

## ON SURJUNCTIVITY OF THE TRANSITION FUNCTIONS OF CELLULAR AUTOMATA ON GROUPS

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**Abstract.** We give a simple proof of the fact that the following property: any injective transition function of a cellular automaton on a group  $G$  is surjective, holds for any group  $G$  approximable by amenable groups. For finitely generated groups this was proved by Gromov in [1]

### 1. INTRODUCTION

According to Gottschalk [2] a map  $f : X \rightarrow X$  is *surjunctive* if it is either surjective or non-injective. Obviously, any self-mapping of a finite set  $X$  is surjunctive as well as any endomorphism of a finite-dimensional vector space. The surjunctivity of a regular self-mapping of a complex algebraic variety was proved in [3]. In [1] the surjunctivity was proved in many other cases that generalize the case of a regular self-mapping of a complex algebraic variety. Among them the surjunctivity of transition functions of cellular automata on a finitely generated group was proved for a huge class of such groups.

The notion of a cellular automaton is due to Ulam [4] and von Neumann [5]. A cellular automaton consists of a finite set  $F$  of states, the space of configurations  $F^{\mathbf{Z}^n}$  and a transition function  $\Lambda : F^{\mathbf{Z}^n} \rightarrow F^{\mathbf{Z}^n}$  that is continuous in Tichonoff's topology on  $F^{\mathbf{Z}^n}$  and commutes with the action of  $\mathbf{Z}^n$  on  $F^{\mathbf{Z}^n}$  by shifts. The surjunctivity of the transition function of an arbitrary cellular automaton was proved in [6].

The generalization of the notion of a cellular automaton on the case when the configuration space is  $F^G$  for a finitely generated group  $G$  was discussed in many papers (see e.g. [1, 7-9]). The transition function  $\Lambda : F^G \rightarrow F^G$  in this case is also continuous in Tichonoff's topology on  $F^G$  and commutes with the action of  $G$  on  $F^G$  by right shifts.

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Let us say that a group  $G$  satisfies the *surjunctivity property* or is an *S-group* if a transition function of any cellular automaton on  $G$  is surjunctive.

In the paper [1] the following two statements about finitely generated  $S$ -groups were proved.

- (A) *If a finitely generated group  $G$  is approximable by finitely generated  $S$ -groups with the same system of generators in the sense of [10] then  $G$  is an  $S$ -group.*
- (B) *Any finitely generated amenable group is an  $S$ -group.*

Statements (A) and (B) imply the surjunctivity of cellular automata defined on all groups approximable by amenable groups. Since all self-mappings of finite sets are surjunctive we immediately obtain from (A) that all groups, approximable by finite groups are  $S$ -groups. The class of groups that are approximable by finite groups was studied in [10], where it was named the class of LEF-groups (groups that are locally embeddable in the class of finite groups). It was proved that this class is a proper subclass of the class of groups that approximable by amenable groups (LEA-groups). So the result about surjunctivity of cellular automata on LEF-groups is weaker than the result about surjunctivity of cellular automata on LEA-groups. On the other hand, it was proved in [11] that there exist non-LEA-groups. More precisely, it was proved that non-amenable finitely presented simple groups (e.g. the R. Thompson's group T) are non-LEA-groups. The problem of characterization of the class of  $S$ -groups is open. In particular, it is not known, whether all groups are  $S$ -groups.

In the present article we give a simple proof of statement (A) using the technique of ultraproducts. This technique allows to prove statement (A) for cellular automata defined not only on groups, but on some more general universal algebras with one binary operation. This gives some new examples of surjunctive transition functions and, hopefully, can help to solve the problems mentioned at the end of the previous paragraph (see discussion in the Section 2).

## 2.

In this section we introduce cellular automata on some generalizations of groups.

**Definition 1.** Let  $S$  and  $H$  be arbitrary sets and  $\circ : S \times H \rightarrow H$ . We say that the triple  $L = \langle S, H; \circ \rangle$  is a configuration system (c. system).

If  $S' \subseteq S$ ,  $H' \subseteq H$ , and

$$\forall s \in S', h \in H' (s \circ h \in H' \wedge \forall x \in H (s \circ x = h \longrightarrow x \in H'))$$

then we say that the c. system  $L' = \langle S', H'; \circ \rangle$  is a c. subsystem of  $L$ .

Now we are going to formulate the definition of a cellular automaton on a c. system  $L$ . Since in this paper we are interested only in the properties of transition functions we identify cellular automata and their transition functions.

**Definition 2.** Let  $F$  be a finite set,  $L = \langle S, H; \circ \rangle$  - a c. system. A map  $\Lambda : F^H \rightarrow F^H$  is called a cellular automaton on  $L$  if there exists a finite set  $D \subseteq S$  and a map  $\Phi : F^D \rightarrow F$  such that for any  $f : H \rightarrow F$  and for any  $g \in H$  holds:

$$(1) \quad \Lambda(f)(g) = \Phi(f_g),$$

where  $f_g : D \rightarrow F$  is defined by the formula:  $f_g(s) = f(s \circ g), \forall s \in D$ .

For example, if  $S = H = G$  and  $(G, \circ)$  is a group, we obtain the usual definition of a cellular automaton on a group. Precisely, the following proposition holds for cellular automata on groups.

**Proposition 1.** Let  $\langle G; \circ \rangle$  be a group. A map  $\Lambda : F^G \rightarrow F^G$  is a cellular automaton on  $L = \langle G, G; \circ \rangle$  iff it commutes with the right shifts and is continuous in the Tichonoff's topology on  $F^G$ .

A proof can be found in [9] (see also [1]).

**Proposition 2.** A cellular automaton  $\Lambda : F^H \rightarrow F^H$  on an arbitrary c. system  $L = \langle S, H; \circ \rangle$  is continuous in the Tichonoff's topology on  $F^H$ .

We omit a simple proof of this proposition.

**Definition 3.** We say that a c. system  $L$  is surjunctive (S-system) if any cellular automaton on  $L$  is surjunctive.

In this paper we study the properties of surjunctive systems. One of important problems here is to find a conditions, when a c. subsystem of a surjunctive system is surjunctive itself <sup>1</sup>. In this section we obtain one sufficient condition.

**Lemma 1.** Let  $L' = \langle S', H'; \circ \rangle$  be a c. subsystem of a c. system  $L = \langle S, H; \circ \rangle$  and a finite set  $D \subseteq S'$ . Then the map  $\Phi : F^D \rightarrow F$  defines by formula (1) the cellular automaton  $\Lambda$  on  $L$  as well as the cellular automaton  $\Lambda'$  on  $L'$ .

If the cellular automaton  $\Lambda$  is surjective then the cellular automaton  $\Lambda'$  is also surjective.

*Proof.* Let  $\pi_{H'} : F^H \rightarrow F^{H'}$  be the projection:  $\pi_{H'}(f) = f \upharpoonright H' \forall f \in F^H$ . It follows from the definitions that the following diagram is commutative:

$$(2) \quad \begin{array}{ccc} F^H & \xrightarrow{\Lambda} & F^H \\ \pi_{H'} \downarrow & & \pi_{H'} \downarrow \\ F^{H'} & \xrightarrow{\Lambda'} & F^{H'} \end{array}$$

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<sup>1</sup>The importance of this problem is discussed in the Section 4

Our lemma follows immediately from the commutativity of diagram (2).  $\blacksquare$

**Definition 4.** Let  $L = \langle S, H; \circ \rangle$  be a c. system. A function  $\phi : H \rightarrow H$  is said to be  $L$ -good if

$$\forall h \in H \exists t \in H \forall s \in S \phi(s \circ h) = s \circ t.$$

**Definition 5.** We say that a configuration system  $L = \langle S, H; \circ \rangle$  is group-like if for every  $h_1, h_2 \in H$  there exists an  $L$ -good map  $\phi_{h_1, h_2} : H \rightarrow H$  such that  $\phi_{h_1, h_2}(h_1) = h_2$ .

Obviously, c. system  $\langle G, G; \cdot \rangle$  for a group  $(G, \cdot)$  is a group-like c.system. Indeed, one can take  $\phi_{h_1, h_2}(x) = x \cdot h_1^{-1} \cdot h_2$  in Definition 5. On the other hand, it is easy to construct a universal algebra  $(G, \circ)$  such that  $G$  is not a groups but the system  $L = \langle G, G; \circ \rangle$  is a group-like c. system.

For example, let  $(G, \cdot)$  be a group,  $\lambda : G \rightarrow G$  - a bijection and  $a \cdot \lambda b = a \cdot \lambda(b)$ . Then  $L = \langle G, G; \cdot \lambda \rangle$  is a group-like configuration system. It is enough to check that the same  $\phi_{h_1, h_2}(x) = x \cdot h_1^{-1} \cdot h_2$  is  $L$ -good. But  $\phi_{h_1, h_2}(s \cdot \lambda x) = s \cdot \lambda \lambda^{-1}(\lambda(x)) \cdot h_1^{-1} \cdot h_2$ .

**Lemma 2.** Let  $L' = \langle S', H'; \circ \rangle$  be a c. subsystem of a group-like c. system  $L = \langle S, H; \circ \rangle$  and a finite set  $D \subseteq S'$ . Then the map  $\Phi : F^D \rightarrow F$  defines by formula (1) the cellular automaton  $\Lambda$  on  $L$  as well as the cellular automaton  $\Lambda'$  on  $L'$ .

If the cellular automaton  $\Lambda'$  is injective then the cellular automaton  $\Lambda$  is also injective.

*Proof.* Assume that  $\Lambda$  is not injective. Then there exist  $f^{(1)}, f^{(2)} \in F^H$  and  $a \in H$  such that  $f^{(1)}(a) \neq f^{(2)}(a)$  but  $\Lambda(f^{(1)})(g) = \Lambda(f^{(2)})(g) \forall g \in H$ . Pick up an arbitrary  $h_0 \in H'$  and an  $L$ -good map  $\phi : H \rightarrow H$  be such that  $\phi(h_0) = a$  (see the Definition 5). Put  $g^{(i)} = f^{(i)}(\phi(\cdot)) \upharpoonright H' \in F^{H'}$ ,  $i = 1, 2$ . Then  $g^{(1)}(h_0) = f^{(1)}(a) \neq f^{(2)}(a) = g^{(2)}(h_0)$ . But for any  $h \in H'$   $d \in D$  holds:

$$g_h^{(i)}(d) = g^{(i)}(d \circ h) = f^{(i)}(\phi(d \circ h)) = f^{(i)}(d \circ t) = f_t^{(i)}(d),$$

where  $t$  is from Definition 4. Hence  $\forall h \in H'$

$$\Lambda'(g^{(i)})(h) = \Phi(g_h^{(i)} \upharpoonright D) = \Phi(f_t^{(i)} \upharpoonright D) = \Lambda(f^{(i)})(t).$$

Therefore,  $\Lambda'(g^{(1)}) = \Lambda'(g^{(2)})$  and  $\Lambda'$  is not injective.  $\blacksquare$

Lemmas 1 and 2 imply the following

**Proposition 3.** A c. subsystem of a surjunctive group-like c. system is surjunctive.

**Corollary 1.** *A subgroup of an  $S$ -group is an  $S$ -group itself. We conclude*

this section with the definition of approximation of a c. system  $L$  by systems that belong to a given class  $\mathcal{K}$  ( $\mathcal{K}$ -systems). This definition is equivalent to a definition of the local embedding of an algebraic system in a class of algebraic systems (see e.g. [12]). The particular case of groups that are locally embedded in the class of finite groups was investigated in details in [10]. Groups approximable by amenable groups were discussed in [11].

**Definition 6.** Let  $\mathcal{K}$  be a class of c. systems. We say that a c. system  $L = \langle S, H; \circ \rangle$  is locally embeddable in the class  $\mathcal{K}$ , or  $L$  is approximable by  $\mathcal{K}$ -systems, if for any finite subsets  $T \subseteq S$  and  $E \subseteq H$  there exists a c. system  $L' = \langle S', H'; \odot \rangle \in \mathcal{K}$  and injective maps  $\varphi : T \rightarrow S'$  and  $\psi : E \rightarrow H'$  such that

$$\forall t \in T, h \in E (t \circ h \in E \implies \varphi(t) \odot \psi(h) = \psi(t \circ h)).$$

3.

In this section we reproduce the notion of an ultraproduct and some related results that can be found in any book on Model Theory (see e.g. [12]).

Let  $I$  be an infinite set,  $\mathcal{P}(I)$  – its power set. Consider a finitely additive measure  $\mu : \mathcal{P}(I) \rightarrow \{0, 1\}$  such that  $\forall i \in I \mu(\{i\}) = 0$ . Using Zorn’s Lemma one can easily show that such  $\mu$  always exists.

Let  $\mathcal{A} = \{A_i \mid i \in I\}$  be a family of non-empty sets. Consider the direct product  $\prod_{i \in I} A_i$  and the equivalence relation  $\sim_\mu$  on  $\prod_{i \in I} A_i$  such that for any  $\langle a_i \rangle, \langle a'_i \rangle \in \prod_{i \in I} A_i$  holds  $\langle a_i \rangle \sim_\mu \langle a'_i \rangle$  iff  $\mu(\{i \mid a_i = a'_i\}) = 1$ . The quotient set  $\mathcal{A}^\mu$  is called an ultraproduct of the family  $\mathcal{A}$  over  $\mu$ . The equivalence class of an element  $\langle a_i \rangle \in \prod_{i \in I} A_i$  is denoted by  $\langle a_i \rangle^\mu$ .

In what follows we assume that  $\mu$  is fixed and consider only ultraproducts over this  $\mu$ .

Let  $\mathcal{B} = \{B_i \mid i \in I\}$  be another family of sets, indexed by elements of  $I$ . It is easy to see that

$$(3) \quad \mathcal{B}^\mu \subseteq \mathcal{A}^\mu \iff \mu(\{i \mid A_i \subseteq B_i\}) = 1$$

Indeed (3) follows from the general Los’s Theorem that will be discussed later in this section.

It is easy to see that not any subset of  $\mathcal{A}^\mu$  can be represented in the form  $\mathcal{B}^\mu$  for an appropriate family  $\mathcal{B}$  of subsets of  $A_i$ . Following the terminology of Nonstandard Analysis we call subsets of ultraproducts that are ultraproducts themselves – *internal subsets*. Non-internal subsets are called *external*. We denote the family of all internal subsets of  $\mathcal{A}^\mu$  by  $\mathcal{P}^{int}(\mathcal{A}^\mu)$ .

**Proposition 4.** *Any finite subset  $E$  of an ultraproduct  $\mathcal{A}^\mu$  is an internal subset.*

This Proposition can be easily proved by induction over the cardinality of  $E$ . ■

The Cartesian product  $\mathcal{A}^\mu \times \mathcal{B}^\mu$  can be obviously identified with the ultraproduct  $\prod_{i \in I} (A_i \times B_i) / \sim_\mu$ .

Consider the logical language  $L$  that contains variables of two types: individual variables denoted by lowercase latin letters and set variables denoted by uppercase latin letters. Atomic formulas are of two types:  $x = \langle y, z \rangle$  and  $x \in X$ . The other formulas of  $L$  are obtained from atomic formulas by application of logic connectives  $\vee, \wedge, \neg, \rightarrow$  and quantifiers over variables of both types.

Notice, that the statement " $F \subseteq X \times Y$  is a (injective, surjective, bijective) map from  $X$  to  $Y$ " can be expressed in the language  $L$  and thus, we may assume that  $L$  contains also atomic formulas of the type  $y = F(x)$ .

A variable  $x$  is called *free* in a formula  $\beta$  if it is not contained in any subformula of the form  $Qx\gamma$ , where  $Q$  is a quantifier and  $\gamma$  is a formula (the same definition works for set variables). A non-free variable is called a *bound variable*. A formula that does not contain free variables is called a *sentence*. A formula that contains only individual bound variables is called an *elementary* formula or a *first order* formula.

It is well-known that any formula is logically equivalent to a formula in the *prenex form*:

$$Q_1\tau_1 \dots Q_n\tau_n\phi(\tau_1, \dots, \tau_n),$$

where each  $Q_i$  is either  $\exists$  or  $\forall$ , each  $\tau_i$  is either an individual or a set variable and  $\phi$  does not contain quantifiers. For any formula  $\phi$  the notation  $\phi(\tau_1, \dots, \tau_n)$  means that the list variables  $\tau_1, \dots, \tau_n$  includes all free variables of  $\phi$  and nothing else.

Let  $\mathcal{U} = \{U_i \mid i \in I\}$  be a family of sets. We interpret the formulas of  $L$  in each of the sets  $U_i$  and in the set  $\mathcal{U}^\mu$ . When we consider the interpretation in  $U_i$ , individual variables run over  $(U_i)_\infty = U_i \cup U_i^2 \cup U_i^3 \cup \dots$  and set variables run over  $\mathcal{P}_\infty(U_i) = \mathcal{P}(U_i) \cup \mathcal{P}(U_i^2) \cup \mathcal{P}(U_i^3) \cup \dots$ . When we consider the interpretation in  $\mathcal{U}^\mu$ , individual variables run over  $\mathcal{U}_\infty^\mu = \mathcal{U}^\mu \cup (\mathcal{U}^\mu)^2 \cup \dots$  and set variables run over  $\mathcal{P}_\infty^{int}(\mathcal{U}^\mu) = \mathcal{P}^{int}(\mathcal{U}^\mu) \cup \mathcal{P}^{int}((\mathcal{U}^\mu)^2) \cup \dots$ . If a formula  $\varphi$  of the language  $L$  is true in an interpretation  $V$  we write  $V \models \varphi$ .

**Theorem 1.** ([1]) *Let  $\varphi(x, \dots, y, X, \dots, Y)$  be a formula of  $L$  with free variables  $x, \dots, y, X, \dots, Y$ ;  $\langle a_i \rangle^\mu, \langle b_i \rangle^\mu \in \mathcal{U}_\infty^\mu$ ,  $\mathcal{A}^\mu, \mathcal{B}^\mu \in \mathcal{P}_\infty^{int}(\mathcal{U}^\mu)$ , where  $\mathcal{A} = \{A_i \mid i \in I\}$ ,  $\mathcal{B} = \{B_i \mid i \in I\}$ . Then*

$$\mathcal{U}^\mu \models \varphi(\langle a_i \rangle^\mu, \dots, \langle b_i \rangle^\mu, \mathcal{A}^\mu, \dots, \mathcal{B}^\mu) \text{ iff } \mu(\{i \mid U_i \models \varphi(a_i, \dots, b_i, A_i, \dots, B_i)\}) = 1.$$

**Remark 1.** Indeed Los's Theorem holds for a much more expressive logical language, but the language  $L$  is enough for our considerations.

**Corollary 2.** *Let  $\mathcal{F} = \{f_i \subseteq A_i \times B_i \mid i \in I\}$ . Then  $\mathcal{F}^\mu$  is a map from  $\mathcal{A}^\mu$  to  $\mathcal{B}^\mu$  iff  $\mu(\{i \mid f_i : A_i \rightarrow B_i\}) = 1$ . In this case  $\mathcal{F}^\mu : \mathcal{A}^\mu \rightarrow \mathcal{B}^\mu$  is injective (surjective) iff  $\mu(\{i \mid f_i : A_i \rightarrow B_i \text{ is injective (surjective)}\}) = 1$ .*

**Definition 7.** A class  $\mathcal{K}$  of c. systems is called *axiomatizable* if there exists a family  $\mathcal{F}$  of formulas of the language  $L$  such that each of these formulas contains only three free variables (say,  $X, Y$  and  $Z$ ) and for any c. system  $\mathcal{L} = \langle S, H; \circ \rangle$  holds:

$$\mathcal{L} \in \mathcal{K} \iff \forall \beta(X, Y, Z) \in \mathcal{F} \ U \models \beta(S, H, \circ),$$

where  $U = S \cup H$ . The set  $\mathcal{F}$  is called the set of axioms of  $\mathcal{K}$ .

If all axioms in  $\mathcal{F}$  are elementary formulas then the class  $\mathcal{K}$  is said to be elementary axiomatizable.

If all axioms in  $\mathcal{F}$  contain only existential quantifiers for set variables in the prenex form (the quantifiers for individual variables may be arbitrary) then we say that  $\mathcal{K}$  is  $\exists^1$ -axiomatizable.

For example the class of groups is elementary axiomatizable and the class of group-like systems is  $\exists^1$ -axiomatizable.

The following statement is an easy corollary of Theorem [1].

**Corollary 3.** *Let  $\mathcal{K}$  be a  $\exists^1$ -axiomatizable (in particular, elementary axiomatizable) class of algebraic systems. Then any ultraproduct of  $\mathcal{K}$ -systems is a  $\mathcal{K}$ -system.*

*In particular, an ultraproduct of groups is a group, an ultraproduct of c. systems is a c. system, an ultraproduct of group-like c. systems is a group-like c. system, etc.*

**Remark 2.** If an axiom for  $\mathcal{K}$  of the form  $\forall X \phi(X)$  is true in an ultraproduct  $\mathcal{U}^\mu$  of  $\mathcal{K}$ -systems then only  $\forall$  internal  $X \phi(X)$  holds in  $\mathcal{U}^\mu$ , while  $\forall X \phi(X)$  may be false in  $\mathcal{U}^\mu$ . That is the reason, why Corollary 3 holds only for  $\exists^1$ -axioms.

**Remark 3.** It is well-known in Model Theory (cf. e.g. [12]) that a class  $\mathcal{K}$  of algebraic systems is elementary axiomatizable iff it is closed under ultraproducts (an ultraproduct of  $\mathcal{K}$ -systems is a  $\mathcal{K}$ -system). A class  $\mathcal{K}$  is closed under approximations (a system approximable by  $\mathcal{K}$ -systems is a  $\mathcal{K}$ -system) iff it is elementary axiomatizable by a family of axioms such that each of these axioms contains only universal quantifiers in the prenex form.

#### 4.

Application of ultraproducts to the problems of surjunctivity is based on the following two theorems.

The first of them can be easily obtained from some results well-known in Model Theory (see e.g. [12]). For the case of groups its proof can be found in [11]. A proof for the case of an arbitrary class of c. systems is similar.

**Theorem 2.** *Let  $\mathcal{K}$  be a class of c. systems. C. system  $L$  is approximable by  $\mathcal{K}$ -systems in the sense of Definition 6 iff  $L$  is isomorphic to a sub c. system of an ultraproduct of  $\mathcal{K}$ -systems.*

**Theorem 3.** *An ultraproduct of  $S$ -systems is an  $S$ -system.*

*Proof.* Let  $\mathcal{L} = \{L_i \mid i \in I\}$  be a family of  $S$ -systems, where  $L_i = \langle S_i, H_i, \circ_i \rangle$ . Consider a system  $L = \mathcal{L}^\mu = \langle S, H, \circ \rangle$ , where  $S = \{S_i\}^\mu$ ,  $H = \{H_i\}^\mu$ ,  $\circ = \{\circ_i\}^\mu$ . Let  $F$  be a finite set. Consider the family  $\mathcal{F} = \{F_i \mid i \in I\}$ , where  $F_i = F \forall i \in I$ . Then  $\mathcal{F}^\mu$  can be naturally identified with  $F$ . Indeed, it is easy to see that if  $F = \{a^{(1)}, \dots, a^{(n)}\}$  then  $\mathcal{F}^\mu = \{\langle a_i^{(1)} \rangle^\mu, \dots, \langle a_i^{(n)} \rangle^\mu\}$ , where  $\forall i \in I, a_i^{(j)} = a^{(j)}$ . So, in what follows we identify  $F$  and  $\mathcal{F}^\mu$ .

Let  $(F^H)^{int}$  be the family of all *internal* mappings  $f : H \rightarrow F$ . Then the following lemma holds:

**Lemma 3.** *The family  $(F^H)^{int}$  is dense in  $F^H$  in the Tichonoff's topology on  $F^H$ .*

*Proof.* Consider an arbitrary  $\alpha : H \rightarrow F$  and a finite set  $E \subseteq H$ . We have to show that there exists an internal mapping  $\beta : H \rightarrow F$  such that  $\alpha \upharpoonright E = \beta \upharpoonright E$ . By Proposition 4 the set  $E$  is internal. Then, by Theorem 1  $H \setminus E = G$  is internal. Pick up an arbitrary  $a \in F$  and put

$$\beta(g) = \begin{cases} \alpha(g), & g \in E \\ s, & g \notin E \end{cases}$$

Then  $\beta$  is internal. Indeed  $\beta = \alpha \upharpoonright E \cup G \times \{s\}$ . The set  $\alpha \upharpoonright E$  is internal since it is finite. The set  $G \times \{s\}$  is internal since it is a Cartesian product of two internal sets. The union of two internal sets is an internal set by Theorem 1. ■

Notice, that  $(F^H)^{int}$  can be naturally identified with  $\{F^{H_i}\}^\mu$  and thus is an internal set.

Let a cellular automaton  $\Lambda : F^H \rightarrow F^H$  be defined by formula (1), where  $D \subseteq S$  is a finite set and  $\Phi : F^D \rightarrow F$  is an arbitrary mapping. Since  $D$  and  $\Phi$  are finite, they are internal, i.e.  $D = \{D_i \subseteq H_i\}^\mu$ ,  $\Phi = \{\Phi_i : F^{D_i} \rightarrow F\}^\mu$ . Let  $\Lambda_i : F^{H_i} \rightarrow F^{H_i}$  be the cellular automata on  $L_i$  defined by  $\Phi_i$  according to formula (1). Then  $\Lambda \upharpoonright (F^H)^{int} = \{\Lambda_i\}^\mu$ .

We have to prove that if  $\Lambda$  is injective then it is surjective. If  $\Lambda$  is injective then  $\Lambda \upharpoonright (F^H)^{int}$  is injective and, thus, by Corollary 2  $\mu(\{i \mid \Lambda_i \text{ is injective}\}) =$

1. Since all  $L_i$  are  $S$ -systems, all cellular automata  $\Lambda_i$  are surjunctive. Hence,  $\mu(\{i \mid \Lambda_i \text{ is surjective}\}) = 1$  and by Theorem 1  $\Lambda \upharpoonright (F^H)^{int}((F^H)^{int}) = (F^H)^{int}$ . The surjunctivity of  $\Lambda$  follows now from the continuity of  $\Lambda$  (Proposition 2) and Lemma 3. ■

Let us say that a class  $\mathcal{K}$  of  $c$ . systems has a *subsystem surjunctivity property* (is an s.s.p. class) if any subsystem of a surjunctive  $\mathcal{K}$ -system is surjunctive.

**Theorem 4.** *Let an axiomatizable class  $\mathcal{K}$  of  $c$ . systems be an s.s.p. class. Then*

1. *any  $c$ . system approximable by surjunctive  $\mathcal{K}$ -systems is surjunctive;*
2. *any  $c$ . system approximable by finite  $\mathcal{K}$ -systems is surjunctive.*

This theorem follows immediately from Theorems 2, 3 and the obvious fact that any finite system is surjunctive. ■

Theorem 4 and Proposition 3 imply the following:

**Corollary 4.** *Any  $c$ . system (in particular, a group) approximable by surjunctive group-like  $c$ . systems (in particular, by  $S$ -groups) is surjunctive.*

This corollary shows that it is interesting to investigate the class of groups approximable by finite group-like systems, especially, its connection with the class of groups approximable by amenable groups that is surjunctive by Gromov's result [1]. The paper [11] contains some results about approximations of actions of amenable groups by actions of finite groups on finite sets.

Another interesting problem connected with the results of the section is following.

It is easy to see that any group is approximable by finite quasigroups (see [13]). Unlikely that the class of quasigroups is an s.s.p. class (if it were so then Theorem 4 would imply the surjunctivity of any group). However, it was shown in [14] that any group is approximable by some very special finite quasigroups that have a lot of group properties. So, it is interesting to investigate whether the class of quasigroups considered in [14] has subsystem surjunctivity property.

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