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BRAIDED CROSSED MODULES AND REDUCED SIMPLICIAL GROUPS

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Abstract. In this paper, we established an equivalence between the category of Braided Crossed Modules of Groups and the category of Simplicial Groups with Moore Complex of length 2.

1. Introduction

R. Brown and N.D. Gilbert introduced in [2] the notion of braided, regular crossed modules for an algebraic models of 3-types. They then showed that this structure is closely related to simplicial groups; they proved that the category of braided, regular crossed modules is equivalent to that of simplicial groups with Moore complex of length 2.

D. Counduché has shown in [6] that the category of simplicial groups with Moore complex of lenght 2 is also equivalent to that of 2-crossed modules. This gives a composite equivalence between the category of 2-crossed module and that of braided regular crossed modules. The related ideas of Counduché (cf.[6]) have been used by Carrasco and Cegarra [5], to study braided categorical groups. The categorical braiding is a commutator degenerate elements and the satisfaction of axioms for a braiding corresponding to the vanishing of certain commutators of intersections of face-map kernels.

In this paper we will concentrate on the reduced case: As crossed modules of groups is regular, we proved that the category of braided crossed modules of groups is equivalent to that of *reduced* simplicial groups with Moore complex of lenght 2 in terms of hypercrossed complex pairings $F_{\alpha,\beta}$ defined in [10].

Thus the braiding group can be given by using hypercrossed complex pairings $F_{\alpha,\beta}$ which gives products of commutators. This is reformulation of a result of Brown and Gilbert (cf. [2]). Our aim is to show the role of the $F_{\alpha,\beta}$ in the structure.

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2. Preliminaries

2.1. Truncated Simplicial Groups

Denoting the usual category of finite ordinals by Δ, we obtain for each $k \geq 0$, a subcategory $\Delta_{\leq k}$ determined by the objects [j] of Δ with $j \leq k$. A simplicial group is a functor from the opposite category Δ^{op} to the category of groups **Grp**. A *reduced* simplicial group is a simplicial group which last component is trivial. A k-truncated simplicial group is a functor from $\Delta^{op}_{\leq k}$ to **Grp**. We denote the category of simplicial groups by **SimpGrp** and the category of k-truncated simplicial groups by $\mathbf{Tr}_k$**SimpGrp** . By a *k-truncation of a simplicial group*, we mean a k-truncated simplicial group $\mathbf{tr}_k\mathbf{G}$ obtained by forgetting dimensions of order $>$ in a simplicial group **G.** This gives a truncation functor $\mathbf{tr}_k$: **SimpGrp**$\rightarrow\mathbf{Tr}_k$**SimpGrp** which admits a right adjoint $\mathbf{cosk}_k$:$\mathbf{Tr}_k$**SimpGrp** $\rightarrow$**SimpGrp** called the *k-coskeleton functor*, and a left adjoint $\mathbf{sk}_k$:$\mathbf{Tr}_k$**SimpGrp** $\rightarrow$**SimpGrp ,** called the *k-skeleton functor.* For explicit constructions of these see [7].

Recall that given a simplicial group **G,** *the Moore complex* (**NG**,∂) of **G** is the normal chain complex defined by

$$(\mathbf{NG})_n = \bigcap_{i=0}^{n-1} \ker d_i^n$$

with $\partial_n : NG_n \rightarrow NG_{n-1}$ induced from d_n^n by restriction. There is an alternative form of Moore complex given by the convention of taking

$$\bigcap_{i=1}^{n} \ker d_i^n$$

and using d_0 instead of d_n as the boundary. They lead to equivalent theories.

The nth *homotopy group* π_n(**G**) of **G** is the nth homology of the Moore complex of **G**, i.e.

$$\begin{aligned}\pi_n(\mathbf{G}) &\cong H_n(\mathbf{NG}, \partial) \\ &= \bigcap_{i=0}^{n} \ker d_i^n / d_{n+1}^{n+1}\Big(\bigcap_{i=0}^{n} \ker d_i^{n+1}\Big).\end{aligned}$$

We say that the Moore complex **NG** of a simplicial group is of *length k* if $NG_n = 1$ for all $n \geq k+1$, so that a Moore complex of length k is also of length l for $l \geq k$.

Corollary 2.1. *Let* $\mathbf{G}'$ *be* $(n-1)$*-truncated simplicial group. Then there is a simplicial group* **G** *with* $\mathbf{tr}_k\mathbf{G}\approxeq\mathbf{G}'$ *if and only if* $\mathbf{G}'$ *satisfies the following property*:

For all nonempty sets of indices $(I \neq J), I, J \subset [n-1]$ *with* $I \cup J = [n-1]$,

$$[\ \bigcap_{i \in I} \mathit{ker} d_i, \bigcap_{j \in J} \mathit{ker} d_j\ ] = 1.$$

2.2. Peiffer Pairings Generate

In the following we will define a normal subgroup N_n of G_n. First of all we adapt ideas from Carrasco [3, 4] to get the construction of a useful family of natural pairings. We define a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$, with respect to lexicographic ordering in $S(n)$ where $\alpha = (i_r, ...i_1), \beta = (j_s, ...j_1) \in S(n)$ The pairings that we will need,

$$\{F_{\alpha,\beta} : NG_{n-\sharp\alpha} \times NG_{n-\sharp\beta} \to NG_n : (\alpha, \beta) \in P(n), n \geq 0\}$$

are given as composites by the diagram

$$\begin{array}{ccc} NG_{n-\sharp\alpha} \times NG_{n-\sharp\beta} & \xrightarrow{F_{\alpha,\beta}} & NG_n \\ {\scriptstyle s_\alpha \times s_\beta}\downarrow & & \uparrow{\scriptstyle p} \\ G_n \times G_n & \xrightarrow[\mu]{} & G_n \end{array}$$

where

$s_\alpha = s_{i_r}, ...s_{i_1} : NG_{n-\sharp\alpha} \to G_n, \quad s_\beta = s_{j_s}, ...s_{j_1} : NG_{n-\sharp\beta} \to G_n,$
$p : G_n \to NG_n$ is defined by composite projections $p(x) = p_{n-1}...p_0(x)$,where

$$p_j(z) = z s_j d_j(z)^{-1} \text{ with } j = 0, 1, ..., n-1 \text{ and}$$

$\mu : G_n \times G_n \to G_n$ is given by commutator map and $\sharp\alpha$ is the number of the elements in the set of α, similarly for $\sharp\beta$. Thus

$$\begin{aligned} F_{\alpha,\beta}(x_\alpha, y_\beta) &= p\mu(s_\alpha \times s_\beta)(x_\alpha, y_\beta) \\ &= p[s_\alpha(x_\alpha), s_\beta(x_\beta)]. \end{aligned}$$

Definition 2.2. Let N_n or more exactly N_n^G be the normal subgroup of G_n generated by elements of the form

$$F_{\alpha,\beta}(x_\alpha, y_\beta),$$

where $x_\alpha \in NG_{n-\sharp\alpha}$ and $y_\beta \in NG_{n-\sharp\beta}$.

This normal subgroup N_n^G depends functorially on G, but we will usually abbreviate N_n^G to N_n, when no change of group is involved. We illustrate this normal subgroup for $n = 2$ and $n = 3$ to show what it looks like.

Example 2.3. For $n = 2$, assume that $\alpha = (1)$ $\beta = (0)$ and $x, y \in NG_1 = \ker d_0$. It follows that

$$\begin{aligned} F_{(1)(0)}(x, y) &= p_1 p_0([s_0 x, s_1 y]) \\ &= p_1[s_0 x, s_1 y] \\ &= [s_0 x, s_1 y][s_1 y, s_1 x]. \end{aligned}$$

is a generating element of the normal subgroup N_2.

For $n = 3$, the possible pairings are the following;

$$\begin{array}{lll} F_{(1,0)(2)}, & F_{(2,0)(1)}, & F_{(0)(2,1)}, \\ F_{(0)(2)}, & F_{(1)(2)}, & F_{(0)(1)}. \end{array}$$

For all $x_1 \in NG_1, y_2 \in NG_2$, the corresponding generators of N_3 are:

$$\begin{aligned} F_{(1,0)(2)}(x_1, y_2) &= [s_1 s_0 x_1, s_2 y_2][s_2 y_2, s_2 s_0 x_1], \\ F_{(2,0)(1)}(x_1, y_2) &= [s_2 s_0 x_1, s_1 y_2][s_1 y_2, s_2 s_1 x_1][s_2 s_1 x_1, s_2 y_2][s_2 y_2, s_2 s_0 x_1] \end{aligned}$$

and for all $x_2 \in NG_2, y_1 \in NG_1$,

$$F_{(0)(2,1)}(x_2, y_1) = [s_0 x_2, s_2 s_1 y_1][s_2 s_1 y_1, s_1 x_2][s_2 x_2, s_2 s_1 y_1]$$

whilst for all $x_2, y_2 \in NG_2$,

$$\begin{aligned} F_{(0)(1)}(x_2, y_2) &= [s_0 x_2, s_1 y_2][s_1 y_2, s_1 x_2][s_2 x_2, s_2 y_2], \\ F_{(0)(2)}(x_2, y_2) &= [s_0 x_2, s_2 y_2], \\ F_{(1)(2)}(x_2, y_2) &= [s_1 x_2, s_2 y_2][s_2 y_2, s_2 x_2]. \end{aligned}$$

3. Braided Crossed Module of Groups

In this section we will show that the descriptions of two equivalent categories: The category of braided crossed modules and the category of simplicial group with Moore complex length 2.

Crossed modules were initially defined by Whitehead in [11] as models for 2-types. A crossed module (M, P, ∂) is a group homomorphism $\partial : M \to P$, together with an action of P on M written m^p for $p \in P$ and $m \in M$, satisfying the following conditions: for all $m, m' \in M, p \in P$,

CM1) : $\partial(m^p) = p^{-1}(\partial m)p$

CM2) : $m^{\partial m'} = m'^{-1} m m'$.

The second condition is called *Peiffer identity*.

A braided crossed modules of group(oid)s were initially defined by Brown and Gilbert in [2].

Definition 3.1. A braided crossed module of groups

$$C_2 \xrightarrow{\delta} C_1$$

is a crossed module with the braiding function $\{,\} : C_1 \times C_1 \to C_2$ satisfying the following axioms:

BC1: $\{x, yy'\} = \{x, y\}^{y'}\{x, y'\}$
BC2: $\{xx', y\} = \{x', y\}\{x, y\}^{x'}$
BC3: $\delta\{x, y\} = [y, x]$
BC4: $\{x, \delta a\} = a^{-1}a^x$
BC5: $\{\delta b, y\} = (b^{-1})^y b$

where $x, x'y, y' \in C_1$ and $a, b \in C_2$.

Proposition 3.2. *Let **G** be a reduced simplicial group with Moore complex **NG**. Then the complex of groups*

$$NG_2/\partial_3(NG_3 \cap D_3) \xrightarrow{\partial_2} NG_1$$

is a braided crossed module of groups. The braiding map can be defined as follows:

$$\begin{array}{rcl} \{\ \ ,\ \ \} : NG_1 \times NG_1 & \longrightarrow & NG_2/\partial_3(NG_3 \cap D_3) \\ (x, y) & \longmapsto & \overline{s_0x^{-1}s_1y^{-1}s_0xs_1x^{-1}s_1ys_1x}. \end{array}$$

Here the right hand side denotes a coset in $NG_2/\partial_3(NG_3 \cap D_3)$ represented by an element in NG_2

The two actions of NG_1 on $NG_2/\partial_3(NG_3 \cap D_3)$ are given by

1. $l^{\partial_1 m}$ corresponds to the action $s_0\ (m)^{-1}l\ s_0(m)$ and conjugation.
2. l^m corresponds to the action $s_1(m)^{-1}ls_1(m)$.

Proof. This is a reformulation of a result of Brown and Gilbert [2]. Our aim is to show the role of the $F_{\alpha,\beta}$ in the structure. We will show that all axioms of a braided crossed module are verified. It is plainly that the morphism $\partial_2 : NG_2/(\partial_3 NG_3 \cap D_3) \longrightarrow NG_1$ is a crossed module of groups. Since every crossed module of groups is regular this construction is regular.

In the following calculations we display the elements omitting the overlines as:

BC1**:** For $x_i \in NG_1$,$(i = 0, 1, 2.)$

$$\begin{aligned}
\{x_0, x_1x_2\} &= s_0x_0^{-1}s_1x_2^{-1}s_1x_1^{-1}s_0x_0s_1x_0^{-1}s_1x_1s_1x_2s_1x_0 \\
&= (s_0x_0^{-1}s_1x_2^{-1}s_1x_1^{-1}s_0x_0s_1x_0^{-1}s_1x_1)(s_1x_0s_0x_0^{-1}s_1x_2s_0x_0) \\
&\quad (s_0x_0^{-1}s_1x_2^{-1}s_0x_0s_1x_0^{-1})s_1x_2s_1x_0 \\
&= (s_0x_0^{-1}s_1x_2^{-1}s_1x_1^{-1}s_0x_0s_1x_0^{-1}s_1x_1s_1x_0s_0x_0^{-1}s_1x_2s_0x_0)\{x_0, x_2\} \\
&= s_0x_0^{-1}s_1x_2^{-1}(s_0x_0s_0x_0^{-1})(s_1x_1^{-1}s_0x_0s_1x_0^{-1}s_1x_1s_1x_0)s_0x_0^{-1}s_1x_2s_0x_0\{x_0, x_2\} \\
&= (s_0x_0^{-1}s_1x_2^{-1}s_0x_0)(s_0x_0^{-1}s_1x_1^{-1}s_0x_0s_1x_0^{-1}s_1x_1s_1x_0)s_0x_0^{-1}s_1x_2s_0x_0\{x_0, x_2\} \\
&= s_0x_0^{-1}s_1x_2^{-1}s_0x_0(\{x_0, x_1\})s_0x_0^{-1}s_1x_2s_0x_0\{x_0, x_2\} \\
&= (s_1x_2^{-1})^{\partial_1x_0}(\{x_0, x_1\})(s_1x_2)^{\partial_1x_0})(\{x_0, x_2\}) \\
&= (s_1x_2^{-1}\{x_0, x_1\}s_1x_2)\{x_0, x_2\} \quad \text{(since } \partial_1\text{=identity)} \\
&= \{x_0, x_1\}^{x_2}\{x_0, x_2\}. \quad \text{(by the (2) action.)}
\end{aligned}$$

BC2**:** For $y_i \in NG_1$,$(i = 0, 1, 2.)$

$$\begin{aligned}
\{y_0y_1, y_2\} &= s_0y_1^{-1}s_0y_0^{-1}s^1y_2^{-1}s_0y_0s_0y_1s_1y_1^{-1}s_1y_0^{-1}s_1y_2s_1y_0s_1y_1 \\
&= s_0y_1^{-1}s_0y_0^{-1}s^1y_2^{-1}s_0y_0s_0y_1s_1y_1^{-1}(s_0y_0^{-1}s_1y_2s_0y_0) \\
&\quad (s_0y_0^{-1}s_1y_2^{-1}s_0y_0)s_1y_0^{-1}s_1y_2s_1y_0s_1y_1. \\
&= s_0y_1^{-1}s_0y_0^{-1}s^1y_2^{-1}s_0y_0s_0y_1s_1y_1^{-1}s_0y_0^{-1}s_1y_2s_0y_0 \\
&\quad (s_0y_0^{-1}s_1y_2^{-1}s_0y_0s_1y_0^{-1}s_1y_2s_1y_0)s_1y_1. \\
&= s_0y_1^{-1}s_0y_0^{-1}s^1y_2^{-1}s_0y_0s_0y_1s_1y_1^{-1}s_0y_0^{-1}s_1y_2s_0y_0 \\
&\quad s_1y_1(s_1y_1^{-1}\{y_0, y_2\}s_1y_1) \\
&= s_0y_1^{-1}(s_1y_2^{-1})^{\partial_1y_0}s_0y_1s_1y_1^{-1}(s_1y_2)^{\partial_1y_0}s_1y_1\{y_0, y_2\}^{y_1} \quad \text{(since the action.)} \\
&= s_0y_1^{-1}s_1y_2^{-1}s_0y_1s_1y_1^{-1}s_1y_2s_1y_1\{y_0, y_2\}^{y_1} \quad \text{(since } \partial_1\text{=identity.)} \\
&= \{y_1, y_2\}\{y_0, y_2\}^{y_1}.
\end{aligned}$$

BC3**:**

$$\begin{aligned}
\overline{\partial}_2\{x, y\} &= d_2(s_0x^{-1}s_1y^{-1}s_0xs_1x^{-1}s_1ys_1x) \\
&= s_0d_1(x^{-1})y^{-1}s_0d_1(x)x^{-1}yx \\
&= (y^{-1})^{\partial_1x}x^{-1}yx \quad \text{(since by action)} \\
&= y^{-1}x^{-1}yx \quad \text{(since } \partial_1\text{=identity)} \\
&= [y, x]
\end{aligned}$$

BC4**:**

For $\alpha = (2, 0)$,$\beta = (1)$ and $x \in NG_2$,$y \in NG_1$ from,

$\partial_3(F_{(2,0)(1)}(y, x)) = [s_0y, s_1d_2x][s_1d_2x, s_1y][s_1y, x][x, s_0y] \in \partial_3(NG_3 \cap D_3)$,

$$\begin{aligned}\{y,\overline{\partial}_2(x)\} &= [s_0y,s_1d_2x][s_1d_2x,s_1y]\\ &\equiv [s_0y,x][x,s_1y] \mod \partial_3(\mathrm{NG}_3\cap \mathrm{D}_3)\\ &= s_0y^{-1}x^{-1}s_0yxx^{-1}s_1y^{-1}xs_1y\\ &= s_0y^{-1}x^{-1}s_0ys_1y^{-1}xs_1y\\ &\equiv s_1s_0d_0y^{-1}x^{-1}s_1s_0d_0ys_1y^{-1}xs_1y \mod \partial_3(\mathrm{NG}_3\cap \mathrm{D}_3)\\ &= (x^{-1})^{\partial_1 y}x^y\\ &= x^{-1}x^y. \text{ (since } \partial_1\text{=identity)}\end{aligned}$$

BC5**:**
For $\alpha=(0)$,$\beta=(2,1)$ and $x\in NG_2$,$y\in NG_1$ from

$$\partial_3(F_{(0)(2,1)}(x,y))=[s_0d_2x,s_1y][s_1y,s_1d_2x][x,s_1y]\in\partial_3(NG_3\cap D_3),$$

$$\begin{aligned}\{\overline{\partial}_2(x),y\} &= s_0d_2x^{-1}s_1y^{-1}s_0d_2xs_1d_2x^{-1}s_1ys_1d_2x\\ &= [s_0d_2x,s_1y][s_1y,s_1d_2x]\\ &\equiv [s_1y,x] \mod \partial_3(\mathrm{NG}_3\cap \mathrm{D}_3)\\ &= s_1y^{-1}x^{-1}s_1yx\\ &= (x^{-1})^yx.\end{aligned}$$

Therefore we show that all axioms of braided crossed module are verified. ■

Theorem 3.3. *The category of braided crossed modules of groups is equivalent to the category of reduced simplicial groups with Moore complex of length 2.*

Proof. Let **G** be a simplicial group with Moore complex of length 2. In the previous proposition, a braided crossed module

$$NG_2\xrightarrow{\partial_2}NG_1$$

has already been constructed. This defines a functor from **SimpGrp** the category of simplicial groups to **BCM** the category of braided crossed modules

$$\theta:\mathbf{SimpGrp}\to\mathbf{BCM}.$$

Conversely we suppose that $L\xrightarrow{\partial}M$ is a braided crossed module of groups. Let $1_M\in M$ be identity element of the group M . Define $G_0=\{1_M\}$ and $G_1=M$. Then $\{G_0,G_1\}$ is 1-truncated simplicial group with trivial homomorphisms.

Since $L \xrightarrow{\partial} M$ is a crossed module (by using an action of M on L) we can create the semidirect product $L \rtimes M$ with

$$(l,m)(l',m') = (l^m l', mm').$$

with

$$^{m}l' = l'\{m^{-1}, \partial(l')\}.$$

where $\{,\}$ is braiding map and $m, m' \in M$, $l, l' \in L$.
An action of $m \in M$ on $(l, m') \in L \rtimes M$ is given by

$$^{m}(l,m') = (l\{m^{-1}, \partial(l)\}, mm'm^{-1}).$$

Then by using this action of M on $L \rtimes M$, we can define $G_2 = (L \rtimes M) \rtimes M$. A multiplication on G_2 is given by

$$(l, m_1, m_2).(l', m_1', m_2') = (ll'\{m_2^{-1}m_1^{-1}, \partial(l')\}, m_1m_2m_1'm_2^{-1}, m_2m_2')$$

We will show that G_2 is a group. It is clear that $(1_L, 1_M, 1_M)$ is identity element of G_2, because

$$\begin{aligned}(l, m_1, m_2).(1_L, 1_M, 1_M) &= (l1_M\{m_2^{-1}m_1^{-1}, \partial(1_L)\}, m_1m_2 1_M m_2^{-1}, m_2 1_M)\\ &= (l\{m_2^{-1}m_1^{-1}, 1_M\}, m_1, m_2)\\ &= (l1_L, m_1, m_2)\\ &= (l, m_1, m_2).\end{aligned}$$

The invers element of $(l, m_1, m_2) \in G_2$ is given by

$$(l, m_1, m_2)^{-1} = (l^{-1}\{m_1m_2, \partial(l^{-1})\}, m_2^{-1}m_1^{-1}m_2, m_2^{-1})$$

Indeed,

$$\begin{aligned}(l, m_1, m_2).(l, m_1, m_2)^{-1} &= (l, m_1, m_2)(l^{-1}\{m_1m_2, \partial(l^{-1})\}, m_2^{-1}m_1^{-1}m_2, m_2^{-1})\\ &= (ll^{-1}\{m_1m_2, \partial(l^{-1})\}\{m_2^{-1}m_1^{-1}, \partial(l^{-1}\{m_1m_2, \partial(l^{-1})\})\},\\ &\quad m_1m_2m_2^{-1}m_1^{-1}m_2m_2^{-1}, m_2m_2^{-1})\\ &= (l(^{m_2^{-1}m_1^{-1}}l^{-1})(^{m_1m_2}l)(^{m_1m_2}(^{m_2^{-1}m_1^{-1}}l^{-1})), 1_M, 1_M)\\ &= (ll^{-1}, 1_M, 1_M)\\ &= (1_L, 1_M, 1_M).\end{aligned}$$

It is clear that the multiplication on G_2 is associative. Indeed, for

$$\begin{aligned}x_0 &= (l, m_1, m_2)\\ x_1 &= (l', m_1', m_2')\\ x_2 &= (l'', m_1'', m_2''),\end{aligned}$$

$$\begin{aligned} x_0(x_1x_2) &= (l,m_1,m_2).[(l',m_1',m_2').(l'',m_1'',m_2'')] \\ &= (l,m_1,m_2).(l'({}^{m_1'm_2'}l''),m_1'm_2'm_1''(m_2')^{-1},m_2'm_2'') \\ &= (l({}^{m_1m_2}(l'{}^{m_1'm_2'}l'')),m_1m_2(m_1'm_2'm_1''(m_2')^{-1})m_2^{-1},m_2m_2'm_2'') \\ &= (l({}^{m_1m_2}l')({}^{m_1m_2m_1'm_2'}l''),m_1m_2m_1'm_2'm_1''(m_2')^{-1}m_2^{-1},m_2m_2'm_2'') \end{aligned}$$

and

$$\begin{aligned} (x_0x_1)x_2 &= [(l,m_1,m_2).(l',m_1',m_2')](l'',m_1'',m_2'') \\ &= [(l({}^{m_1m_2}l'),m_1m_2m_1'm_2^{-1},m_2m_2')].(l'',m_1'',m_2'') \\ &= [(l({}^{m_1m_2}l')({}^{m_1m_2m_1'm_2^{-1}m_2m_2}l''),m_1m_2m_1'm_2^{-1}m_2m_2'm_1'' \\ &\quad (m_2m_2')^{-1},m_2m_2'm_2'')] \\ &= (l({}^{m_1m_2}l')({}^{m_1m_2m_1'm_2'}l''),m_1m_2m_1'm_2'm_1''(m_2')^{-1}m_2^{-1},m_2m_2'm_2''). \end{aligned}$$

Therefore we have

$$(x_0x_1)x_2 = x_0(x_1x_2).$$

We have homomorphisms

$$\begin{aligned} d_0^2(l,m_1,m_2) &= m_1 \\ d_1^2(l,m_1,m_2) &= m_1m_2 \\ d_2^2(l,m_1,m_2) &= m_2 \\ s_0^1(m_1) &= (1_L,m_1,1_M) \\ s_1^1(m_1) &= (1_L,1_M,m_1) \end{aligned}$$

These maps satisfy the simplicial identities.

Thus we have a 2-truncated simplicial group

$$\{G_0,G_1,G_2\}.$$

There is a **cosk**$_2$ functor from the category of 2- truncated simplicial groups to that of simplicial groups.

We have that $\partial_3(NG_3\cap D_3)$ is the product $[\ker d_2,\ker d_0\cap\ker d_1]$, $[\ker d_1,\ker d_0\cap\ker d_2]$, $[\ker d_1\cap\ker d_2,\ker d_0]$, $[\ker d_0\cap\ker d_2,\ker d_0\cap\ker d_1]$, $[\ker d_0\cap\ker d_1,\ker d_1\cap\ker d_2]$ and

$[\ker d_0\cap\ker d_1,\ker d_1\cap\ker d_2]$. Now we show that $\partial_3(NG_3)=$identity. For this we use the functions $F_{\alpha,\beta}$.

Step 1 : For $\alpha = (1,0)$, $\beta = (2)$ and $x = m \in NG_1, y \in$ker$d_0\cap$kerd_1, where $y = (l, 1_M, 1_M)$. Then

$$\begin{aligned}
\partial_3(F_{(1,0)(2)}(x,y)) &= [s_1s_0d_1m, (l,1_M,1_M)][(l,1_M,1_M), s_0m] \\
&= (l,1_M,1_M)(l,m,1_M)(l^{-1},1_M,1_M)(l,m^{-1},1_M) \\
&= (l,m,1_M)(l^{-1},m^{-1},1_M) \\
&= (l^m l^{-1}, 1_M, 1_M) \in [\ker d_2, \ker d_0 \cap \ker d_1].
\end{aligned}$$

Step 2 : For $\alpha = (2,0)$, $\beta = (1)$ and $x = m \in NG_1, y \in$ker$d_0\cap$kerd_1, where $y = (l, 1_M, 1_M)$. Then

$$\begin{aligned}
\partial_3(F_{(2,0)(1)}(x,y)) &= [s_0m, s_1d_2(l,1_M,1_M)][s_1d_2(l,1_M,1_M), s_1m] \\
&\quad\ [s_1m, (l,1_M,1_M)][(l,1_M,1_M), s_0m] \\
&= ({}^m l, 1_M, m)(1_L, m, m^{-1})(l^{-1}, m^{-1}, 1_M) \\
&= ({}^m l, 1_M, m)(l^{-1}, 1_M, m^{-1}) \\
&= (1_L, 1_M, 1_M) \in [\ker d_1, \ker d_0 \cap \ker d_2].
\end{aligned}$$

Step 3 : For $\alpha = (2,1), \beta = (0)$ and $x = m \in NG_1, y = (l, 1_M, 1_M) \in NG_2$. Then

$$\begin{aligned}
\partial_3(F_{(2,1)(0)}(x,y)) &= [s_1m, s_0d_2(l,1_M,1_M)][s_1d_2(l,1_M,1_M), s_1m][s_1m, (l,1_M,1_M)] \\
&= (1_L,1_M,m)(l,1_M,1_M)(1_L,1_M,m^{-1})(l^{-1},1_M,1_M) \\
&= ({}^m l, 1_M, m)({}^{m^{-1}} l^{-1}, 1_M, m^{-1}) \\
&= ({}^m l l^{-1}, 1_M, 1_M) \in [\ker d_1 \cap \ker d_2, \ker d_0].
\end{aligned}$$

Step 4 : For $\alpha = (2), \beta = (1)$,let $x = (l', 1_M, 1_M), y = (l, 1_M, 1_M)$ be any elements of NG_2. Then

$$\begin{aligned}
\partial_3(F_{(2)(1)}(x,y)) &= [x, s_1d_2y][y,x] \\
&= (l,1_M,1_M)(l',1_M,1_M)(l^{-1},1_M,1_M)(l'^{-1},1_M,1_M) \\
&= (ll'l^{-1}l'^{-1}, 1_M, 1_M) \in [\ker d_0 \cap \ker d_2, \ker d_0 \cap \ker d_1].
\end{aligned}$$

Step 5 : For $\alpha = (2), \beta = (0)$,let $x = (l', 1_M, 1_M), y = (l, 1_M, 1_M)$ be any elements of NG_2. Then

$$\begin{aligned}
\partial_3(F_{(2)(0)}(x,y)) &= [x, s_0d_2y] \\
&= (l',1_M,1_M)(l'^{-1},1_M,1_M) \\
&= (1_L,1_M,1_M) \in [\ker d_0 \cap \ker d_1, \ker d_1 \cap \ker d_2].
\end{aligned}$$

Step 6 : For $\alpha = (1), \beta = (0)$,let $x = (l', 1_M, 1_M), y = (l, 1_M, 1_M)$ be any elements of NG_2. Then

$$\begin{aligned}\partial_3(F_{(1)(0)}(x,y)) &= [s_1d_2x, s_0d_2y][s_1d_2y, s_1d_2x][x,y] \\ &= (l', 1_M, 1_M)(l, 1_M, 1_M)(l'^{-1}, 1_M, 1_M)(l^{-1}, 1_M, 1_M) \\ &= (l'll'^{-1}l^{-1}, 1_M, 1_M) \in [\ker d_0 \cap \ker d_1, \ker d_1 \cap \ker d_2].\end{aligned}$$

Since

$$\begin{aligned}\partial_3(NG_3) = &[\ker d_2, \ker d_0 \cap \ker d_1][\ker d_1, \ker d_0 \cap \ker d_2] \\ &[\ker d_1 \cap \ker d_2, \ker d_0][\ker d_0 \cap \ker d_2, \ker d_0 \cap \ker d_1] \\ &[\ker d_0 \cap \ker d_2, \ker d_1 \cap \ker d_2][\ker d_0 \cap \ker d_1, \ker d_1 \cap \ker d_2].\end{aligned}$$

Therefore for $\tau \in \partial_3(NG_3)$,

$$\begin{aligned}\tau &= (l^m l^{-1}, 1_M, 1_M)(1_L, 1_M, 1_M)({}^m l l^{-1}, 1_M, 1_M) \\ &= (ll'l^{-1}l'^{-1}, 1_M, 1_M)(1_L, 1_M, 1_M)(l'll'^{-1}l^{-1}, 1_M, 1_M) \\ &= (l^m(l^{-1})^m l l^{-1}, 1_M, 1_M)(ll'l^{-1}l'^{-1}l'll'^{-1}l^{-1}, 1_M, 1_M) \\ &= (1_L, 1_M, 1_M).\end{aligned}$$

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