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ON A HARDY-CARLEMAN'S TYPE INEQUALITY

Bicheng Yang

Abstract. In this paper, we prove that the constant factor in the Hardy-Carleman's type inequality is the best possible. A related integral inequality with a best constant factor is considered.

1. Introduction

If $p > 1$, $a_n \geq 0\,(n \in N)$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$, then the famous Hardy's inequality is

$$\sum_{n=1}^{\infty} (\frac{1}{n}\sum_{k=1}^{n} a_k^{1/p})^p < (\frac{p}{p-1})^p \sum_{n=1}^{\infty} a_n, \tag{1.1}$$

where the constant factor $\left(\frac{p}{p-1}\right)^p$ is the best possible, and (1.1) may be reduced to the following Carleman's inequality when $p \to \infty$ (see [1, Ch. 9.12]):

$$\sum_{n=1}^{\infty} (a_1 a_2 \ldots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n, \tag{1.2}$$

where the constant factor e is still the best possible, and the left-hand side of (1.2) is related to the sum of geometric average. Inequalities (1.1) and (1.2) are important in analysis and its applications (see [2]).

Recently, we have proved the following two distinct strengthened versions of (1.2) (see Yang et al. [3, 4]):

$$\sum_{n=1}^{\infty} (a_1 a_2 \ldots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left[1 - \frac{1}{2(n+1)}\right] a_n; \tag{1.3}$$

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$$\sum_{n=1}^{\infty}(a_1a_2\dots a_n)^{1/n} < e\sum_{n=1}^{\infty}\big(1-\frac{1-2/e}{n}\big)a_n. \tag{1.4}$$

Another strengthened version of (1.2) was built by [5].

If we set $p = 1/r$ in (1.1), equivalently, we have $0 < r < 1$ and

$$\sum_{n=1}^{\infty}\big(\frac{1}{n}\sum_{k=1}^{n}a_k^r\big)^{1/r} < \big(\frac{1}{1-r}\big)^{1/r}\sum_{n=1}^{\infty}a_n, \tag{1.5}$$

where the constant factor $\big(\frac{1}{1-r}\big)^{1/r}$ is the best possible, and the left-hand sides of (1.5) is related to the sum of generalized arithmetic average. Thanh et al. [6] prove that (1.5) is still true for $r \in [-1,0),$ and for $r \in (-\infty,-1),$ it follows from [6] that

$$\sum_{n=1}^{\infty}\big(\frac{1}{n}\sum_{k=1}^{n}a_k^r\big)^{1/r} < \frac{2^{(r-1)/r}r}{r-1}\sum_{n=1}^{\infty}a_n. \tag{1.6}$$

In particular, for $r = -1,$ we reduce (1.5) as

$$\sum_{n=1}^{\infty}\frac{n}{\sum_{k=1}^{n}a_k^{-1}} < 2\sum_{n=1}^{\infty}a_n. \tag{1.7}$$

More recently, Thanh et al. [7] gave a strengthened version of (1.7) as:

$$\sum_{n=1}^{\infty}\frac{n}{\sum_{k=1}^{n}a_k^{-1}} < 2\sum_{n=1}^{\infty}\big(1-\frac{1}{3n+1}\big)a_n. \tag{1.8}$$

For $r \in [-1,0)$ in (1.5), replace r by -r and equivalently we have

$$\sum_{n=1}^{\infty}\big(\frac{n}{\sum_{k=1}^{n}a_k^{-r}}\big)^{1/r} < (1+r)^{1/r}\sum_{n=1}^{\infty}a_n \;\; (r \in (0,1]). \tag{1.9}$$

And for $r \in (-\infty,-1)$ in (1.6), still replace r by -r, and equivalently we have

$$\sum_{n=1}^{\infty}\big(\frac{n}{\sum_{k=1}^{n}a_k^{-r}}\big)^{1/r} < \frac{2^{(1+r)/r}r}{1+r}\sum_{n=1}^{\infty}a_n \;\; (r \in (1,\infty)). \tag{1.10}$$

Both of the left-hand sides of (1.9) and (1.10) are related to the sum of the generalized Harmonic average. We call (1.9) and (1.10) the Hardy-Carleman's type inequalities.

The main objective of this paper is to prove that the constant factor in (1.9) is the best possible. For $r > 0$, a related integral inequality of (1.9) and (1.10) with a best constant factor is considered.

2. Lemma and Main Result

Lemma 2.1. *If* $o_n = o(1)\ (n \to \infty)$, *then we have*

$$\frac{\sum_{n=1}^{N} \frac{o_n}{n}}{\sum_{n=1}^{N} \frac{1}{n}} = o(1)\ (N \to \infty). \tag{2.1}$$

Proof. For any $\varepsilon > 0$, there exists $N_0 > 1$, such that for any $n > N_0$, $|o_n| < \varepsilon/2$. Setting $M = max\{|o_1|, |o_2|, \cdots, |o_{N_0}|\}$, since we find

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N_0} \frac{M}{n}}{\sum_{n=1}^{N} \frac{1}{n}} = 0,$$

there exists $N_1 > N_0$, such that for any $N > N_1$,

$$\frac{\sum_{n=1}^{N_0} \frac{M}{n}}{\sum_{n=1}^{N} \frac{1}{n}} < \frac{\varepsilon}{2}.$$

Then for any $N > N_1$, that makes

$$\begin{aligned} |\frac{\sum_{n=1}^{N} \frac{o_n}{n}}{\sum_{n=1}^{N} \frac{1}{n}}| &\le \frac{\sum_{n=1}^{N} \frac{|o_n|}{n}}{\sum_{n=1}^{N} \frac{1}{n}} = \frac{\sum_{n=1}^{N_0} \frac{|o_n|}{n} + \sum_{n=N_0+1}^{N} \frac{|o_n|}{n}}{\sum_{n=1}^{N} \frac{1}{n}} \\ &< \frac{\sum_{n=1}^{N_0} \frac{M}{n} + \frac{\varepsilon}{2}\sum_{n=N_0+1}^{N} \frac{1}{n}}{\sum_{n=1}^{N} \frac{1}{n}} < \frac{\sum_{n=1}^{N_0} \frac{M}{n}}{\sum_{n=1}^{N} \frac{1}{n}} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Hence we have (2.1). The lemma is proved.

Theorem 2.1. *If* $0 < r \le 1$, $a_n \ge 0\ (n \in N)$, *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$, *then the constant factor* $(1+r)^{1/r}$ *in (1.9) is the best possible.*

Proof. We set $\tilde{a}_n$ as

$$\tilde{a}_n = \frac{1}{n},\ for\ n = 1, 2, \cdots, N;\ \tilde{a}_n = 0,\ for\ n = N+1, N+2, \cdots.$$

If the constant factor $(1+r)^{1/r}$ in (1.9) is not the best possible, then there exists $r \in (0, 1]$, and a positive number K, with $K < (1+r)^{1/r}$, such that (1.9) is still valid if we replace $(1+r)^{1/r}$ by K. In particular, we have

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sum_{k=1}^{n} \tilde{a}_k^{-r}}\right)^{1/r} < K \sum_{n=1}^{\infty} \tilde{a}_n, \tag{2.2}$$

and then

$$\sum_{n=1}^{N}\Big(\frac{n}{\sum_{k=1}^{n}k^r}\Big)^{1/r} < K\sum_{n=1}^{N}\frac{1}{n}. \tag{2.3}$$

Since for $r \in (0,1]$, we have

$$\sum_{k=1}^{n}k^r < \int_0^{n+1}x^r dx = \frac{1}{r+1}(n+1)^{r+1},$$

then we find

$$\sum_{n=1}^{N}\Big(\frac{n}{\sum_{k=1}^{n}k^r}\Big)^{1/r} > (1+r)^{1/r}\sum_{n=1}^{N}\frac{n^{1/r}}{(n+1)^{1+1/r}}$$
$$= (1+r)^{1/r}\sum_{n=1}^{N}\frac{1}{n}\Big(1+\frac{1}{n}\Big)^{-1-1/r} = (1+r)^{1/r}\sum_{n=1}^{N}\frac{1+o_n}{n},$$

where $o_n = \big(1+\frac{1}{n}\big)^{-1-1/r} - 1 = o(1)\ (n\to\infty)$. Hence by (2.3), we find

$$K > (1+r)^{1/r}\Big(1+\frac{\sum_{n=1}^{N}\frac{o_n}{n}}{\sum_{n=1}^{N}\frac{1}{n}}\Big),$$

and for $N\to\infty$ by (2.1), it follows that $K \geq (1+r)^{1/r}$. This contracts the fact that $K < (1+r)^{1/r}$. Hence the constant factor $(1+r)^{1/r}$ in (1.9) is the best possible. The theorem is proved.

3. A Related Integral Inequality

To show a related integral inequality of (1.9) and (1.10), we need the following $H\ddot{o}lder's$ inequality:

If $p<0, \frac{1}{p}+\frac{1}{q}=1,\ f(t), g(t)\geq 0$, and $f\in L^p(E),\ g\in L^q(E)$, then (see [8, p.29])

$$\int_E f(t)g(t)dt \geq \Big(\int_E f^p(t)dt\Big)^{1/p}\Big(\int_E g^q(t)dt\Big)^{1/q}, \tag{3.1}$$

where the equality holds only if there exist real numbers a and b, such that $a^2+b^2 > 0$,

$$af^p(t) = bg^q(t),\ a.e.\ in\ E.$$

Theorem 3.1. *If* $r > 0,\ f(x) \geq 0\ (x \in (0,\infty))$, *and* $0 < \int_0^\infty f(x)dx < \infty$, *then we have*

$$\int_0^\infty \big(\frac{x}{\int_0^x f^{-r}(t)dt}\big)^{1/r} dx < (1+r)^{1/r} \int_0^\infty f(x)dx, \tag{3.2}$$

where the constant factor $(1+r)^{1/r}$ *is the best possible.*

Proof. For $x > 0$, setting $p = -\frac{1}{r}$ and $E = (0,x)$ in (3.1), we find $r > 0$, and

$$\big(\int_0^x f(t)g(t)dt\big)^{-1/r} \leq \big(\int_0^x f^{-1/r}(t)dt\big)\big(\int_0^x g^{1/(1+r)}(t)dt\big)^{-(1+r)/r}. \tag{3.3}$$

Hence by (3.3), we have

$$\begin{aligned}\big(\int_0^x f^{-r}(t)dt\big)^{-1/r} &= \big(\int_0^x (t^{1+r}f(t))^{-r}(t^{(1+r)r})dt\big)^{-1/r} \\ &\leq \big(\int_0^x t^{1+r}f(t)dt\big)\big(\int_0^x t^r dt\big)^{-(1+r)/r} \\ &= (r+1)^{(1+r)/r}x^{-(1+r)^2/r}\int_0^x t^{1+r}f(t)dt.\end{aligned} \tag{3.4}$$

If for any $x > 0$, (3.4) takes the form of equality, setting $x \to \infty$, then by (3.1), there exists real numbers a and b, such that $a^2 + b^2 > 0$,

$$at^{1+r}f(t) = bt^r,\ a.e.\ in\ (0,\infty).$$

It follows that $af(t) = bt^{-1},\ a.e.\ in\,(0,\infty)$. Since $a^2+b^2 > 0$, we have $a \neq 0$, and then $f(t) = \frac{b}{a}t^{-1},\ a.e.\ in\,(0,\infty)$. It contracts the face that $0 < \int_0^\infty f(x)dx < \infty$. Hence, there exists $x_0 > 0$, such that (3.4) takes the form of strict inequality . Then for $x > x_0$, we have

$$\begin{aligned}&\big(\frac{x}{\int_0^x f^{-r}(t)dt}\big)^{1/r} < (r+1)^{(1+r)/r}x^{-r-2}\int_0^x t^{1+r}f(t)dt,\ and \\ &\qquad\int_0^\infty \big(\frac{x}{\int_0^x f^{-r}(t)dt}\big)^{1/r}dx < (r+1)^{(1+r)/r}\int_0^\infty x^{-r-2}\int_0^x t^{1+r}f(t)dtdx \\ &= (r+1)^{(1+r)/r}\int_0^\infty\big(\int_t^\infty x^{-r-2}dx\big)t^{1+r}f(t)dt = (1+r)^{1/r}\int_0^\infty f(t)dt,\end{aligned}$$

which shows (3.2).

For $0 < \varepsilon < 1$, we set $f_\varepsilon(x)$ as

$$f_\varepsilon(x) = x^{\varepsilon-1},\ for\ x \in (0,1]; f_\varepsilon(x) = 0,\ for\ x \in (1,\infty).$$

If there exists $r > 0$, such that the constant factor $(r+1)^{1/r}$ in (3.2) is not the best possible, then there exists a positive number k,with $k < (r+1)^{1/r}$, such that (3.2) is still valid if we replace $(r+1)^{1/r}$ by k. In particular, we have

$$\int_0^\infty \left(\frac{x}{\int_0^x f_\varepsilon^{-r}(t)dt}\right)^{1/r} dx < k \int_0^\infty f_\varepsilon(x)dx, \ and$$

$$\int_0^1 \left(\frac{x}{\int_0^x t^{(1-\varepsilon)r}dt}\right)^{1/r} dx < k \int_0^1 x^{\varepsilon-1}dx = \frac{k}{\varepsilon}.$$

We obtain that $[r(1-\varepsilon)+1]^{1/r} < k$, and for $\varepsilon \to 0^+$, it follows that $(r+1)^{1/r} \le k$. This contradicts the fact that $k < (r+1)^{1/r}$. Hence the constant factor $(r+1)^{1/r}$ in (3.2) is the best possible. The theorem is proved.

Remark 3.2. (a) It is interesting that the left-hand sides of inequalities (1.2), (1.5), and (1.9) are related to three different kinds of the sum of averages.

(b) We still can't show that the constant factor in (1.10) is the best possible or not, even if we find that for $r > 1$ the related integral inequality of (1.10) is still (3.2).

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Bicheng Yang
Department of Mathematics,
Guangdong Education College,
Guangzhou,
Guangdong 510303,
People's Republic of China.
E-mail: bcyang@pub.guangzhou.gd.cn