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ON HIGHER ORDER EIGENVALUES OF THE SPHERICAL LAPLACIAN OPERATOR

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Abstract. In this paper, we use the boundary measurements of normalized eigenfunctions to estimate the variation of the corresponding eigenvalues. With this form, we can show that some eigenvalues, as functions of domain, possess monotonicity as the domain varies according some constraints.

1. Introduction

The study of eigenvalues for Laplacian operator is a quite interested and classical subject. Many mathematicians studied this problem using rearrangement methods, variational principle and integral inequalities, but these techniques are not efficient for the investigation for higher order eigenvalues. Instead of these techniques, we use the technique of shape derivatives (please refer [3] for details) to study some eigenvalue problems of spherical Laplacian operator. Let S^2 denote the unit sphere in $\mathbb{R}^3$, and

$$X(\theta,\phi) = (\cos\theta\sin\phi,\ \sin\theta\sin\phi, \cos\phi),\ 0 \le \phi \le \pi, 0 \le \theta < 2\pi,$$

the Euler coordinate for S^2 and

$$C(\phi_0) = \{X(\theta,\phi) | 0 \le \phi \le \phi_0,\ 0 \le \theta \le 2\pi\},$$

for $0 < \phi_0 < \pi$, a spherical cap. For $0 < \phi_0 < \phi_1$, we denote the spherical band

$$B(\phi_0,\phi_1) = \mathrm{the closure of}\{C(\phi_1) \setminus C(\phi_0)\}.$$

We will concentrate on the study of the following eigenvalue value problem

$$\begin{cases} \triangle_{S^2} u + \lambda u & = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} & = 0. \end{cases} \tag{1.1}$$

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where $\Omega = B(\phi_0, \phi_1)$. In particular, we want to investigate the behaviors of eigenvalues when the domain $B(\phi_0, \phi_1)$ varies under some constraints.

Note that the Laplace operator Δ_{S^2} on S^2 can be written as

$$\Delta_{S^2} u\,(\phi, \theta) = \frac{1}{\sin\phi}\left[\frac{\partial}{\partial\phi}(\sin\phi u_\phi) + \frac{\partial}{\partial\theta}\left(\left(\frac{u_\theta}{\sin\phi}\right)\right]. \tag{1.2}$$

For the case $\Omega = B(\phi_0, \phi_1)$, with the separation of variables, we know the eigenvalues $\{\lambda_n(B(\phi_0, \phi_1))\}_{n=1}^{\infty}$ of (1.1) corresponding to $\Omega = B(\phi_0, \phi_1)$ consists of all the union $\cup_{k=0}^{\infty}\{\lambda_n^k(\phi_0, \phi_1)\}_{n=1}^{\infty}$ of the eigenvalues

$$v''(\phi) + \cot\phi v'(\phi) + \left(\lambda - \frac{k^2}{\sin^2\phi}\right)v(\phi) = 0,\ v(\phi_0) = v(\phi_1) = 0. \tag{1.3}$$

In [2], the author showed that

Theorem 1.1. *The second eigenvalue $\lambda_2(B(\phi_0, \phi_1))$ of (1.1) corresponding to $\Omega = B(\phi_0, \phi_1)$ is equal to the first eigenvalue $\lambda_1^1(\phi_0, \phi_1)$ of (1.3) corresponding to $k = 1$.*

This theorem implies that

Corollary 1.2. *The second eigenvalue $\lambda_2(B(\phi_0, \phi_1))$ of (1.1) is equal to the first eigenvalue of the following boundary value problem*

$$\begin{cases} \triangle_{S^2} + \lambda \quad u = 0 \text{ in } B_H(\phi_0, \phi_1) \\ \qquad\qquad u = 0 \text{ on } \partial B_H(\phi_0, \phi_1) \end{cases} \tag{1.4}$$

where $B_H(\phi_0, \phi_1) = \{X(\theta, \phi) | 0 \le \theta \le \pi,\ \phi_0 \le \phi \le \phi_1\}$.

Moreover, the author also proved

Theorem 1.3. *For $0 < A < 1$, let B_0 be the spherical band symmetric to the equator and with area $2\pi A$. Then,for a spherical bands B with given area $2\pi A$, we have $\lambda_2(B_0) \ge \lambda_2(B)$. "=" holds if and only if $B = B_0$.*

However, the restriction "$0 < A < 1$" is added because of the technical reasons, which is not natural. In section 2, we will generalize the theorem for higher order eigenvalues under some different constraints, in the case, we can drop the area restriction and obtain a monotonicity property of the second eigenvalue (as a function of domains), moreover, we provide an evidence to show that the restriction $0 < A < 1$ in theorem 1.3 can be also dropped and some improvements can be made.

2. Extremal Problems for Eigenvalues Among Spherical Bands

In this section, we will use the technique of shape derivatives to study the extremal problems of eigenvalues among spherical bands with fixed area. For $0 < A < 2$, $0 < \xi < f(\xi) = \cos^{-1}(\cos\xi - A) < \pi$, denote the band $B(\xi, f(\xi))$ by $B(\xi; A)$, then $B(\xi; A)$ has area $2\pi A$. In [1], the authors used the variational principle and some integral inequalities to show that

Theorem 2.1. (Theorem 1, [1]) *For* $0 < A < 2$, *let* $\Omega = B(\xi; A)$ *in* **(??)** *and* $\lambda_1(\xi)$, *defined on* $(0, \pi - \cos^{-1}(1-A))$, *the corresponding first eigenvalue on* $B(\xi, f(\xi))$. *Then* $\lambda_1(\xi)$ *is increasing on* $(0, \cos^{-1}(A/2))$ *and attains its maximum when* $B(\xi; A)$ *is symmetric to the equator, i.e.,* $\xi = \cos^{-1}(A/2)$.

Now we are going to reprove this fact as an example to show how the technique of shape derivatives works for our problems. To prove this we need the following lemma and corollaries.

Lemma 2.2. *Let* $p(x)$ *be a continuous function defined on* $[-1, 1]$ *and* $p(x) \geq p(-x)$ *for* $-1 < x < 0$. *Denote* $y(x)$ *a nontrivial and nonnegative solution (if such a solution exists) of the differential equation*

$$y^{''}(x) + p(x)y(x) = 0,\ y(-1) = y(1) = 0. \tag{2.1}$$

Then $|y^{'}(1)|^2 \leq |y^{'}(-1)|^2$.

Proof. Denote $w(x) = y(-x)$, then we have

$$\begin{cases} y^{''}(x) + p(x)y(x) = 0, \\ w^{''}(x) + p(-x)w(x) = 0, \end{cases} \tag{2.2}$$

for $-1 < x < 1$. From (2.2), we have

$$w^{'}y - wy^{'}|_{-1}^{x} = \int_{-1}^{x} [p(t) - p(-t)]y(t)w(t)\,dt,$$

for $-1 < x < 0$, i.e.,

$$[w^{'}(x)y(x) - w(x)y^{'}(x)] = \int_{-1}^{x} [p(t) - p(-t)]y(t)w(t)\,dt \geq 0,$$

hence

$$\frac{w^{'}(x)y(x) - w(x)y^{'}(x)}{y^2(x)} = \left(\frac{w(x)}{y(x)}\right)^{'} \geq 0,\ \text{for } -1 < x \leq x_0 \to 0.$$

This implies $\lim_{x\to -1}\frac{w(x)}{y(x)} = \frac{w^{'}(-1)}{y^{'}(-1)} \leq \frac{w(0)}{y(0)} \leq 1$, therefore

$$|w^{'}(-1)|^2 = |y^{'}(1)|^2 \leq |y^{'}(-1)|^2.$$

Note the equality holds only if $p(x) \equiv p(-x)$. ■

Lemma 2.3. *Suppose that* $0 < \phi_0 < \phi < \phi_1 < \pi$. *Then the following eigenvalue problem*

$$\text{(2.3)} \qquad \begin{cases} (\sin\phi y^{'}(\phi))^{'} + \left(\lambda \sin\phi - \dfrac{k^2}{\sin\phi}\right) y(\phi) = 0, \ \phi_0 < \phi < \phi_1, \\ y(\phi_0) = y(\phi_1) = 0, \end{cases}$$

is equivalent to the following eigenvalue problem

$$\text{(2.4)} \qquad \begin{cases} u^{''}(\phi) + \left(\lambda + \dfrac{1}{4} - \dfrac{k^2 - \frac{1}{4}}{\sin^2(\phi)}\right) u(\phi) = 0, \\ u(\phi_0) = u(\phi_1) = 0, \end{cases}$$

where $u(\phi) = (\sin\phi)^{1/2} y(\phi)$.

Proof. Apply the Liouville transformation on (2.3), we can directly derive (2.4), and this can be done easily by Mathematica or Maple, we omit the details here. ■

Corollary 2.4. $0 < \phi_0 < \phi_1 \leq \pi - \phi_0$. *Let* $y_1(x)$ *denote the first eigenfunction corresponding to the first eigenvalue* λ_1 *of*

$$\text{(2.5)} \qquad \begin{cases} (\sin(\phi) y^{'}(\phi))^{'} + \lambda \sin(\phi) y(\phi) = 0, \ \phi_0 < \phi < \phi_1 \\ y(\phi_0) = y(\phi_1) = 0. \end{cases}$$

Then $(\sin\phi_0)(y_1^{'}(\phi_0))^2 \geq \sin(\phi_1)(y_1^{'}(\phi_1))^2$, *equality holds if and only if* $\phi_1 = \pi - \phi_0$.

Proof. Let $z(\phi) = (\sin(\phi))^{1/2} y_1(\phi)$, then apply corollary 2.3, (2.5) can be transformed to

$$\text{(2.6)} \qquad \begin{cases} z^{''} + (\lambda_1 + \dfrac{1}{2} + \dfrac{1}{4}(\cot(\phi))^2) z(\phi) = 0, \ \phi_0 < \phi < \phi_1 \\ z(\phi_0) = z(\phi_1) = 0. \end{cases}$$

Since $\phi_0 < \pi - \phi_1$, we have $\lambda + \frac{1}{2} + \frac{1}{4}(\cot(\phi))^2 > \lambda + \frac{1}{2} + \frac{1}{4}(\cot(\phi_1 + \phi_0 - \phi))^2$, for $\phi_0 \leq \phi \leq (\phi_0 + \phi_1)/2$. Take $z(\phi)$ the first eigenfunction of (2.6) and apply Lemma (2.2), we have $(z^{'}\phi_0)^2 \geq (z^{'}(\phi_1))^2$, i.e., $(\sin\phi_0)(y_1^{'}(\phi_0))^2 \geq \sin(\phi_1)(y_1^{'}(\phi_1))^2$. ■

Corollary 2.5. $0 < \phi_0 < \phi_1 < \pi - \phi_0,\ k \geq 1$. *Let* $y_1(x)$ *denote the first eigenfunction corresponding to the first eigenvalue* $\lambda_{\,1}$*of*

$$\text{(2.7)}\qquad \begin{cases} (\sin(\phi)y^{'}(\phi))^{'} + \left(\lambda\sin(\phi) - \dfrac{k^2}{\sin\phi}\right)y(\phi) = 0,\ \phi_0 < \phi < \phi_1 \\ y(\phi_0) = y(\phi_1) = 0. \end{cases}$$

Then $(\sin\phi_0)(y_1^{'}(\phi_0))^2 \leq \sin(\phi_1)(y_1^{'}(\phi_1))^2$. *Equality holds if and only if* $\phi_{\,1} = \pi - \phi_0$.

Proof. The situation of this corollary is different from Corollary 2.4, but the proof is the same. Take $u(\phi) = \sin^{1/2}\phi y_1(\phi)$, then, by Lemma 2.3, we have

$$\text{(2.6)}\qquad \begin{cases} u^{''}(\phi) + \left(\lambda_1 + \dfrac{1}{4} - \dfrac{k^2 - \frac{1}{4}}{\sin^2(\phi)}\right)u(\phi) = 0, \\ u(\phi_0) = u(\phi_1) = 0. \end{cases}$$

Note that

$$\lambda_1 + \frac{1}{4} - \frac{k^2 - \frac{1}{4}}{\sin^2(\phi)} \leq \lambda_1 + \frac{1}{4} - \frac{k^2 - \frac{1}{4}}{sin^2(\phi_1 - \phi)}$$

for $\phi_0 \leq \phi \leq \phi_1$. Hence we can apply Lemma 2.2 to conclude our assertion. ∎

Alternative Proof of Theorem 2.1. For $0 < \phi_0 < \phi_1 < \pi$, let $w_1(\theta, \phi; s)$ denote the positive normalized eigenfunction (actually $w_1(\theta, \phi; s) = w_1(\phi)$ is independent of θ) corresponding to the first eigenvalue $\nu_1(s)$ of

$$\text{(2.8)}\qquad \begin{cases} \triangle_{S^2} w_1 + \nu_1(s)w = 0, & \text{in } B(\phi_0 + s, \phi_1 + s), \\ w\,|_{\partial B(\phi_0+s,\phi_1+s)} = 0, & \end{cases}$$

for $0 < s < \pi - \phi_1$. Hence

$$\text{(2.9)}\qquad \begin{cases} \triangle_{S^2} w_1 + \nu_1 w = 0, & \text{in } B(\phi_0 + s, \phi_1 + s), \\ w_1\,|_{\partial B(\phi_0+s,\phi_1+s)} = 0, & \end{cases}$$

If we differentiate (2.9) and the boundary conditions $w_1(\theta, \phi_i + s, s) = 0$, for $i = 1$ and 2, with respect to s, we obtain

$$\text{(2.10)}\qquad \begin{cases} \triangle_{S^2} w_1 s + \nu_1(s)w_{1s} &= -\dot{\nu}(s)\,w_1,\ \text{in } B(\phi_0 + s, \phi_1 + s), \\ -w_{1n} + w_{1s} &= 0,\ \text{for } \phi = \phi_0 + s, \\ w_{1n} + w_{1s} &= 0,\ \text{for } \phi = \phi_1 + s, \end{cases}$$

Combine (2.9), (2.12) and (2.10), we have

$$\begin{aligned}\dot{\nu}_1(s) &= \iint_{B(\phi_0+s,\phi_1+s)} [w_1\triangle w_{1s} - w_{1s}\triangle w_1]\, dA \\ &= \int_{\partial B(\phi_0+s,\phi_1+s)} -w_{1s}\frac{\partial w_1}{\partial n}\, ds \\ &= \int_{\partial C(\phi_1+s)} \left(\frac{\partial w_1}{\partial n}\right)^2 ds - \int_{\partial C(\phi_0+s)} \left(\frac{\partial w_1}{\partial n}\right)^2 ds, \\ &= \pi\, \sin(\phi_0+s)(w_{1\phi}(\phi_0+s))^2 - \pi \sin(\phi_1+s)(w_{1\phi}(\phi_1+s))^2,\end{aligned} \tag{2.11}$$

since

$$\begin{cases} w_{1\phi} = -\dfrac{\partial w_1}{\partial n} & \text{for } \phi = \phi_0 + s, \\ w_{1\phi} = \dfrac{\partial w_1}{\partial n} & \text{for } \phi = \phi_1 + s. \end{cases} \tag{2.12}$$

Note the $w_1(\phi)$ is also the first eigenfunction of

$$\begin{cases} (\sin(\phi)y'(\phi))' + \lambda \sin(\phi)y(\phi) = 0,\ \phi_0 < \phi < \phi_1, \\ y(\phi_0) = y(\phi_1) = 0. \end{cases} \tag{2.13}$$

By Corollary 2.4, $\nu_1(s)$ is increasing. Now take $\phi_0 = \xi$ and $\phi_1 = f(\xi)$. We only treat the case $0 < \xi < \cos^{-1}(A/2)$ since for $\xi > \cos^{-1}(A/2)$ we can apply the property of symmetry. For

$$0 < \xi < f(\xi) + s < \cos^{-1}(A/2),$$

we can easily see that $f(\xi + s) < f(\xi) + s$, this means

$$B(\xi + s; A) \subset B(\xi + s, f(\xi) + s).$$

Hence

$$\lambda_1(\xi) = \nu_1(0) < \nu_1(s) < \lambda_1(\xi + s),\ \text{if } f(\xi) + s < \cos^{-1}(A/2).$$

This completes the proof. ∎

Remark. The equation (2.11) can be also derived directly from (2.13).

With the same arguments above, we may easily generalize Theorem 3.1 of [2] for higher eigenvalues as follows

Theorem 2.6. $0 < \phi_0 < \phi_1 < \pi - \phi_0$. *Let* $B(s) = B(\phi_0 + s, \phi_1 + s)$ *for* $0 \le s \le \pi - \phi_1$, *where* $B(a, b)$ *is as that defined in section 1. Let* $\nu_2(s)$

denote the second eigenvalue of (1.1) *with* $\Omega = B(s)$. *Then* $\nu_2(s)$ *is decreasing in* $(0, (\phi_1 - \phi_0)/2))$ *and attains its minimum when the band* $B(s)$ *is symmetric to the equator.*

Proof. By Corollary 1.2, we know that $\nu_2(s)$ is equal to the first eigenvalue of

$$\begin{cases} \triangle_{S^2} z + \mu z = 0, \text{ in } B_H(s), \\ z|_{\partial B_H(s)} = 0, \end{cases} \tag{2.14}$$

where $B_H(s) = B_H(\phi_0 + s, \phi_1 + s)$ which is defined in Section 1. What remains is to repeat the arguments in the alternative proof for Theorem 2.1. Let $z_1(\theta, \phi; s)$ denote the positive normalized eigenfunction corresponding to the first eigenvalues $\mu_1(s)$ of

$$\begin{cases} \triangle_{S^2} z_1 + \mu_1 z_1 = 0, \text{ in } B_H(s), \\ z_1|_{\partial B_H(s)} = 0. \end{cases} \tag{2.15}$$

Denote

$$\mathcal{T}_0 = \{X(\theta, \phi_0 + s) | 0 \le \theta \le \pi\}$$

and

$$\mathcal{T}_1 = \{X(\theta, \phi_1 + s) | 0 \le \theta \le \pi\},$$

then

$$\begin{cases} z_{1\phi} = -z_{1n} \text{ on } \mathcal{T}_0, \\ z_{1\phi} = z_{1n} \text{ on } \mathcal{T}_1. \end{cases} \tag{2.16}$$

Moreover if we differentiate (2.15), we obtain

$$\begin{cases} \triangle_{S^2} z_{1s} + \mu_1(s) z_{1s} &= -\dot{\mu}_1(s) z_{1s}, \\ z_{1s} &= z_{1n} \text{ on } \mathcal{T}_0, \\ z_{1s} &= -z_{1n} \text{ on } \mathcal{T}_1, \\ z_{1s} &= 0 \text{ on } \mathcal{S}_0 \cup \mathcal{S}_1, \end{cases} \tag{2.17}$$

where

$$\mathcal{S}_0 = \{X(0, \phi) | \phi_0 + s \le \phi \le \phi_1 + s\}$$

and

$$\mathcal{S}_1 = \{X(\pi, \phi) | \phi_0 + s \le \phi \le \phi_1 + s\}.$$

Combine (2.15) and (2.17), we have, as that in (2.11),

$$\begin{aligned}\dot{\nu}_1(s) &= \iint_{B_H(s)} [z_1 \triangle z_{1s} - z_{1s} \triangle z_1]\, dA \\ &= \int_{\mathcal{T}_0} (\frac{\partial z_1}{\partial n})^2\, ds - \int_{\mathcal{T}_1} (\frac{\partial z_1}{\partial n})^2\, ds, \\ &= \frac{\pi}{2}\sin(\phi_0+s)(y_{1\phi}(\phi_0+s))^2 - \frac{\pi}{2}\sin(\phi_1+s)(y_{1\phi}(\phi_1+s))^2,\end{aligned} \tag{2.18}$$

where we can take $z_1(\theta,\phi) = C\sin\theta y_1(\phi)$ and $y_1(\phi)$ is the normalized first eigenfunction of

$$\begin{cases} (\sin\phi y'(\phi))' + \left(\lambda\sin\phi - \dfrac{1}{\sin\phi}\right) y(\phi) = 0,\ \phi\in(\phi_0+s, \phi_1+s), \\ y(\phi_0+s) = y(\phi_1+s) = 0. \end{cases} \tag{2.19}$$

Then apply Corollary 2.5, we have $\dot{\nu}_2(s) = \dot{\mu}_1(s) < 0$, for $0 < s < (\phi_1 - \phi_0)/2$. This shows that $\nu_2(s)$ is decreasing in $(0, \frac{\phi_1-\phi_0}{2})$, and by the symmetry, $\nu_2(s)$ attains minimum when the $B(s)$ is symmetric about the equator. ■

Since we know the eigenvalues $\{\lambda_n(B(\phi_0,\phi_1))\}_{n=1}^{\infty}$ of (1.1) with $\Omega = B(\phi_0,\phi_1)$ consist of the eigenvalues $\cup_{k=0}^{\infty}\{\lambda_n^k(\phi_0,\phi_1)\}_{n=1}^{\infty}$ of (1.3), we can easily conclude that

Corollary 2.7. $0 < \phi_0 < \phi_1 < \pi$. *Let* $\nu_n(s)$ *be an eigenvalue of (1.1) with* $\Omega = B(s) = B(\phi_0+s, \phi_1+s)$, *which is of the form* $\lambda_1^k(\phi_0+s, \phi_1+s)$, $k = 1,2,3,\ldots$. *Then* $\nu_n(s)$ *is decreasing in* $0 < s \le (\phi_1-\phi_0)/2$ *and attains its minimum when* $B(s)$ *is symmetric about the equator.*

In the rest of this section, we want to investigate the behavior of higher order eigenvalues of the spherical bands with fixed area. This problems is more sophisticated. We use the same techniques as above. For $0 < A < 2$ and $0 < \xi$, let $f(\xi)$ and $B(\xi; A)$ be as that defined in the beginning of this section. Denote $\lambda_n(\xi)$ be the n^{th} eigenvalue of (1.1) with $\Omega = B(\xi; A)$. Note that if $\lambda_n(\xi)$ is of the type $\lambda_1^k(\xi, f(\xi))$, Then $\lambda_n(\xi)$ is the first eigenvalue $\lambda_1^k(\xi, f(\xi))$ of the boundary value problem

$$\begin{cases} (\sin\phi y'(\phi))' + \left(\lambda\sin\phi - \dfrac{k^2}{\sin\phi}\right) y(\phi) = 0,\ \xi < \phi < f(\xi), \\ y(\xi) = y(f(\xi)) = 0, \end{cases} \tag{2.20}$$

$k = 1,2,3,4,\ldots$. For simplicity, we treat the case $\lambda_2(\xi) = \lambda_1^1(\xi, f(\xi))$ first, for $k \ge 2$, the argument is similar.

Immediately, we have

Lemma 2.8. *Let* $\lambda_2(\xi)$ *be defined as above. Then*

$$\dot{\lambda}_2(\xi) = \sin\xi[(y)'(\xi;\xi))^2 - (y'(f(\xi);\xi))^2], \tag{2.21}$$

where $y(\phi;\xi)$ *is the normalized positive eigenfunction of (2.20) corresponding to* $\lambda_1^1(\xi, f(\xi)), i.e. \int_\xi^{f(\xi)} \sin\phi y^2(\phi;\xi)\,d\phi = 1.$

Proof. As we mentioned above that $\lambda_2(\xi) = \lambda_1^1(\xi, f(\xi))$. Differentiate (2.20) with k=1, we have

$$\begin{cases} (\sin\phi\dot{y}'(\phi;\xi))' + \left(\lambda_2\sin\phi - \dfrac{1}{\sin\phi}\right)\dot{y}(\phi;\xi) = -\dot{\lambda}\sin\phi y(\phi;\xi), \\ y'(\xi,\xi) + \dot{y}(\xi,\xi) = 0, \\ y'(f(\xi);\xi)f'(\xi) + \dot{y}(f(\xi);\xi) = 0, \end{cases} \tag{2.22}$$

since $y(\xi;\xi) = y(f(\xi);\xi) = 0$. Also note that $\cos(\xi) - \cos(f(\xi)) = A$, Hence

$$f'(\xi) = \sin\xi/\sin(f(\xi)). \tag{2.23}$$

Combine (2.20), (2.22) and (2.23), we obtain

$$\begin{aligned} \dot{\lambda}_2(\xi) &= \sin(f(\xi))y'(f(\xi);\xi)\dot{y}((f(\xi);\xi) - \sin\xi y'(\xi;\xi)\dot{y}(\xi;\xi) \\ &= \sin\xi[(y'(\xi,\xi))^2 - (y'(f(\xi);\xi))^2] \end{aligned} \tag{2.24}$$

∎

Moreover, using the same arguments, we have

Corollary 2.9. *Suppose that* $\lambda_n(\xi)$ *is of the type* $\lambda_1^k(\xi, f(\xi))$*,* $k \geq 1$*, we write* $\lambda_n(\xi) = \lambda_1^k(\xi, f(\xi))$*. Let* $y_k(\phi,\xi)$ *be the normalized first eigenfunction of* (2.20) *corresponding to* $\lambda_1^k(\xi, f(\xi))$*. Then*

$$\dot{\lambda}_n(\xi) = \sin\xi[(y_k'(\xi;\xi))^2 - (y_k'(f(\xi);\xi))^2]. \tag{2.25}$$

The author used to study the behaviors of eigenvalues by variational principle and integral inequalities, but via this approach, the investigation of the higher-order eigenvalues will become quite complicated. Instead of variational principle, we use the boundary measurements of the normalized eigenfunction to represent the derivative of eigenvalue and this is the main theme of this paper. These method can be also applied on the same problems on surfaces of revolution.

For one who concerns the behavior of eigenvalues, this method is quite efficient for numerical estimations. The author got many numerical results (One can just use shooting method to estimate $(y_k'(\xi;\xi))^2 - (y_k'(f(\xi);\xi))^2)$ to support the following conjecture

Conjecture 1. $0 < \phi_0 < \phi_1 < \pi$*. Let* $y_k(\phi;\lambda)$ *denote the solution of the initial value problem*

$$\begin{cases} (\sin\phi y'(\phi))' + \left(\lambda\sin\phi - \dfrac{k^2}{\sin\phi}\right)y(\phi) = 0,\ \phi_0 < \phi < \phi_1, \\ y(\phi_0) = 0,\ y'(\phi_0) = 1, \end{cases} \tag{2.26}$$

where λ *is a positive parameter which is large enough such that* $y_k(\phi)$ *has zeros in* (ϕ_0, π). *Denote* ϕ_1 *be the first zero of* y_k *in* (ϕ_0, π). *Then* $(y_k'(\phi_1; \lambda))^2 > 1$ *if* $\phi_1 > \pi - \phi_0$ *and* $(y_k'(\phi_1; \lambda))^2 < 1$ *if* $\phi_1 < \pi - \phi_0$.

Once this conjectured can be proved, we can obtain, by Corollary (2.9), that

Conjecture 2. $0 < A < 2$. $B(\xi; A)$ *is as that defined in the beginning of this section for* $0 < \xi < \pi - \cos^{-1}(1-A)$. *Let* $\lambda_n(\xi)$ *denote the* n^{th} *Dirichlet eigenvalue which is of the type* $\lambda_1^k(\xi, f(\xi))$. *Then* $\lambda_n(\xi)$ *is increasing in* $(0, \cos^{-1}(A/2)$ *and attains its maximum at* $\cos^{-1}(A/2)$.

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