shaped $\psi(y)$ defined by

$$\psi(y) = \begin{cases} K_1 y - K_1 a_2, & \text{if } y \ge a_2, \\ K_2 y - K_2 a_2, & \text{if } a_3 \le y < a_2, \\ K_3 y, & \text{if } 0 \le y < a_3, \\ K_4 y, & \text{if } a_4 \le y < 0, \\ K_5 y - K_5 a_5, & \text{if } a_5 \le y < a_4, \\ K_6 y - K_6 a_5, & \text{if } y < a_5. \end{cases}$$

Its graph is depicted in Figure 15.

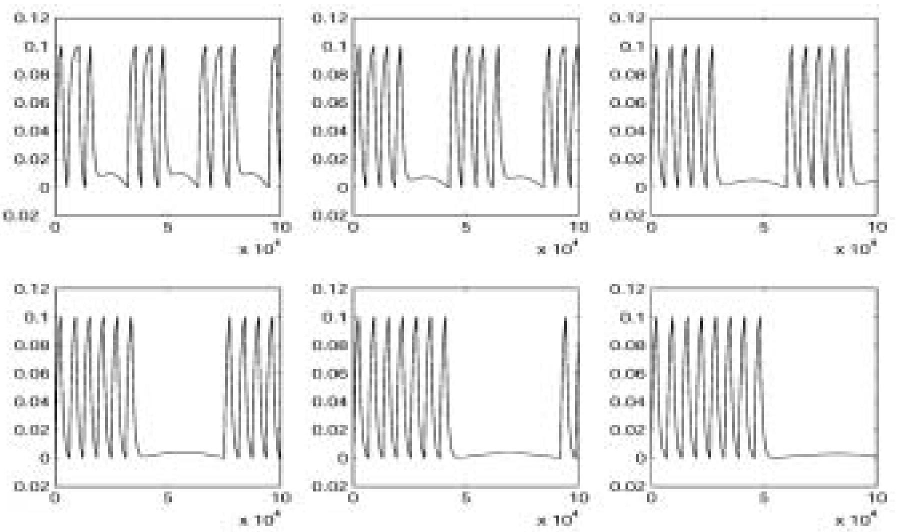

Fig. 14. V vs t for various choices of ω.

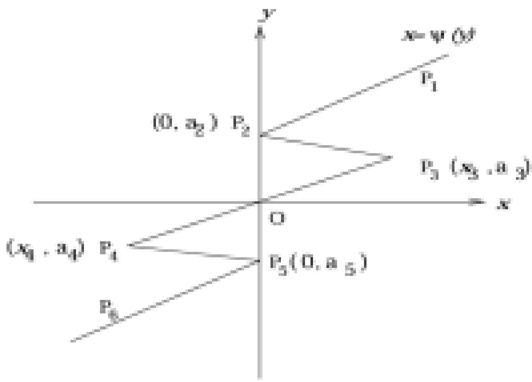

Fig. 15. The double S-shaped ψ.

For this characteristic function the solution may have spikes both above or below the time axis. We thus define the upper and lower spike solutions accordingly. We give the definition below according to the variable y. From the phase analysis and computational results we will see that the spikes count in terms of the current y is more clear than that in terms of the voltage x.

By the phase plane analysis described in §2, it is easy to see that any solution of the system (2.1) approaches the upper cycle $\Gamma_1 : OP_3P_1P_2$ when $f(t)$ stays long enough in the interval $[a_3, a_2]$, and approaches the lower cycle $\Gamma_2 : OP_4P_6P_5$ when

$f(t)$ stays long enough in the interval $[a_5, a_4]$ (See Figure 16). One upper cycle of the phase plane solution corresponds to one upper spike of the solution y in the y-t plane. One lower cycle of the phase plane solution corresponds to one lower spike of the solution y in the y-t plane. There would be no any solution cycle when $f(t)$ stays in other regions. Figure 17 shows a typical upper and lower spike solution (2 spikes) with a sinusoidal input.

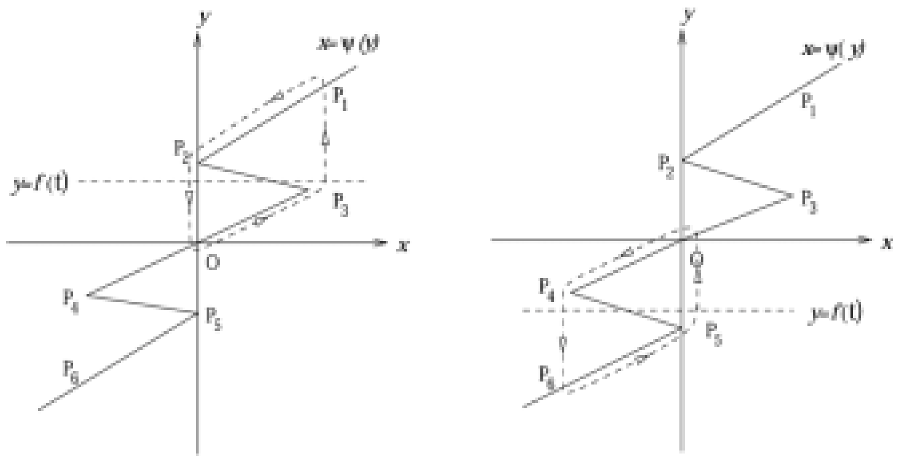

Fig. 16. Solution cycles in the phase plane.

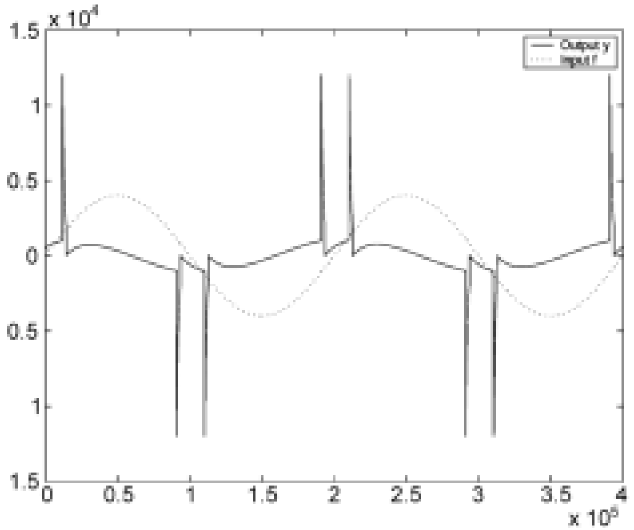

Fig. 17. Output y vs. t and input f vs. t.

The spike count based on $x(t)$ is different for the double S-shaped characteristic function. From the phase plane analysis when the solution moves from the upper cycle to the lower cycle (P_3 to P_4) or from the lower cycle to the upper cycle (P_4 to P_3) an extra peak and an extra valley will appear in the solution $x(t)$ (See Figure 18 with the same parameters as in Figure 17). Obviously the spike count is more clear in the y-t curve. So we suggest to use the current (i.e. y) versus time curve to examine the spike signal in practice.

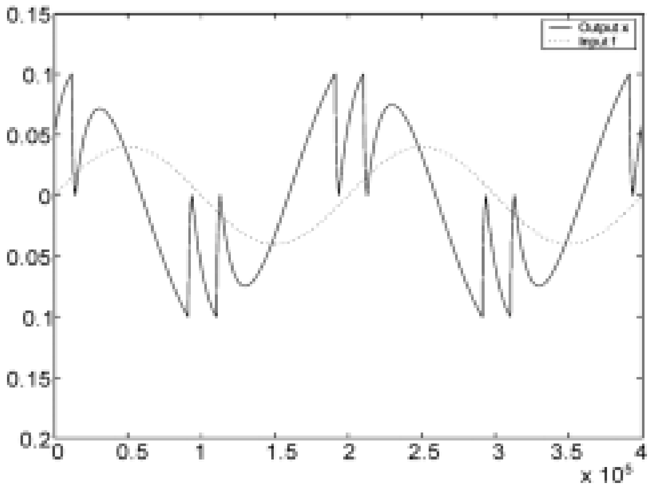

Fig. 18. Output x vs. t and input f vs. t.

Again our interest is to compute the period of solution cycles Γ_1 and Γ_2 in order to derive formulas to compute the number of spikes in spike solutions associated with the system. For simplicity we will only provide the zeroth order approximation for the double S case. Higher order approximations can be obtained similarly as we did in Section 3.

Let $\varepsilon = 0$ in the system (2.1). We then obtain the degenerated system

$$\begin{cases} \dot{x} &= f(t) - y, \\ 0 &= x - \psi(y). \end{cases} \tag{5.1}$$

The general solution of the system (5.1) is

$$\begin{cases} x &= \psi(y), \\ y &= e^{-\frac{t}{K}}\left(c + \displaystyle\int_0^t \frac{1}{K} e^{\frac{\xi}{K}} f(\xi) \mathrm{d}\xi\right), \end{cases} \tag{5.2}$$

where K is the slope of the piecewise linear $\psi(y)$ in each corresponding interval and c is an arbitrary constant.

The zeroth order approximate period T_1' of the cycle Γ_1 and T_2' of the cycle Γ_2

It is easy to see that

$$T_1' = p_1 + p_2,$$

where p_1 is the time traveling from P_1 to P_2, and p_2 is the time traveling from O to P_3. In the line P_1P_2, $K = K_1$. By (5.2), the solution of the degenerated system (5.1) starting at the point P_1 is

$$\begin{cases} x &= K_1 y - K_1 a_2, \\ y &= e^{-\frac{t}{K_1}}\left(a_3 + \int_0^t \frac{1}{K_1} e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi\right). \end{cases}$$

So the time p_1 traveling from P_1 to P_2 can be obtained from the nonlinear equation:

$$a_2 = e^{-\frac{p_1}{K_1}}\left(a_3 + \int_0^{p_1} \frac{1}{K_1} e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi\right),$$

or

$$\frac{1}{K_1}\int_0^{p_1} e^{\frac{\xi}{K_1}} f(\xi)\mathrm{d}\xi = a_2 e^{\frac{p_1}{K_1}} - a_3. \tag{5.3}$$

In the line OP_3, $K = K_3$. By (5.2), the solution of the degenerated system (5.1) starting at the point O is

$$\begin{cases} x &= K_3 y, \\ y &= e^{-\frac{t}{K_3}} \int_0^t \frac{1}{K_3} e^{\frac{\xi}{K_3}} f(\xi)\mathrm{d}\xi. \end{cases}$$

So the time p_2 traveling from O to P_3 can be obtained from

$$a_3 = e^{-\frac{p_2}{K_3}} \int_0^{p_2} \frac{1}{K_3} e^{\frac{\xi}{K_3}} f(\xi)\mathrm{d}\xi,$$

or

$$\frac{1}{K_3}\int_0^{p_2} e^{\frac{\xi}{K_3}} f(\xi)\mathrm{d}\xi = a_3 e^{\frac{p_2}{K_3}}. \tag{5.4}$$

Similarly, we have

$$T_2' = p_3 + p_4,$$

where p_3 is the time traveling from O to P_4 and p_4 is the time traveling from P_6 to P_5. p_3 and p_4 can be obtained from the following two nonlinear equations:

$$\frac{1}{K_4}\int_0^{p_3} e^{\frac{\xi}{K_4}} f(\xi)\mathrm{d}\xi = a_4 e^{\frac{p_3}{K_4}} \tag{5.5}$$

and

$$\frac{1}{K_6}\int_0^{p_4} e^{\frac{\xi}{K_6}} f(\xi)\mathrm{d}\xi = a_5 e^{\frac{p_4}{K_6}} - a_4, \tag{5.6}$$

respectively.

Formulas for special periodic inputs

(1) Periodic piecewise linear inputs

$$f(t) = \begin{cases} k_0 \in (a_4, a_3), & t \in [0, T_0]. \\ k_1 \in (a_3, a_2), & t \in [T_0, T_1], \\ k_2 \in (a_5, a_4), & t \in [T_1, T_2]. \end{cases}$$

By (5.3)-(5.6), we have

$$p_1 = K_1 \ln \frac{k_1 - a_3}{k_1 - a_2}, \quad p_2 = K_3 \ln \frac{k_1}{k_1 - a_3},$$
$$p_3 = K_4 \ln \frac{k_2}{k_2 - a_4}, \quad p_4 = K_6 \ln \frac{k_2 - a_4}{k_2 - a_5}.$$

So

$$\begin{aligned} T_1' &= p_1 + p_2 = K_1 \ln \frac{k_1 - a_3}{k_1 - a_2} + K_3 \ln \frac{k_1}{k_1 - a_3}, \\ T_2' &= p_3 + p_4 = K_4 \ln \frac{k_2}{k_2 - a_4} + K_6 \ln \frac{k_2 - a_4}{k_2 - a_5}. \end{aligned}$$

Hence, the number of spikes can be roughly determined.

(2) Sinusoidal inputs

$$f(t) = k \sin \omega t.$$

By (5.3)-(5.6), p_1, p_2, p_3 and p_4 can be obtained from the following nonlinear equations:

$$\begin{aligned} K_1\omega k + k\big(\sin\omega p_1 - K_1\omega\cos\omega p_1\big)e^{\frac{p_1}{K_1}} &= a_2(1 + K_1^2\omega^2)e^{\frac{p_1}{K_1}} \\ &\quad - a_3(1 + K_1^2\omega^2), \\ K_3\omega k + k\big(\sin\omega p_2 - K_3\omega\cos\omega p_2\big)e^{\frac{p_2}{K_3}} &= a_3(1 + K_3^2\omega^2)e^{\frac{p_2}{K_3}}, \\ K_4\omega k + k\big(\sin\omega p_3 - K_4\omega\cos\omega p_3\big)e^{\frac{p_3}{K_4}} &= a_4(1 + K_4^2\omega^2)e^{\frac{p_3}{K_4}}, \\ K_6\omega k + k\big(\sin\omega p_4 - K_6\omega\cos\omega p_4\big)e^{\frac{p_4}{K_6}} &= a_5(1 + K_6^2\omega^2)e^{\frac{p_4}{K_6}} \\ &\quad - a_4(1 + K_6^2\omega^2). \end{aligned}$$

Then T'_1 and T'_2 can be obtained afterwards.

Numerical experiments

We now do some numerical computations to see various spike patterns for the double S-shaped characteristic function. The parameters we first choose are: $K_1 = K_6 = 10^3$, $K_2 = K_5 = -10^4$ and $K_3 = K_4 = -K_2$. The input function $f(t) = A\sin(2\pi Cft)$ with $C = 10^{-5}$ and $f = 0.5$. Figures 19-21 show the output current y for various input amplitudes $A = 0.6$, 0.5 and 0.35, respectively.

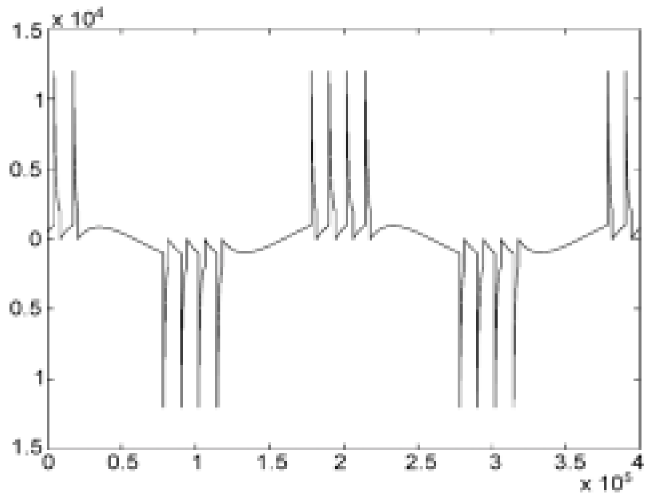

Fig. 19. Four upper spikes and four lower spikes.

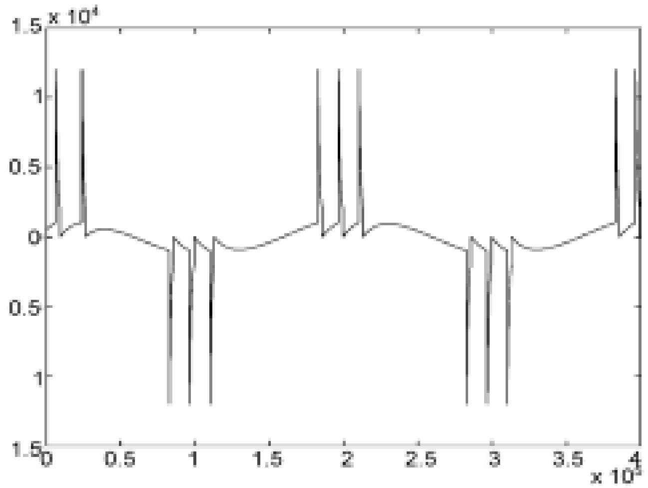

Fig. 20. Three upper spikes and three lower spikes.

Choosing different input amplitudes A and slopes of the double S-shaped characteristic function ψ we can see other spike signal patterns. By changing the slope K_4 to 0.00015 we obtain a four upper and three lower spike pattern depicted in Figure 22. By changing the amplitude of the input, Figure 23 depicts a similar case as in Figure 7 for the single S-shaped characteristic. By changing slopes of ψ we can easily obtain various patterns of spike-wave solutions corresponding to various numbers of upper and lower spikes.

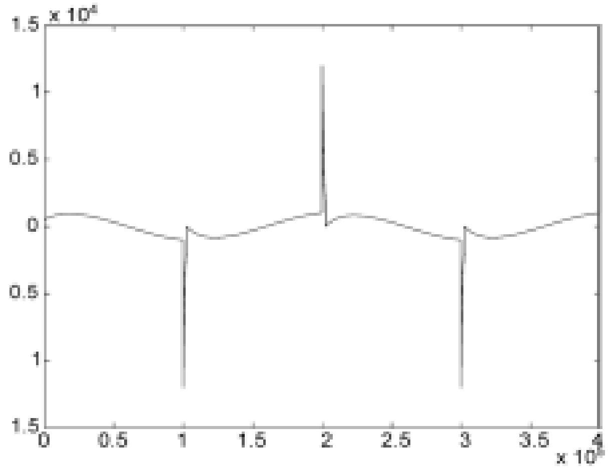

Fig. 21. One upper spike and one lower spike.

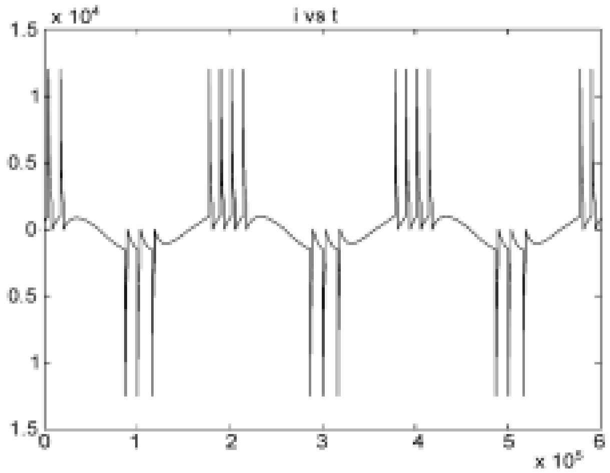

Fig. 22. Four upper spikes and three lower spikes.

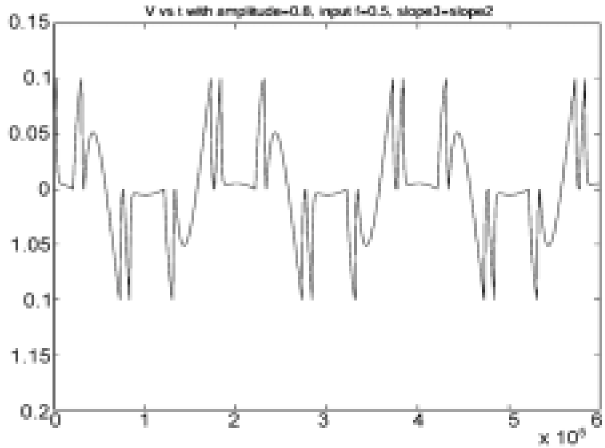

Fig. 23. The spike pattern $x(t)$ with input $A = 0.8$ and $K_4 = 0.0001$.

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