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ON A SYSTEM OF DIRAC-KLEIN-GORDON TYPE IN 1+1 DIMENSIONS

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Abstract. We establish a local and a global existence results for a Dirac-Klein-Gordon type system in 1+1 dimensions with a pseudoscalar bilinear form.

1. Introduction

In this paper, we consider the Cauchy problem for the type of Dirac-Klein-Gordon equations

$$\begin{cases} \mathcal{D}\psi = \phi\psi; \qquad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^1, \\ \Box\phi = \overline{\psi}\gamma^5\psi\,; \\ \psi(0) = \psi_0, \quad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1, \end{cases} \tag{1.1}$$

where the vector function ψ takes values in $\mathbb{C}^4$, the scalar function ϕ takes values in $\mathbb{R}^1$, the Dirac operator $\mathcal{D} := -i\gamma^\mu\partial_\mu$, $\mu = 0, 1, 2, 3$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and γ^μ are the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

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the wave operator $\Box = -\partial_{tt} + \partial_{xx}$, and $\overline{\psi} = \psi^\dagger \gamma^0$, where † is the complex conjugate transpose .

Chadam showed that the Cauchy problem for DKG equations has a global unique solution if $\psi_0 \in H^1$, $\phi_0 \in H^1$, and $\phi_1 \in L^2$ in 1973, see [4]. In 1993 ,Zheng proved that there exists a global weak solution with $\psi_0 \in L^2$, $\phi_0 \in H^1$, and $\phi_1 \in L^2$, see [9]. In 2000, Bournaveas gave a new proof of a global existence for the DKG equations, by using a null form estimate, if $\psi_0 \in L^2$, $\phi_0 \in H^1$, and $\phi_1 \in L^2$, see [3]. In 2003, Fang obtained a simple direct proof for the problem and the result is parallel to that of Bournaveas, see [5].

Our purpose of this paper is to demonstrate a proof of a variant null form estimate, see [5]. The motivation for studying this type of nonlinearity, $\overline{\psi}\gamma^5\psi$, comes from the fact that it is a bilinear covariant of Lorentz transformation, see [2]. Notice that the quadratic term can be rewritten in the following form:

$$\overline{\psi}\gamma^5\psi = 2\Im(\psi_1\overline{\psi_3} + \psi_2\overline{\psi_4}), \tag{1.2}$$

i.e., the imaginary part of $2(\psi_1\overline{\psi_3} + \psi_2\overline{\psi_4})$, where the ψ_j, $j = 1, 2, 3, 4$, are the components of the vector ψ. We give an interpretation of the null form structure different from that in [3]. The nonlinear term has the null form structure, see [6, 7]. Notice that the Dirac-Klein-Gordon equations in one space dimension can be decoupled into two similar subsystems, in other words, ψ can be taken as 2-spinors, instead of 4-spinors, see [4, 9].

We adopt the approach and ideas in [3, 5] and make necessary modification. First, we derive the conservation law of charge,

$$\int |\psi(t)|^2 dx = \text{constant}, \tag{1.3}$$

which can be applied to derive the global solution existence for the DKG-type equations. Next, we write down the direct solution representation and use it to estimate the nonlinear form $\overline{\psi}\gamma^5\psi$, and the derivations of some necessary estimates become straight forward. Finally we can prove the local and global existence results of DKG-type equations with data $\psi_0 \in L^2$, $\phi_0 \in H^1$, and $\phi_1 \in L^2$, which are called charge class solutions.

Theorem 1. (Global Existence) *If the initial data of* (1.1), $\psi_0 \in L^2, \phi_0 \in H^1, \phi_1 \in L^2$, *then there is a unique global solution* (ψ, ϕ) *for* (1.1) *and* $(\psi, \phi) \in C([0,\infty) \times L^2) \times \Big(C^1([0,\infty) \times H^1) \times C^0([0,\infty) \times L^2)\Big)$.

2. Solution Representation

Consider the Dirac equation,

(2.4)
$$\begin{cases} \mathcal{D}\psi = G; (t,x) \in \mathbb{R}^1 \times \mathbb{R}^1, \\ \psi(0) = \psi_0. \end{cases}$$

Using the equation

$$I_{4\times 4}\Box\psi = \mathcal{D}\mathcal{D}\psi = \mathcal{D}G,$$

the solution is

(2.5)
$$\begin{aligned} 2\psi(t,x) =& \Big[\psi_0(x+t) + \psi_0(x-t)\Big] + \int_{x-t}^{x+t} \gamma^1\gamma^0\partial_x\psi_0(y)dy + i\gamma^0\int_0^t G(s,x+t-s) \\ &+G(s,x-t+s)ds + i\gamma^1\int_0^t\int_{x-t+s}^{x+t-s}\partial_x G(s,y)dyds \\ =& (\gamma^0+\gamma^1)\gamma^0\psi_0(x+t) + (\gamma^0-\gamma^1)\gamma^0\psi_0(x-t) \\ &+i\int_0^t(\gamma^0+\gamma^1)G(s,x+t-s)ds + i\int_0^t(\gamma^0-\gamma^1)G(s,x-t+s)ds. \end{aligned}$$

Recall that , for the wave equation

(2.6)
$$\begin{cases} \Box\phi = F, (t,x) \in \mathbb{R}^1 \times \mathbb{R}^1, \\ \phi(0) = \phi_0, \partial_t(0) = \phi_1, \end{cases}$$

the solution of the equation is

(2.7)
$$\begin{aligned} 2\phi(t,x) = \phi_0(x-t) + \phi_0(x+t) + \int_{x-t}^{x+t}\phi_1(y)dy \\ + \int_0^t\int_{x-t+s}^{x+t-s} F(s,y)dyds. \end{aligned}$$

3. ESTIMATES

Lemma 1. *We have the law of conservation of charge*, *i.e.*,

$$\int |\psi(t)|^2 dt = \text{constant}.$$

Proof. The equation $\mathcal{D}\psi = \phi\psi$ implies

(3.8)
$$\psi_t + \gamma^0\gamma^1\psi_x = i\gamma^0\phi\psi.$$

Multiplying (3.8) by $\psi^\dagger$ and it's complex conjugate transpose by ψ, and then summing up, we get

$$\partial_t(|\psi|^2) + \partial_x(\psi^\dagger\gamma^0\gamma^1\psi) = 0.$$

This completes the proof. ∎

Lemma 2. *For the solution of the Dirac equation*, *we have*

$$||\psi(t)||_{L^2} \leqq C\Big(||\psi_0(t)||_{L^2} + \int_0^T ||G(s)||_{L^2} ds\Big). \tag{3.9}$$

This can be shown straightforward, using the solution representation (2.3), so we skip the proof.

Consider the Dirac equations

$$\begin{cases} \mathcal{D}\psi_j = G_j, \quad j = 1, 2 \\ \psi_j(0) = \psi_{0j}. \end{cases}$$

Lemma 3. (*Null Form Estimate*)

$$\begin{aligned} &||\overline{\psi_1}\gamma^5\psi_2||_{L^2([0,T),L^2)} \\ &\leqslant C\Big(||\psi_{01}||_{L^2} + \int_0^T ||G_1(s)||_{L^2} ds\Big)\Big(||\psi_{02}||_{L^2} + \int_0^T ||G_2(s)||_{L^2} ds\Big) \end{aligned} \tag{3.10}$$

Proof. For simplicity, we prove a special case when $\psi_1 = \psi_2$, and then the general case will follow. Consider the linear Dirac equation and write its solution as

$$2\psi(t,x) = U_+ + U_- + iV_+ + iV_-, \tag{3.11}$$

where

$$U_\pm(t,x) = (\gamma^0 \pm \gamma^1)\gamma^0\psi_0(x \pm t), \tag{3.12}$$

$$V_\pm(t,x) = \int_0^t (\gamma^0 \pm \gamma^1) G(s, x \pm (t-s)) ds. \tag{3.13}$$

Through some elementary calculations, we get

$$\overline{U}_\pm\gamma^5 U_\pm = \overline{V}_\pm\gamma^5 V_\pm = \overline{U}_\pm\gamma^5 V_\pm = \overline{V}_\pm\gamma^5 U_\pm = 0. \tag{3.14}$$

Thus

$$||\overline{U}_\pm\gamma^5 U_\mp||_{L^2([0,T),L^2)} \leq C||\psi_0||^2_{L^2}, \tag{3.15}$$

$$(3.16) \qquad ||\overline{V}_{\pm}\gamma^5 U_{\mp}||_{L^2([0,T),L^2)} \le C||\psi_0||_{L^2}\int_0^T ||G(s)||_{L^2}ds,$$

$$(3.17) \qquad ||\overline{U}_{\pm}\gamma^5 V_{\mp}||_{L^2([0,T),L^2)} \le C||\psi_0||_{L^2}\int_0^T ||G(s)||_{L^2}ds,$$

$$(3.18) \qquad ||\overline{V}_{\pm}\gamma^5 V_{\mp}||_{L^2([0,T),L^2)} \le C\Big(\int_o^T ||G(s)||_{L^2}ds\Big)^2.$$

The calculations for these cases are analogous. Among these cases, we only demonstrate the case of $\overline{U}_+\gamma^5 U_-$, $\overline{V}_+\gamma^5 U_-$, and $\overline{V}_+\gamma^5 V_-$. For convenience, we denote $\gamma = (\gamma^0-\gamma^1)\gamma^0\gamma^5(\gamma^0-\gamma^1)$. Since

$$\begin{aligned}||\overline{U}_+\gamma^5 U_-||_{L^2([0,T),L^2)} &= \Big(\int_0^T\int |\psi_0^\dagger(x+t)\gamma^0\gamma\gamma^0\psi_0(x-t)|^2dxdt\Big)^{\frac{1}{2}} \\ &\le C\Big(\int_0^T\int |\psi_0^\dagger(x+t)|^2|\psi_0(x-t)|^2dxdt\Big)^{\frac{1}{2}} = C||\psi_0||^2_{L^2}.\end{aligned}$$

If we use Minkovski inequality, we get

$$\begin{aligned}&||\overline{V}_+\gamma^5 U_-||_{L^2([0,T),L^2)} \\ &\quad= \Big(\int_0^T\int\Big|\int_0^t G^\dagger(s,x+t-s)\gamma\gamma^0\psi_0(x-t)ds\Big|^2dxdt\Big)^{\frac{1}{2}} \\ &\quad\le C\int_0^T\Big(\int_0^T\int |G^\dagger(s,x+t-s)|^2|\psi_0(x-t)|^2dxdt\Big)^{\frac{1}{2}}ds \\ &\quad\le C||\psi_0||_{L^2}\int_0^T ||G(s)||_{L^2}ds.\end{aligned}$$

Finally, for the nonhomogeneous term, we have

$$\begin{aligned}&||\overline{V}_+\gamma^5 V_-||_{L^2([0,T),L^2)} \\ &\quad= \Big(\int_0^T\int\Big(\int_0^t\int_0^t |G^\dagger(s,x+t-s)\gamma G(r,x-t+r)|drds\Big)^2dxdt\Big)^{\frac{1}{2}} \\ &\quad\le C\int_0^T\int_0^T\Big(\int_0^T\int |G^\dagger(s,x+t-s)|^2|G(r,x-t+r)|^2dxdt\Big)^{\frac{1}{2}}drds \\ &\quad\le C\int_0^T\int_0^T ||G(s)||_{L^2}||G(r)||_{L^2}drds \\ &\quad\le C\Big(\int_0^T ||G(s)||_{L^2}ds\Big)^2.\end{aligned}$$

This completes the proof of the lemma. ■

Lemma 4. *For the wave equation, we have the energy estimate*

$$||\phi(t)||_{H^1} + ||\phi_t(t)||_{L^2} \le C(T)\Big(||\phi_0||_{H^1} + ||\phi_1||_{L^2} + \int_0^T ||F(s)||_{L^2} ds\Big). \tag{3.19}$$

Proof. For the solution of wave equation (1.9), we have

$$2\phi(t,x) = \phi_0(x+t) + \phi_0(x-t) + \int_{x-t}^{x+t} \phi_1(y)dy + \int_0^t \int_{x-t+s}^{x+t-s} F(s,y)dyds.$$

Differentiating $\phi(t,x)$ with respect to t and x, respectively, give

$$\begin{aligned} 2\partial_x\phi(t,x) &= \partial_x\phi_0(x-t) + \partial_x\phi_0(x+t) + (\phi_1(x+t) - \phi_(x-t)) \\ &\quad + \int_0^t F(s,x+t-s) + F(s,x-t+s)ds \\ 2\partial_t\phi(t,x) &= \partial_t\phi_0(x-t) + \partial_t\phi_0(x+t) + \phi_1(x+t) + \phi_1(x-t) \\ &\quad + \int_0^t F(s,x+t-s) + F(s,x-t+s)ds). \end{aligned}$$

The above equations imply (3.19). ∎

4. Existence

Let (ψ, ϕ) and (ψ', ϕ') be two charge class solutions of the DKG-type equations. We define the following quantities:

$$J(0) = ||\psi_0||_{L^2} + ||\phi_0||_{H^1} + ||\phi_1||_{L^2} \tag{4.20}$$

$$J'(0) = ||\psi_0'||_{L^2} + ||\phi_0'||_{H^1} + ||\phi_1'||_{L^2} \tag{4.21}$$

$$J(T) = \sup_{[0,T)}(||\psi(t)||_{L^2} + ||\phi(t)||_{H^1} + ||\phi(t)||_{L^2}) \tag{4.22}$$

$$J'(T) = \sup_{[0,T)}(||\psi'(t)||_{L^2} + ||\phi'(t)||_{H^1} + ||\phi'(t)||_{L^2}) \tag{4.23}$$

$$\Delta(0) = ||\psi_0 - \psi_0'||_{L^2} + ||\phi_0 - \phi_0'||_{H^1} + ||\phi_1 - \phi_1'||_{L^2} \tag{4.24}$$

$$\Delta(T) = \sup_{[0,T)}(||\psi(t) - \psi'(t)||_{L^2} + ||\phi(t) - \phi'(t)||_{H^1} + ||\phi_t(t) - \phi_t'(t)||_{L^2}). \tag{4.25}$$

Lemma 5. *For the equations* (1.1), *we have*

$$||\phi(t)||_{L^\infty(\ )} \leq C(T, J(0)). \tag{4.26}$$

Proof. Write $\phi = \phi_L + \phi_N$ where ϕ_L is the solution of

$$\Box \phi_L = 0,\ \phi_L(0,x) = \phi_0,\ \partial_t \phi_L(0,x) = \phi_1,$$

and ϕ_N is a solution of

$$\Box \phi_N = \overline{\psi}\gamma^5\psi,\ \phi_N(0,x) = 0,\ \partial_t \phi_N(0,x) = 0.$$

Apply the standard energy estimate and the Sobolev inequality to get

$$\begin{aligned} ||\phi_L(t)||_{L^\infty(\ )} &\leq C||\phi_L(t)||_{H^1(\ )} \\ &\leq C(T)(||\phi_0||_{H^1(\ )} + ||\phi_1||_{L^2(\ )}) \leq C(T)J(0). \end{aligned}$$

We use the law of conservation of charge here to get

$$\begin{aligned} |\phi_N(t,x)| &\leq C\left| \int_0^t \int_{x-t+s}^{x+t-s} \overline{\psi}(s,y)\gamma^5\psi(s,y)dyds \right| \\ &\leq C\int_0^t \int_{-\infty}^{\infty} |\psi(s,y)|^2 dyds \\ &\leq C\int_0^T ||\psi(s)||^2_{L^2(\ )} ds \leq CT||\psi_0||^2_{L^2(\ )}. \end{aligned}$$

Since $\phi_N + \phi_L = \phi$, we have

$$||\phi(t)||_{L^\infty(\ )} \leq C(T, J(0)). \qquad \blacksquare$$

Lemma 6. *Let $T > 0$ and let (ψ, ϕ) be a charge class solution of the DKG equations. Then there exists a constant $C > 0$, depending only on T and $J(0)$, such that $J(T) \leq C(T, J(0))$.*

Proof. Since $||\psi(t,x)||_{L^2} = ||\psi(0,x)||_{L^2}$ and

$$||\phi(t)||_{H^1} + ||\phi_t(t)||_{L^2} \leq C(T)\left(||\phi_0||_{H^1} + ||\phi_1||_{L^2} + \int_0^T ||F(s)||_{L^2} ds \right),$$

we compute

$$\begin{aligned} \int_0^T ||F(s)||_{L^2} ds &= \int_0^T ||\overline{\psi}\gamma^5\psi(s)||_{L^2} ds \leq T^{\frac{1}{2}}||\overline{\psi}\gamma^5\psi||_{L^2([0,T],L^2)} \\ &\leq CT^{\frac{1}{2}}\left(||\psi_0||_{L^2} + \int_0^T ||\phi(s)\psi(s)||_{L^2} ds \right)^2 \\ &\leq T^{\frac{1}{2}}\left(J(0) + \int_0^T ||\phi(s)||_{L^\infty} ||\psi(s)||_{L^2} ds \right)^2 \\ &\leq C(T, J(0))T^{\frac{1}{2}}. \end{aligned} \tag{4.27}$$

Now we can get

$$\begin{aligned}||\phi(t)||_{H^1} + ||\phi_t(t)||_{L^2} &\leq C(T)\Big(J(0) + \int_0^T ||F(s)||_{L^2} ds\Big) \\ &\leq C(T, J(0)).\end{aligned} \tag{4.28}$$

This completes the proof of the lemma. ∎

Lemma 7. *Let $T > 0$ and (ψ, ϕ) and (ψ', ϕ') be two charge class solutions of the DKG equations. Then there exists a constant $\epsilon > 0, C > 0$, depending only on T and $J(0)$ and $J'(0)$, such that if $T \leq \epsilon$, then*

$$\Delta(T) \leq C\Delta(0). \tag{4.29}$$

Proof. Consider the difference of the two solutions, we have

$$\begin{aligned}\mathcal{D}(\psi - \psi') &= (\phi - \phi')\psi + \phi'(\psi - \psi') \\ \Box(\phi - \phi') &= \overline{\psi - \psi'}\gamma^5\psi + \overline{\psi'}\gamma^5(\psi - \psi').\end{aligned} \tag{4.30}$$

Recall that the quantity

$$\Delta(T) = \sup_{[0,T)}(||\psi(t) - \psi'(t)||_{L^2} + ||\phi(t) - \phi'(t)||_{H^1} + ||\phi_t(t) - \phi'_t||_{L^2})$$

We compute the first term of $\Delta(T)$. Since

$$||\mathcal{D}(\psi - \psi')(s)||_{L^2} \tag{4.31}$$

$$\leq ||\phi(s) - \phi'(s)||_{L^\infty}||\psi(s)||_{L^2} + ||\phi'(s)||_{L^\infty}||\psi(s) - \psi'(s)||_{L^2} \tag{4.32}$$

$$\leq ||\phi(s) - \phi'(s)||_{H^1}||\psi_0||_{L^2} + ||\phi'(s)||_{L^\infty}||\psi(s) - \psi'(s)||_{L^2} \tag{4.33}$$

$$\leq C(T, J(0), J'(0))\Delta(T), \tag{4.34}$$

we can get

$$\begin{aligned}||\psi(t) - \psi'(t)||_{L^2} &\leq C\Big(||\psi_0 - \psi'_0||_{L^2} + \int_0^T ||\mathcal{D}(\psi - \psi')(s)||_{L^2} ds\Big) \\ &\leq C(\Delta(0) + C(T, J(0), J'(0))T\Delta(T).\end{aligned} \tag{4.35}$$

For the other two terms, by invoking Lemma 3, we first calculate

(4.36)
$$||\overline{(\psi-\psi')}\gamma^5\psi||_{L^2} \le C\Big(||\psi_0-\psi_0'||_{L^2} + \int_0^T ||\mathcal{D}(\psi-\psi')(s)||_{L^2}ds\Big)\cdot \Big(||\psi_0||_{L^2} + \int_0^T ||\mathcal{D}(\psi)(s)||_{L^2}ds\Big),$$

and the calculation for $||\overline{\psi'}\gamma^5(\psi-\psi')||_{L^2}$ is the same. Then we compute

(4.37)
$$\begin{aligned}&\int_0^T ||\Box(\phi-\phi')(s)||_{L^2}ds\\ &\le \int_0^T ||\overline{(\psi-\psi')}\gamma^5\psi(s)||_{L^2} + ||\overline{\psi'}\gamma^5(\psi-\psi')(s)||_{L^2}ds\\ &\le T^{\frac12}(||\overline{(\psi-\psi')}\gamma^5\psi||_{L^2([0,T];L^2)} + ||\overline{\psi'}\gamma^5(\psi-\psi')||_{L^2([0,T];L^2)}).\end{aligned}$$

Therefore, applying (3.19), we get

(4.38)
$$\begin{aligned}&||\phi(t)-\phi'(t)||_{H^1} + ||\phi_t(t)-\phi_t'(t)||_{L^2}\\ &\le C(T)(||\phi_0-\phi_0'||_{H^1} + ||\phi_1-\phi_1'||_{L^2} + \textstyle\int_0^T ||\Box(\phi-\phi'(s))||_{L^2}ds)\\ &\le C(T)(\Delta(0) + T^{\frac12}C(T,J(0),J'(0))(\Delta(0)+T\Delta(T))\\ &\le C(T,J(0),J'(0))(\Delta(0)+T\Delta(T)).\end{aligned}$$

Now we assume $T<1$ to get

(4.39)
$$C(T,J(0),J'(0)) \le C(J(0),J'(0)) = C(1,J(0),J'(0))$$

and

(4.40)
$$\Delta(T) \le C(J(0),J'(0))(\Delta(0)+T\Delta(T)).$$

This concludes the proof. ■

Theorem 8. (Local Existence) *Let* $\psi_0 \in L^2(\mathbb{R}), \phi_0 \in H^1(\mathbb{R}), \phi_1 \in L^2(\mathbb{R})$ *then there exists a* $T>0$*,dependent only on* $J(0)$ *and a unique charge class solution of DKG-type equations defined on* $[0,T)\times\mathbb{R}$

Proof. Let (ψ^0,ϕ^0) be the solution of

(4.41)
$$\begin{cases}\mathcal{D}\psi = 0; \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^1\\ \Box\phi = 0;\\ \psi(0)=\psi_0,\ \phi(0)=\phi_0,\ \partial_t\phi(0)=\phi_1,\end{cases}$$

(ψ^1, ϕ^1) be the solution of

$$\text{(4.42)} \qquad \begin{cases} \mathcal{D}\psi = \phi^0\psi^0; \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^1 \\ \Box\phi = \overline{\psi^0}\gamma^5\psi^0; \\ \psi(0) = \psi_0, \quad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1, \end{cases}$$

and (ψ^{k+1}, ϕ^{k+1}) be the solution of

$$\text{(4.43)} \qquad \begin{cases} \mathcal{D}\psi = \phi^k\psi^k; \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^1 \\ \Box\phi = \overline{\psi^k}\gamma^5\psi^k; \\ \psi(0) = \psi_0, \quad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1, \end{cases}$$

where $k = 1, 2, 3, \cdots$. Thus we have

$$\begin{aligned} ||\psi^k(t) - \psi^{k-1}(t)||_{L^2} &\le T^{k-1}C(T, J(0)) + T^{k-2}C(T, J(0)), \quad k \in \mathbb{N}, \\ ||\phi^k(t) - \phi^{k-1}(t)||_{H^1} &\le T^{k-1}C(T, J(0)) + T^{k-2}C(T, J(0)), \quad k \in \mathbb{N}, \\ ||\partial_t\phi^k(t) - \partial_t\phi^{k-1}(t)||_{L^2} &\le T^{k-1}C(T, J(0)) + T^{k-2}C(T, J(0)), \quad k \in \mathbb{N}. \end{aligned}$$

So we can get, for $m > n$

$$\begin{aligned} &||\psi^m(t) - \psi^n(t)||_{L^2} + ||\phi^m(t) - \phi^n(t)||_{H^1} + ||\partial_t\phi^m(t) - \partial_t\phi^n(t)||_{L^2} \\ &\qquad \le \tfrac{T^{m-1}(1-T^{m-n})}{1-T}(C(T, J(0))). \end{aligned}$$

Since $T < 1$, we get $||\psi^m - \psi^n||_{L^2} \to 0$ as $m, n \to \infty$. We obtain that $\{\psi^k\}$ is a Cauchy sequence in L^2, thus its limiting function ψ is the solution. Similarly we can get $\{\phi^k\}$ to be a Cauchy sequence in H^1 and its limiting function ϕ is the solution. Thus if $0 < T < 1$, then there exists a solution (ψ, ϕ) for (1.1). Finally Lemma 7 gives us the uniqueness of the solution. ■

The existence of the global solution is ensured by the law of the conservation of the charge.

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