TAIWANESE JOURNAL OF MATHEMATICS
Vol. 9, No. 4, pp. 531-550, December 2005
This paper is available online at http://www.math.nthu.edu.tw/tjm/

STRONG CONVERGENCE THEOREMS FOR COMMUTATIVE SEMIGROUPS OF CONTINUOUS LINEAR OPERATORS ON BANACH SPACES

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Abstract. We first prove a strong convergence theorem of Mann's type for a commutative family of continuous linear operators in a Banach space by using strongly regular sequences of means on commutative semigroups. Using this, we obtain various strong convergence theorems for continuous linear operators in a Banach space.

1. Introduction

In 1938, Yosida [23] proved the following mean ergodic theorem for continuous linear operators: Let E be a real Banach space and let T be a linear operator of E into itself such that there exists a constant C with $\|T^n\| \leq C$ for $n = 1, 2, 3, \ldots$, and T is weakly completely continuous, i.e., T maps the closed unit ball of E into a weakly compact subset of E. Then the Cesàro means

$$S_n x = \frac{1}{n} \sum_{k=1}^{n} T^k x$$

converge strongly as $n \to \infty$ to a fixed point of T for each $x \in E$. Mann [11] also discussed an iterative sequence as follows: $x_0 = x \in E$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n,$$

where $\{\alpha_n\} \subseteq [0, 1]$ and T is a continuous linear operator on E. Kido and Takahashi [9] extended Yosida's theorem to amenable semigroups of continuous linear

Received June 19, 2003.
Communicated by Sze-Bi Hsu.
2000 *Mathematics Subject Classification*: 47A35, 47D03, 43A07.
Key words and phrases: Banach space, Continuous linear operator, Commutative semigroup, Ergodic theorem, Iterative process, Mean, Strongly regular sequence.

operators in a Banach space. On the other hand, in 1975, Baillon [4] proved the following nonlinear ergodic theorem: Let C be a bounded closed convex subset of a real Hilbert space H and let T be a nonexpansive mapping of C into itself. Then the Cesàro means

$$S_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x$$

converge weakly as $n \to \infty$ to a fixed point of T for each $x \in C$. Lau, Shioji and Takahashi [10] extended Baillon's theorem to amenable semigroups of nonexpansive mappings in a uniformly convex Banach space whose norm is Fréchet differentiable. In 1997, Shimizu and Takahashi [13] also found an iteration scheme of obtaining a common fixed point of a family of nonexpansive mappings in a Hilbert space. Then, many authors have studied such iterative schemes for families of nonlinear mappings in a Hilbert space or a Banach space; see, for instance, [1-3, 13-20].

In this paper, we introduce an iterative scheme of finding a common fixed point of a commutative family of continuous linear operators in a Banach space. Then, using Kido and Takahashi's result [9], we prove a strong convergence theorem for such a family of operators in a Banach space. Further, we apply this result to obtain various strong convergence theorems for continuous linear operators in a Banach space.

2. Preliminaries

Let E be a real Banach space with norm $\| \ \|$. We denote by E^* the dual of E. For $x \in E$ and $x^* \in E^*$, we denote by $\langle x, x^* \rangle$ the value of x^* at x. For a continuous linear operator T on E, we denote by T^* the adjoint of T, i.e., for every $x^* \in E^*$, $T^* x^* \in E^*$ is defined by $\langle x, T^* x^* \rangle = \langle Tx, x^* \rangle$ for any $x \in E$. For a net $\{x_\alpha\}$, we write $x_\alpha \to x$ if $\{x_\alpha\}$ converges strongly to a point x. For a subset A of E, we denote by $\overline{\mathrm{co}}\, A$ the closure of convex hull of A. We denote by $\mathbb{Z}_+$ the set of nonnegative integers, and denote by $\mathbb{R}_+$ the set of nonnegative real numbers. Let S be a nonempty set. Then we denote by $l^\infty(S)$ the Banach space of all bounded real-valued functions of S with supremum norm. For a topological space S, we denote by $C(S)$ the set of all $f \in l^\infty(S)$ such that f is continuous. If, S has the discrete topology, $C(S) = l^\infty(S)$ holds. Let S be a topological space. Then μ is said to be a *mean on* $C(S)$ if $\mu \in C(S)^*$ and $\mu(1_S) = \|\mu\| = 1$, where $1_S(s) = 1$ for every $s \in S$. It is well-known that a linear function μ of $C(S)$ into reals is a mean on $C(S)$ if and only if $\inf_{s \in S} f(s) \le \mu(f) \le \sup_{s \in S} f(s)$ for every $f \in C(S)$; see [21, Theorem 1.4.1]. A semigroup S is said to be a *semitopological semigroup* if S is a Hausdorff space and for every $a \in S$, the mappings $s \mapsto sa$ and $s \mapsto as$ of S into itself are continuous. For $\mu \in C(S)^*$ and $f \in C(S)$, we write $\mu_s f(s)$ instead of $\mu(f)$. For example, if $f(s) = \langle T_s x, x^* \rangle$ for every $s \in S$, where $x \in E$

and $x^* \in E^*$, we write $\mu_s\langle T_s x, x^*\rangle$ instead of $\mu(f)$. Let S be a semitopological semigroup. For $a \in S$ and $f \in C(S)$, we define $l_a f \in C(S)$ by $(l_a f)(s) = f(as)$ for every $s \in S$, and define $r_a f \in C(S)$ by $(r_a f)(s) = f(sa)$ for every $s \in S$. For $a \in S$ and a mean μ on $C(S)$, we define $l_a^*\mu \in C(S)^*$ by $(l_a^*\mu)(f) = \mu(l_a f)$ for every $f \in C(S)$, and define $r_a^*\mu \in C(S)^*$ by $(r_a^*\mu)(f) = \mu(r_a f)$ for every $f \in C(S)$. In this case, $l_a^*\mu$ and $r_a^*\mu$ are also means on $C(S)$. A mean μ on $C(S)$ is *left invariant* (*right invariant*) if $l_s^*\mu = \mu$ ($r_s^*\mu = \mu$) for every $s \in S$, respectively. A mean μ is *invariant* if μ is left invariant and right invariant; see [7, 12, 21] for more details. For an operator T on E, we denote by $F(T)$ the set of fixed points of T. For a family $\mathcal{S} = \{T_s : s \in S\}$ of operators on E, we denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S} = \{T_s : s \in S\}$, i.e., $F(\mathcal{S}) = \bigcap_{s\in S} F(T_s)$. Let E be a real Banach space and let S be a semitopological semigroup with identity e. Let $\mathcal{S} = \{T_s : s \in S\}$ be a family of continuous linear operators on E satisfying the following conditions:

- (C1) There exists $C \geq 0$ such that $\|T_s\| \leq C$ for each $s \in S$;
- (C2) for every $x \in E$, $\{T_s x : s \in S\}$ is relatively weakly compact on E;
- (C3) $T_{st} = T_s T_t$ for each $s, t \in S$ and $T_e x = x$ for every $x \in E$;
- (C4) for every $x \in E$ and $x^* \in E^*$, the mapping $s \mapsto \langle T_s x, x^*\rangle$ is continuous.

Then such a family $\mathcal{S} = \{T_s : s \in S\}$ is called a *C-semigroup of continuous linear operators on E* [9, 21]. From (C1), $\{T_s x : s \in S\}$ is bounded for every $x \in E$. Let μ be a mean on $C(S)$ and let $\mathcal{S} = \{T_s : s \in S\}$ be a C-semigroup of continuous linear operators on E. For every $x \in E$, it follows from [9] that there exists a unique element $T_\mu x \in \overline{\text{co}}\{T_s x : s \in S\}$ such that $\langle T_\mu x, x^*\rangle = \mu_s\langle T_s x, x^*\rangle$ for any $x^* \in E^*$. Further, we know the following results; see, for instance, Kido and Takahashi [9].

Lemma 2.1. *Let μ be a mean on $C(S)$. Then T_μ is a continuous linear operator on E such that $\|T_\mu\| \leq C$.*

Lemma 2.2. *Let μ be a left invariant mean on $C(S)$. Then $T_\mu x \in F(\mathcal{S})$ for every $x \in E$.*

Theorem 2.3. *Let $\mathcal{S} = \{T_s : s \in S\}$ be a C-semigroup of continuous linear operators on E such that $C(S)$ has an invariant mean. Then for each $x \in E$, $\overline{\text{co}}\{T_s x : s \in S\} \cap F(\mathcal{S})$ consists of one point.*

Theorem 2.4. *Let $\mathcal{S} = \{T_s : s \in S\}$ be a C-semigroup of continuous linear operators on E. Let $\{\mu_\alpha\}$ be a net of means on $C(S)$ such that for each $s \in S$ and $f \in C(S)$, $\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0$ and $\|\mu_\alpha - r_s^*\mu_\alpha\| \to 0$. Then the following hold:*

(1) *There exists a unique continuous linear operator Q of E onto $F(\mathcal{S})$ such that $QT_s = T_sQ = Q$ for each $s \in S$ and $Qx \in \overline{\mathrm{co}}\{T_t x : t \in S\}$ for each $x \in E$;*

(2) *for every $x \in E$, the net $\{T_{\mu_\alpha} x\}$ converges strongly to Qx.*

Remark 2.5. Such a Q satisfies $Q^2 = Q$. In fact, for $x \in E$ and $\varepsilon > 0$, it follows from $Qx \in \overline{\mathrm{co}}\{T_s x : s \in S\}$ that there exist finite elements $\{s_i\}_{i \in I}$ of S and nonnegative numbers $\{\lambda_i\}_{i \in I}$ with $\sum_{i \in I} \lambda_i = 1$ such that $\left\| Qx - \sum_{i \in I} \lambda_i T_{s_i} x \right\| < \varepsilon$. Then we have

$$\begin{aligned} \|Q^2 x - Qx\| &\leq \left\| Q\left(Qx - \sum_{i \in I} \lambda_i T_{s_i} x \right) \right\| + \left\| Q\left(\sum_{i \in I} \lambda_i T_{s_i} x \right) - Qx \right\| \\ &\leq \|Q\|\varepsilon + \left\| \sum_{i \in I} \lambda_i Q T_{s_i} x - Qx \right\| \\ &= \|Q\|\varepsilon + \left\| \sum_{i \in I} \lambda_i Qx - Qx \right\| \\ &= \|Q\|\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $Q^2 = Q$.

If a semitopological semigroup S is commutative, by Markov and Kakutani's fixed point theorem, there exists an invariant mean on $C(S)$; see [21] for the proof. Let $\{\mu_\alpha\}$ be a net of means on $C(S)$. Then, $\{\mu_\alpha\}$ is said to be *strongly regular* if $\|\mu_\alpha - r_s^* \mu_\alpha\| \to 0$; see, for instance, [8]. From Theorems 2.3 and 2.4, we have the following theorem for commutative semigroups of continuous linear operators. We shall use this result for the proof of our main result (Theorem 3.3).

Theorem 2.6. *Let E be a real Banach space, let S be a commutative semitopological semigroup with identity and let $\mathcal{S} = \{T_s : s \in S\}$ be a C-semigroup of continuous linear operators on E. Then there exists a unique continuous linear projection Q of E onto $F(\mathcal{S})$ such that $QT_s = T_sQ = Q$ for each $s \in S$ and $Qx \in \overline{\mathrm{co}}\{T_t x : t \in S\}$ for each $x \in E$. Further, if $\{\mu_\alpha\}$ is a strongly regular net of means on $C(S)$, then $\{T_{\mu_\alpha} x\}$ converges strongly to Qx for every $x \in E$.*

3. Main Result

In this section, we proved a strong convergence theorem of Mann's type for a commutative semigroup of continuous linear operators on a real Banach space. Before proving the theorem, we need the following two lemmas.

Lemma 3.1. *Let S be a commutative semitopological semigroup with identity, let $\mathcal{S} = \{T_s : s \in S\}$ be a C-semigroup of continuous linear operators on E and let μ be a mean on $C(S)$. Then $T_s T_\mu = T_\mu T_s$ for every $s \in S$.*

Proof. Let $s \in S$. For every $x \in E$ and $x^* \in E^*$, we have

$$\begin{aligned}\langle T_s T_\mu x, x^* \rangle &= \langle T_\mu x, T_s^* x^* \rangle \\ &= \mu_t \langle T_t x, T_s^* x^* \rangle \\ &= \mu_t \langle T_s T_t x, x^* \rangle \\ &= \mu_t \langle T_t T_s x, x^* \rangle \\ &= \langle T_\mu T_s x, x^* \rangle .\end{aligned}$$

Hence we get $T_s T_\mu = T_\mu T_s$. ∎

Lemma 3.2. *Let $\{\alpha_n\} \subseteq [0,1]$ satisfy $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$, and let $\{b_n\}, \{\varepsilon_n\} \subseteq [0,\infty)$ be sequences such that*

$$b_{n+1} \le \alpha_n b_n + (1-\alpha_n)\varepsilon_n, \quad n = 0, 1, 2, \ldots,$$

and $\lim_{n\to\infty} \varepsilon_n = 0$. Then $\lim_{n\to\infty} b_n = 0$.

Proof. Fix $\varepsilon > 0$. Then there exists an n_0 such that $\varepsilon_n \le \varepsilon$ for each $n \ge n_0$. So, for every $n \ge n_0$, we have

$$b_{n+1} \le \alpha_n b_n + (1-\alpha_n)\varepsilon$$

and hence

$$b_{n+1} - \varepsilon \le \alpha_n (b_n - \varepsilon).$$

Consequently, we have

$$\begin{aligned}b_n - \varepsilon &\le \alpha_{n-1}(b_{n-1} - \varepsilon) \\ &\le \alpha_{n-1}\alpha_{n-2}(b_{n-2} - \varepsilon) \\ &\ldots \\ &\le \alpha_{n-1}\alpha_{n-2}\ldots\alpha_{n_0}(b_{n_0} - \varepsilon)\end{aligned}$$

and hence

$$b_n \le \varepsilon + (b_{n_0} - \varepsilon)\prod_{k=n_0}^{n-1}\alpha_k.$$

Note that $\prod_{k=n_0}^{\infty}\alpha_k = 0$. In fact, from $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$, it follow that

$$0 \le \prod_{k=n_0}^{n-1}\alpha_k \le \prod_{k=n_0}^{n-1}\exp(\alpha_k - 1) = \exp\left(-\sum_{k=n_0}^{n-1}(1-\alpha_k)\right) \to 0$$

as $n \to \infty$. Then we have

$$\limsup_{n\to\infty} b_n \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\lim_{n\to\infty} b_n = 0$. ∎

Theorem 3.3. *Let E be a real Banach space, let S be a commutative semi-topological semigroup with identity and let $\mathcal{S} = \{T_s : s \in S\}$ be a C-semigroup of continuous linear operators on E. Let $\{\mu_n\}_{n=0}^{\infty}$ be a strongly regular sequence of means on $C(S)$. Consider the following iteration scheme:*

$$\begin{aligned} x_0 &= x \in E; \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, \quad n = 0, 1, 2, \dots, \end{aligned}$$

where $\{\alpha_n\} \subseteq [0,1]$ satisfies $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. Then the sequence $\{x_n\}$ converges strongly to $Qx \in F(\mathcal{S})$, where $Qx = \lim_{n\to\infty} T_{\mu_n} x$.

Proof. Put $K_0(x) = \overline{\mathrm{co}}\{T_s x : s \in S\}$. First, we show that $\{x_n\} \subseteq K_0(x)$. Assume $x_n \in K_0(x)$. Then $T_{\mu_n} x_n \in K_0(x)$. In fact, suppose $T_{\mu_n} x_n \notin K_0(x)$. Then, by the separation theorem, there exists $x^* \in E^*$ such that

$$\begin{aligned} \langle T_{\mu_n} x_n, x^* \rangle &< \inf\{\langle z, x^* \rangle : z \in K_0(x)\} \\ &= \inf\{\langle T_s x, x^* \rangle : s \in S\}. \end{aligned}$$

So we have

$$\begin{aligned} \langle T_{\mu_n} x_n, x^* \rangle &= (\mu_n)_s \langle T_s x_n, x^* \rangle \\ &\ge \inf\{\langle T_s x_n, x^* \rangle : s \in S\} \\ &= \inf\{\langle x_n, T_s^* x^* \rangle : s \in S\} \\ &\ge \inf\{\langle z, T_s^* x^* \rangle : s \in S,\ z \in K_0(x)\} \\ &= \inf\{\langle T_t x, T_s^* x^* \rangle : s, t \in S\} \\ &= \inf\{\langle T_s T_t x, x^* \rangle : s, t \in S\} \\ &\ge \inf\{\langle T_s x, x^* \rangle : s \in S\} \\ &> \langle T_{\mu_n} x_n, x^* \rangle . \end{aligned}$$

This is a contradiction. Since x_n and $T_{\mu_n} x_n$ are in $K_0(x)$, we have $x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n \in K_0(x)$. Then, by the induction, we have $\{x_n\} \subseteq K_0(x)$. Next we show that $\|T_{\mu_n} x_n - Qx\| \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be arbitrary. Since $T_{\mu_n} x \to Qx$ by Theorem 2.6, there exists n_0 such that

$$(*) \qquad C \cdot \|T_{\mu_n} x - Qx\| < \varepsilon \quad \text{for any } n \ge n_0.$$

Fix $n \geq n_0$. Since $x_n \in K_0(x)$, there exist finite elements $\{s_i\}_{i \in I}$ of S and nonnegative numbers $\{\lambda_i\}_{i \in I}$ with $\sum_{i \in I} \lambda_i = 1$ such that

$$C \cdot \left\| x_n - \sum_{i \in I} \lambda_i T_{s_i} x \right\| < \varepsilon.$$

Then, by Lemma 3.1 and from $(*)$, we have

$$\begin{aligned} \|T_{\mu_n} x_n - Qx\| &\leq \left\| T_{\mu_n} x_n - T_{\mu_n}\left(\sum_{i \in I} \lambda_i T_{s_i} x \right) \right\| + \left\| T_{\mu_n}\left(\sum_{i \in I} \lambda_i T_{s_i} x \right) - Qx \right\| \\ &\leq \|T_{\mu_n}\| \cdot \left\| x_n - \sum_{i \in I} \lambda_i T_{s_i} x \right\| + \left\| \sum_{i \in I} \lambda_i T_{\mu_n} T_{s_i} x - Qx \right\| \\ &< \varepsilon + \sum_{i \in I} \lambda_i \|T_{s_i} T_{\mu_n} x - T_{s_i} Qx\| \\ &\leq \varepsilon + \sum_{i \in I} \lambda_i \|T_{s_i}\| \, \|T_{\mu_n} x - Qx\| \\ &< \varepsilon + \sum_{i \in I} \lambda_i \varepsilon = 2\varepsilon. \end{aligned}$$

It follows that $\|T_{\mu_n} x_n - Qx\| \to 0$ as $n \to \infty$. On the other hand, we have

$$\begin{aligned} \|x_{n+1} - Qx\| &= \|\alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n - Qx\| \\ &= \|\alpha_n (x_n - Qx) + (1 - \alpha_n)(T_{\mu_n} x_n - Qx)\| \\ &\leq \alpha_n \|x_n - Qx\| + (1 - \alpha_n)\|T_{\mu_n} x_n - Qx\|. \end{aligned}$$

Put $b_n = \|x_n - Qx\|$ and $\varepsilon_n = \|T_{\mu_n} x_n - Qx\|$. Then we have

$$b_{n+1} \leq \alpha_n b_n + (1 - \alpha_n)\varepsilon_n, \quad n = 0, 1, 2, \dots,$$

and $\lim_{n \to \infty} \varepsilon_n = 0$. By Lemma 3.2, it follows that $b_n = \|x_n - Qx\| \to 0$ as $n \to \infty$. ∎

4. Strongly Regular Sequences of Means

In this section, we deal with examples of strongly regular sequences of means. A family $\{q(n,j)\}_{n,j=0}^{\infty}$ of real numbers is said to be a *strongly regular summation method* [5,6] if it satisfies the following:

(S1) $q(n,j) \geq 0$;

(S2) $\sum_{j=0}^{\infty} q(n,j) = 1$ for every n;

(S3) $\lim_{n\to\infty} q(n,j) = 0$ for every j;

(S4) $\lim_{n\to\infty} \sum_{j=0}^{\infty} |q(n,j+1) - q(n,j)| = 0$.

In particular, if $q(n,0) \geq q(n,1) \geq q(n,2) \geq \cdots$ for every n, then (S1)–(S3) imply (S4). In fact, fix $n \geq 0$. From (S2), we have

$$\begin{aligned}\sum_{j=0}^{\infty} |q(n,j+1) - q(n,j)| &= \sum_{j=0}^{\infty} (q(n,j) - q(n,j+1)) \\ &= q(n,0) - \lim_{j\to\infty} q(n,j) \\ &= q(n,0).\end{aligned}$$

Therefore (S3) implies (S4).

Remark 4.1. (S1), (S2) and (S4) imply (S3). In fact, for every j, we have

$$\begin{aligned}0 \leq q(n,j) &= \lim_{k\to\infty} q(n,k) + \sum_{k=j}^{\infty} (q(n,k) - q(n,k+1)) \\ &\leq \sum_{k=0}^{\infty} |q(n,k) - q(n,k+1)| \to 0\end{aligned}$$

as $n \to \infty$. Hence (S3) holds.

Theorem 4.2. *Let $\{q(n,j)\}_{n,j=0}^{\infty}$ be a family of real numbers. For every $n \geq 0$ and $f \in l^{\infty}(\mathbb{Z}_+)$, define $\mu_n(f) = \sum_{j=0}^{\infty} q(n,j) f(j)$. Then the following are equivalent:*

(1) *$\{\mu_n\}_{n=0}^{\infty}$ is a strongly regular sequence of means on $l^{\infty}(\mathbb{Z}_+)$;*

(2) *$\{q(n,j)\}_{n,j=0}^{\infty}$ is a strongly regular summation method.*

Proof. $(2) \Rightarrow (1)$: For every $n \geq 0$, we show that μ_n is a mean on $l^{\infty}(\mathbb{Z}_+)$. In fact, we have, for each $f \in l^{\infty}(\mathbb{Z}_+)$,

$$\mu_n(f) = \sum_{j=0}^{\infty} q(n,j) f(j) \leq \sum_{j=0}^{\infty} q(n,j) \cdot \left(\sup_j f(j) \right) = \sup_j f(j).$$

Similarly, we have $\mu_n(f) \geq \inf_j f(j)$. Then μ_n is a mean on $l^{\infty}(\mathbb{Z}_+)$. Next, we show that $\{\mu_n\}$ is strongly regular. In fact, for every $f \in l^{\infty}(\mathbb{Z}_+)$, we have

$$
\begin{aligned}
|(\mu_n - r_1^*\mu_n)(f)| &= |\mu_n(f) - \mu_n(r_1 f)| \\
&= \left| \sum_{j=0}^{\infty} q(n,j)f(j) - \sum_{j=0}^{\infty} q(n,j)f(j+1) \right| \\
&= \left| q(n,0)f(0) + \sum_{j=0}^{\infty} q(n,j+1)f(j+1) - \sum_{j=0}^{\infty} q(n,j)f(j+1) \right| \\
&\leq |q(n,0)f(0)| + \sum_{j=0}^{\infty} |q(n,j+1) - q(n,j)|\,|f(j+1)| \\
&\leq |q(n,0)|\,\|f\| + \sum_{j=0}^{\infty} |q(n,j+1) - q(n,j)|\,\|f\|.
\end{aligned}
$$

Then, from (S3) and (S4), we have

$$
\|\mu_n - r_1^*\mu_n\| \leq |q(n,0)| + \sum_{j=0}^{\infty} |q(n,j+1) - q(n,j)| \to 0
$$

as $n \to \infty$. Fix $k \geq 1$. Then we have

$$
\begin{aligned}
\|\mu_n - r_k^*\mu_n\| &= \sup_{\|f\|\leq 1} |\mu_n(f) - \mu_n(r_k f)| \\
&= \sup_{\|f\|\leq 1} \left| \sum_{j=0}^{k-1} (\mu_n(r_j f) - \mu_n(r_{j+1} f)) \right| \\
&\leq \sup_{\|f\|\leq 1} \sum_{j=0}^{k-1} |\mu_n(r_j f) - \mu_n(r_1(r_j f))| \\
&\leq \sup_{\|f\|\leq 1} \sum_{j=0}^{k-1} \|\mu_n - r_1^*\mu_n\|\,\|r_j f\| \\
&\leq \sum_{k=0}^{k-1} \|\mu_n - r_1^*\mu_n\| \to 0
\end{aligned}
$$

as $n \to \infty$. Hence $\{\mu_n\}$ is a strongly regular sequence of means on $l^\infty(\mathbb{Z}_+)$.

$(1) \Rightarrow (2)$: For every $j, k \in \mathbb{Z}_+$, we define

$$
\delta_j(k) = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}
$$

Then $\delta_j \in l^\infty(\mathbb{Z}_+)$ for every j. Now, for every n and j, we have

$$q(n,j) = \sum_{k=0}^{\infty} q(n,k)\delta_j(k) = \mu_n(\delta_j) \geq \inf_{k\in\mathbb{Z}_+} \delta_j(k) = 0.$$

Hence (S1) holds. Since μ_n is a mean on $l^\infty(\mathbb{Z}_+)$ for every n, we have

$$= \sum_{j=0}^{\infty} q(n,j)\cdot 1_{\mathbb{Z}_+}(j) = \sum_{j=0}^{\infty} q(n,j).$$

Hence (S2) holds. We show (S4). Fix $n \geq 0$ and define f_n by $f_n(0) = 0$ and

$$f_n(j) = \begin{cases} 1, & \text{if } q(n,j) \geq q(n,j-1), \\ -1, & \text{if } q(n,j) < q(n,j-1), \end{cases}$$

for each $j \geq 1$. It is clear that $f_n \in l^\infty(\mathbb{Z}_+)$ and $\|f_n\| = 1$. Since $\{\mu_n\}$ is strongly regular, we also have

$$\begin{aligned}
&\sum_{j=0}^{\infty} |q(n,j+1) - q(n,j)| \\
&= \sum_{j=0}^{\infty} \big(q(n,j+1) - q(n,j)\big) f_n(j+1) \\
&= \sum_{j=0}^{\infty} q(n,j+1) f_n(j+1) - \sum_{j=0}^{\infty} q(n,j) f_n(j+1) \\
&= q(n,0) f_n(0) + \sum_{j=1}^{\infty} q(n,j) f_n(j) - \sum_{j=0}^{\infty} q(n,j) f_n(j+1) \\
&= \mu_n(f_n) - \mu_n(r_1 f_n) \\
&\leq \|\mu_n - r_1^*\mu_n\| \|f_n\| \\
&= \|\mu_n - r_1^*\mu_n\| \to 0
\end{aligned}$$

as $n \to \infty$. Hence (S4) holds. ∎

A family $\{q(n,i,j)\}_{n,i,j=0}^{\infty}$ of real numbers is called a *strongly regular bi-summation method* if it satisfies the following:

(B1) $q(n,i,j) \geq 0$;

(B2) $\sum_{i,j=0}^{\infty} q(n,i,j) = 1$ for every n;

(B3a) $\lim_{n\to\infty} \sum_{i,j=0}^{\infty} |q(n,i+1,j) - q(n,i,j)| = 0$;

(B3b) $\lim_{n\to\infty} \sum_{i,j=0}^{\infty} |q(n,i,j+1) - q(n,i,j)| = 0$.

Theorem 4.3. *Let* $\{q(n,i,j)\}_{n,i,j=0}^{\infty}$ *be a family of real numbers. For every* $n \geq 0$ *and* $f \in l^{\infty}(\mathbb{Z}_+ \times \mathbb{Z}_+)$, *define* $\mu_n(f) = \sum_{i,j=0}^{\infty} q(n,i,j)f(i,j)$. *Then the following are equivalent:*

(1) $\{\mu_n\}_{n=0}^{\infty}$ *is a strongly regular sequence of means on* $l^{\infty}(\mathbb{Z}_+ \times \mathbb{Z}_+)$;

(2) $\{q(n,i,j)\}_{n,i,j=0}^{\infty}$ *is a strongly regular bi-summation method.*

Proof. $(2) \Rightarrow (1)$: For every $n \geq 0$, we show that μ_n is a mean on $l^{\infty}(\mathbb{Z}_+ \times \mathbb{Z}_+)$. In fact, for every $f \in l^{\infty}(\mathbb{Z}_+ \times \mathbb{Z}_+)$, we have

$$\mu_n(f) = \sum_{i,j=0}^{\infty} q(n,i,j)f(i,j) \leq \sum_{i,j=0}^{\infty} q(n,i,j) \cdot \left(\sup_{(i,j)} f(i,j) \right) = \sup_{(i,j)} f(i,j).$$

Similarly, we have $\mu_n(f) \geq \inf_{(i,j)} f(i,j)$. Then μ_n is a mean on $l^{\infty}(\mathbb{Z}_+ \times \mathbb{Z}_+)$. Note that

$$(**) \qquad \sum_{j=0}^{\infty} q(n,0,j) \leq \sum_{i,j=0}^{\infty} |q(n,i,j) - q(n,i+1,j)|$$

for every $n \geq 0$. In fact, for every $k \geq 1$, we have

$$\begin{aligned}
\sum_{j=0}^{\infty} q(n,0,j) &= \sum_{j=0}^{\infty} \left(q(n,k,j) + \sum_{i=0}^{k-1} (q(n,i,j) - q(n,i+1,j)) \right) \\
&\leq \sum_{j=0}^{\infty} q(n,k,j) + \sum_{i,j=0}^{\infty} |q(n,i,j) - q(n,i+1,j)|.
\end{aligned}$$

It follows from (B2) that $\lim_{k\to\infty} \sum_{j=0}^{\infty} q(n,k,j) = 0$, and hence we have $(**)$. Next, we show that $\{\mu_n\}$ is strongly regular. For every $f \in l^{\infty}(\mathbb{Z}_+ \times \mathbb{Z}_+)$, from $(**)$, we have

$$\begin{aligned}
&|(\mu_n - r^*_{(1,0)}\mu_n)(f)| \\
&= \left| \sum_{i,j=0}^{\infty} q(n,i,j)f(i,j) - \sum_{i,j=0}^{\infty} q(n,i,j)f(i+1,j) \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{j=0}^{\infty} q(n,0,j)f(0,j) + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} q(n,i,j)f(i,j) - \sum_{i,j=0}^{\infty} q(n,i,j)f(i+1,j) \right| \\
&\le \sum_{j=0}^{\infty} |q(n,0,j)f(0,j)| + \sum_{i,j=0}^{\infty} |q(n,i+1,j) - q(n,i,j)|\,|f(i+1,j)| \\
&\le \sum_{j=0}^{\infty} q(n,0,j)\,\|f\| + \sum_{i,j=0}^{\infty} |q(n,i+1,j) - q(n,i,j)|\,\|f\| \\
&\le 2 \sum_{i,j=0}^{\infty} |q(n,i+1,j) - q(n,i,j)|\,\|f\|.
\end{aligned}
$$

Then we have

$$
\|\mu_n - r^*_{(1,0)}\mu_n\| \le 2 \sum_{i,j=0}^{\infty} |q(n,i+1,j) - q(n,i,j)| \to 0
$$

as $n \to \infty$. Similarly, we have $\|\mu_n - r^*_{(0,1)}\mu_n\| \to 0$ as $n \to \infty$. Then for every $(k,l) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, we have $\|\mu_n - r^*_{(k,l)}\mu_n\| \to 0$ as $n \to \infty$.

(1) $\Rightarrow$ (2): As in the proof of Theorem 4.2, we can show (B1) and (B2). Fix $n \ge 0$ and define f_n by $f_n(0,j) = 0$ for each $j \ge 0$ and

$$
f_n(i,j) = \begin{cases} 1, & \text{if } q(n,i,j) \ge q(n,i-1,j) \\ -1, & \text{if } q(n,i,j) < q(n,i-1,j) \end{cases}
$$

for each $i \ge 1$ and $j \ge 0$. It is clear that $f_n \in l^\infty(\mathbb{Z}_+ \times \mathbb{Z}_+)$ and $\|f_n\| = 1$. Then we have

$$
\begin{aligned}
&\sum_{i,j=0}^{\infty} |q(n,i+1,j) - q(n,i,j)| \\
&= \sum_{i,j=0}^{\infty} q(n,i+1,j)f_n(i+1,j) - \sum_{i,j=0}^{\infty} q(n,i,j)f_n(i+1,j) \\
&= \sum_{j=0}^{\infty} q(n,0,j)f_n(0,j) + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} q(n,i,j)f_n(i,j) - \sum_{i,j=0}^{\infty} q(n,i,j)f_n(i+1,j) \\
&= \mu_n(f_n) - \mu_n(r_{(1,0)}f_n) \\
&\le \|\mu_n - r^*_{(1,0)}\mu_n\| \to 0
\end{aligned}
$$

as $n \to \infty$. Hence (B3a) holds. Similarly, we have (B3b). ■

5. Applications

Now, using Theorem 3.3, we can prove some strong convergence theorems for continuous linear operators on a Banach space. A linear operator T on a real Banach space E is said to be *uniformly C-Lipschitz* if there exists $C \geq 1$ such that $\|T^n\| \leq C$ for every $n \geq 0$.

Theorem 5.1. *Let T be a uniformly C-Lipschitz linear operator on a real Banach space E and suppose that $\{T^j x : j \geq 0\}$ is relatively weakly compact for every $x \in E$. Let $\{q(n,j)\}_{n,j=0}^{\infty}$ be a strongly regular summation method. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in E by the following iteration scheme:*

$$x_0 = x \in E;$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) \sum_{j=0}^{\infty} q(n,j) T^j x_n, \quad n = 0, 1, 2, \ldots,$$

where $\{\alpha_n\} \subseteq [0,1]$ satisfies $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. Then the sequence $\{x_n\}$ converges strongly to some $p \in F(T)$.

Proof. It is clear that $\mathcal{S} = \{T^j : j \in \mathbb{Z}_+\}$ is a C-semigroup of continuous linear operators on E. For every $n \geq 0$ and $f \in l^\infty(\mathbb{Z}_+)$, define $\mu_n(f) = \sum_{j=0}^{\infty} q(n,j) f(j)$. Then, from Theorem 4.2, it follows that $\{\mu_n\}$ is a strongly regular sequence of means on $l^\infty(\mathbb{Z}_+)$. Next we show $T_{\mu_n} y = \sum_{j=0}^{\infty} q(n,j) T^j y$ for every $y \in E$. In fact, for every $x^* \in E^*$, we have

$$\begin{aligned} \langle T_{\mu_n} y, x^* \rangle &= (\mu_n)_j \langle T^j y, x^* \rangle \\ &= \sum_{j=0}^{\infty} q(n,j) \left\langle T^j y, x^* \right\rangle \\ &= \left\langle \sum_{j=0}^{\infty} q(n,j) T^j y, x^* \right\rangle. \end{aligned}$$

Hence we have $T_{\mu_n} y = \sum_{j=0}^{\infty} q(n,j) T^j y$. Then, from Theorem 3.3, $\{x_n\}$ converges strongly to $Qx \in F(\mathcal{S}) = F(T)$.

Corollary 5.2. *Let T be a uniformly C-Lipschitz linear operator on a real Banach space E and suppose that $\{T^j x : j \in \mathbb{Z}_+\}$ is relatively weakly compact for every $x \in E$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in E by the following iteration scheme:*

$$x_0 = x \in E;$$
$$x_{n+1} = \alpha_n x_n + \frac{1-\alpha_n}{n+1} \sum_{j=0}^{n} T^j x_n, \quad n = 0, 1, 2, \ldots,$$

where $\{\alpha_n\} \subseteq [0,1]$ *satisfies* $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to some* $p \in F(T)$.

Proof. For every $n, j \geq 0$, we define

$$q(n,j) = \begin{cases} 1/(n+1), & \text{if } j \leq n, \\ 0, & \text{if } j > n. \end{cases}$$

(S1)–(S3) are clear. Since $q(n,0) \geq q(n,1) \geq q(n,2) \geq \cdots$ for every $n \geq 0$, it follows that $\{q(n,j)\}$ is a strongly regular summation method. So, from Theorem 5.1, the sequence $\{x_n\}$ converges strongly to some $p \in F(T)$.

Corollary 5.3. *Let* T *be a uniformly* C*-Lipschitz linear operator on a real Banach space* E *and suppose that* $\{T^j x : j \geq 0\}$ *is relatively weakly compact for every* $x \in E$. *Let* $\{a_n\} \subseteq (0,1)$ *be a sequence converging to* 1. *Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* E *by the following iteration scheme:*

$$x_0 = x \in E;$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)(1-a_n)\sum_{j=0}^{\infty}(a_n)^j T^j x_n, \quad n = 0,1,2,\ldots,$$

where $\{\alpha_n\} \subseteq [0,1]$ *satisfies* $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to some* $p \in F(T)$.

Proof. For every $n, j \geq 0$, we define $q(n,j) = (1-a_n)(a_n)^j$. (S1) and (S3) are clear. (S2): For any $n \geq 0$, we have

$$\sum_{j=0}^{\infty} q(n,j) = (1-a_n)\sum_{j=0}^{\infty}(a_n)^j = (1-a_n)\cdot\frac{1}{1-a_n} = 1.$$

Since $q(n,0) \geq q(n,1) \geq q(n,2) \geq \cdots$ for every $n \geq 0$, we get (S4). So, it follows that $\{q(n,j)\}$ is a strongly regular summation method. So, from Theorem 5.1, the sequence $\{x_n\}$ converges strongly to some $p \in F(T)$. ∎

The following corollary is connected with Yoshimoto [22].

Corollary 5.4. *Let* T *be a uniformly* C*-Lipschitz linear operator on a real Banach space* E *and suppose that* $\{T^j x : j \geq 0\}$ *is relatively weakly compact for every* $x \in E$. *Let* $\delta > 0$ *and let* $\{\lambda_j\}_{j=0}^{\infty}$ *be a sequence of nonnegative numbers such that* $\lambda_{j+1} \geq \lambda_j + \delta$ *for every* j. *Let* $\{t_n\} \subseteq (0,\infty)$ *be a sequence converging to* 0. *Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* E *by the following iteration scheme:*

$$x_0 = x \in E;$$

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{\sum_{j=0}^{\infty} e^{-\lambda_j t_n} T^j x_n}{\sum_{j=0}^{\infty} e^{-\lambda_j t_n}}, \quad n = 0,1,2,\ldots,$$

where $\{\alpha_n\} \subseteq [0,1]$ *satisfies* $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to some* $p \in F(T)$.

Proof. For every $t \geq 0$, we define $g(t) = \sum_{j=0}^{\infty} e^{-\lambda_j t}$. Note that $g(t)$ is well-defined. In fact, since $\lambda_j \geq \lambda_0 + \delta j$, we have

$$g(t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \leq \sum_{j=0}^{\infty} e^{-\lambda_0 t}(e^{-\delta t})^j = \frac{e^{-\lambda_0 t}}{1-e^{-\delta t}} < \infty.$$

We also have $\lim_{t\downarrow 0} g(t) = \infty$. In fact, let $N \geq 1$. Since $\lim_{t\downarrow 0} e^{-\lambda_j t} = 1$ for every j, there exists $t_0 > 0$ such that $e^{-\lambda_j t} > 1/2$ for every $t < t_0$ and $j = 0,1,\ldots,N-1$. Then we have $g(t) \geq \sum_{j=0}^{N-1} e^{-\lambda_j t} > N/2$ for every $t < t_0$. So, we have $\lim_{t\downarrow 0} g(t) = \infty$. Next, for every $n, j \geq 0$, we define $q(n,j) = e^{-\lambda_j t_n}/g(t_n)$. Now we show that $\{q(n,j)\}$ is a strongly regular summation method. In fact, (S1) is clear. (S2): For every n, we have

$$\sum_{j=0}^{\infty} q(n,j) = \sum_{j=0}^{\infty} \frac{e^{-\lambda_j t_n}}{g(t_n)} = \frac{g(t_n)}{g(t_n)} = 1.$$

(S3): Fix $j \geq 0$. Since $\lim_{t\downarrow 0} g(t) = \infty$, we have

$$q(n,j) = \frac{e^{-\lambda_j t_n}}{g(t_n)} \leq \frac{1}{g(t_n)} \to 0$$

as $n \to \infty$. Since $q(n,0) \geq q(n,1) \geq q(n,2) \geq \cdots$, we get (S4). Then $\{q(n,j)\}$ is a strongly regular summation method. So, from Theorem 5.1, the sequence $\{x_n\}$ converges strongly to some $p \in F(T)$. ∎

Theorem 5.5. *Let* S *and* T *be uniformly* C*-Lipschitz linear operators on a real Banach space* E *satisfying* $ST = TS$ *and suppose* $\{S^i T^j x : i,j \in \mathbb{Z}_+\}$ *is relatively weakly compact for every* $x \in E$. *Let* $\{q(n,i,j)\}_{n,i,j=0}^{\infty}$ *be a strongly regular bi-summation method. Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* E *by the following iteration scheme:*

$$x_0 = x \in E;$$

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i,j=0}^{\infty} q(n,i,j) S^i T^j x_n, \quad n = 0,1,2,\ldots,$$

where $\{\alpha_n\} \subseteq [0,1]$ *satisfies* $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to some* $p \in F(S) \cap F(T)$.

Proof. Define $U(i,j) = S^iT^j$ for every $(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. It is clear that $\mathcal{S} = \{U(i,j) : (i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+\}$ is a C^2-semigroup of continuous linear operators on E. For every $n \geq 0$ and $f \in l^\infty(\mathbb{Z}_+ \times \mathbb{Z}_+)$, we define $\mu_n(f) = \sum_{i,j=0}^{\infty} q(n,i,j)f(i,j)$. By Theorem 4.3, it follows that $\{\mu_n\}$ is a strongly regular sequences of means on $l^\infty(\mathbb{Z}_+ \times \mathbb{Z}_+)$. Fix $y \in E$. Then, for every $n \geq 0$ and $x^* \in E^*$, we have

$$\begin{aligned}
\langle U_{\mu_n} y, x^* \rangle &= (\mu_n)_{(i,j)} \langle U(i,j)y, x^* \rangle \\
&= \sum_{i,j=0}^{\infty} q(n,i,j) \langle U(i,j)y, x^* \rangle \\
&= \sum_{i,j=0}^{\infty} q(n,i,j) \left\langle S^iT^j y, x^* \right\rangle \\
&= \left\langle \sum_{i,j=0}^{\infty} q(n,i,j) S^iT^j y, x^* \right\rangle.
\end{aligned}$$

So we have

$$U_{\mu_n} y = \sum_{i,j=0}^{\infty} q(n,i,j) S^iT^j y$$

for every $y \in E$ and $n \geq 0$. From Theorem 3.3, it follows that the sequence $\{x_n\}$ converges strongly to some $p \in F(\mathcal{S}) = F(S) \cap F(T)$. ∎

Corollary 5.6. *Let* S *and* T *be uniformly* C*-Lipschitz linear operators on a real Banach space* E *satisfying* $ST = TS$ *and suppose* $\{S^iT^jx : i,j \in \mathbb{Z}_+\}$ *is relatively weakly compact for every* $x \in E$. *Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* E *by the following iteration scheme:*

$$\begin{aligned}
x_0 &= x \in E; \\
x_{n+1} &= \alpha_n x_n + \frac{1-\alpha_n}{(n+1)^2} \sum_{i,j=0}^{n} S^iT^j x_n, \quad n = 0, 1, 2, \ldots,
\end{aligned}$$

where $\{\alpha_n\} \subseteq [0,1]$ *satisfies* $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. *Then the sequence* $\{x_n\}$ *converges strongly to some* $p \in F(S) \cap F(T)$.

Proof. This is a special case of Theorem 5.5. For every $n, i, j \geq 0$, we define

$$q(n,i,j) = \begin{cases} 1/(n+1)^2, & \text{if } i,j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

(B1) and (B2) are clear. Since

$$\sum_{i,j=0}^{\infty} |q(n,i+1,j)-q(n,i,j)| = \sum_{j=0}^{\infty} |q(n,n+1,j)-q(n,n,j)|$$
$$= \sum_{j=0}^{n} \frac{1}{(n+1)^2} = \frac{1}{n+1} \to 0$$

as $n \to \infty$, we have (B3a). Similarly, we have (B3b). So, $\{q(n,i,j)\}$ is a strongly regular bi-summation method. So, from Theorem 5.5, the sequence $\{x_n\}$ converges strongly to some $p \in F(S) \cap F(T)$. ■

Corollary 5.7. *Let S and T be uniformly C-Lipschitz linear operators on a real Banach space E satisfying $ST = TS$ and suppose $\{S^iT^jx : i,j \in \mathbb{Z}_+\}$ is relatively weakly compact for every $x \in E$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in E by the following iteration scheme:*

$$x_0 = x \in E;$$
$$x_{n+1} = \alpha_n x_n + \frac{2(1-\alpha_n)}{(n+1)(n+2)} \sum_{i+j\le n} S^iT^jx_n, \quad n = 0,1,2,\ldots,$$

where $\{\alpha_n\} \subseteq [0,1]$ satisfies $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. Then the sequence $\{x_n\}$ converges strongly to some $p \in F(S) \cap F(T)$.

Proof. This is also a special case of Theorem 5.5. For every $n,i,j \ge 0$, we define

$$q(n,i,j) = \begin{cases} 2/(n+1)(n+2), & \text{if } i+j \le n, \\ 0, & \text{if } i+j > n. \end{cases}$$

(B1) is clear. Since

$$\sum_{i,j=0}^{\infty} q(n,i,j) = \sum_{k=0}^{\infty}\sum_{i+j=k} q(n,i,j) = \sum_{k=0}^{n} \frac{2(k+1)}{(n+1)(n+2)} = 1,$$

we have (B2). Since

$$\sum_{i,j=0}^{\infty} |q(n,i+1,j)-q(n,i,j)| = \sum_{k=0}^{\infty}\sum_{i+j=k} |q(n,i+1,j)-q(n,i,j)|$$
$$= \sum_{i+j=n} |q(n,i+1,j)-q(n,i,j)|$$
$$= \sum_{i+j=n} \frac{2}{(n+1)(n+2)} = \frac{2}{n+2} \to 0$$

as $n \to \infty$, we have (B3a). Similarly, we have (B3b). It follows that $\{q(n,i,j)\}$ is a strongly regular bi-summation method. So, from Theorem 5.5, the sequence $\{x_n\}$ converges strongly to some $p \in F(S) \cap F(T)$. ∎

Finally, we obtain two strong convergence theorems of Mann's type for one-parameter semigroups of continuous linear operators on a Banach space.

Corollary 5.8. *let $\mathcal{S} = \{T_t : t \in \mathbb{R}_+\}$ be a C-semigroup of continuous linear operators on a real Banach space E. Let $\{\lambda_n\} \subseteq (0,\infty)$ be a sequence with $\lambda_n \to \infty$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in E by the following iteration scheme:*

$$x_0 = x \in E;$$
$$x_{n+1} = \alpha_n x_n + \frac{1-\alpha_n}{\lambda_n} \int_0^{\lambda_n} T_t x_n \, dt, \quad n = 0, 1, 2, \ldots,$$

where $\{\alpha_n\} \subseteq [0,1]$ satisfies $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to some $p \in F(\mathcal{S})$.

Proof. For every $f \in C(\mathbb{R}_+)$, we define $\mu_n(f) = (1/\lambda_n)\int_0^{\lambda_n} f(t)\,dt$. Then $\{\mu_n\}$ is a strongly regular sequence of means on $C(\mathbb{R}_+)$ and, for every $y \in E$, we have $T_{\mu_n} y = (1/\lambda_n)\int_0^{\lambda_n} T_t y\,dt$; see [21, Theorem 3.5.2]. From Theorem 3.3, It follows that the sequence $\{x_n\}$ converges strongly to some $p \in F(\mathcal{S})$. ∎

Corollary 5.9. *Let $\mathcal{S} = \{T_t : t \in \mathbb{R}_+\}$ be a C-semigroup of continuous linear operators on a real Banach space E. Let $\{\lambda_n\} \subseteq (0,\infty)$ be a sequence with $\lambda_n \to 0$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ by the following iteration scheme:*

$$x_0 = x \in E;$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\lambda_n \int_0^{\infty} e^{-\lambda_n t} T_t x_n \, dt, \quad n = 0, 1, 2, \ldots,$$

where $\{\alpha_n\} \subseteq [0,1]$ satisfies $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to some $p \in F(\mathcal{S})$.

Proof. For every n and $f \in C(\mathbb{R}_+)$, define $\mu_n(f) = \lambda_n \int_0^{\infty} e^{-\lambda_n t} f(t)\,dt$. Then $\{\mu_n\}$ is a strongly regular sequence of means on $C(\mathbb{R}_+)$ and, for every $y \in E$, we have $T_{\mu_n} y = \lambda_n \int_0^{\infty} e^{-\lambda_n t} T_t y\,dt$; see [21, Theorem 3.5.3]. From Theorem 3.3, it follows that the sequence $\{x_n\}$ converges strongly to some $p \in F(\mathcal{S})$. ∎

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