

INTERPOLATION OF WEIGHTED ℓ^q SEQUENCES BY H^p FUNCTIONS

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Abstract. Let (z_n) be a sequence of points in the open unit disc D and $\rho_n = \prod_{m \neq n} |(z_n - z_m)(1 - \bar{z}_m z_n)^{-1}| > 0$. Let $a = (a_j)_{j=1}^\infty$ be a sequence of positive numbers and $\ell^s(a) = \{(w_j); (a_j w_j) \in \ell^s\}$ where $1 \leq s \leq \infty$. When $1 \leq p \leq \infty$ and $1/p + 1/q = 1$, we show that $\{(f(z_n)); f \in H^p\} \supset \ell^s(a)$ if and only if there exists a finite positive constant γ such that $\{\sum_{n=1}^\infty (a_n \rho_n)^{-t} (1 - |z_n|^2)^t |f(z_n)|^t\}^{1/t} \leq \gamma \|f\|_q$ ($f \in H^q$), where $1/s + 1/t = 1$. As results, we show that $\{(f(z_j)); f \in H^p\} \supset \ell^1(a)$ if and only if $\sup_n (a_n \rho_n)^{-1} (1 - |z_n|^2)^{1/p} < \infty$, and $\{(f(z_n)); f \in H^1\} \supset \ell^\infty(a)$ if and only if $\sum_n (a_n \rho_n)^{-1} (1 - |z_n|^2) \delta_{z_n}$ is finite measure on D . These are also proved in the case of weighted Hardy spaces.

1. INTRODUCTION

Let H^p ($0 < p \leq \infty$) denote the usual Hardy space in the open unit disc D . In this paper, we assume that a sequence (z_j) in D satisfies that $\sum_{j=1}^\infty (1 - |z_j|) < \infty$, that is, there exists a Blaschke product

$$B(z) = \prod_{j=1}^\infty \frac{\bar{z}_j}{|z_j|} \frac{z - z_j}{1 - \bar{z}_j z}.$$

Let

$$\rho_{k,n} = \prod_{\substack{j=1 \\ j \neq k}}^n \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right|, \quad 1 \leq k \leq n,$$

$$\rho_k = \prod_{\substack{j=1 \\ j \neq k}}^\infty \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right|.$$

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Then $\rho_{k,n} \geq \rho_{k,n+1}$ and $\lim_{n \rightarrow \infty} \rho_{k,n} = \rho_k$ for $k \geq 1$. We assume that $\rho_k > 0$ for $k = 1, 2, \dots$.

For a positive sequence $a = (a_j)$, $\ell^s(a)$ denotes $\{(w_j); w_j \in \mathcal{C} \text{ and } \sum_{j=1}^{\infty} (a_j |w_j|)^s < \infty\}$ and $\ell^\infty(a)$ denotes $\{(w_j); w_j \in \mathcal{C} \text{ and } \sup_{1 \leq j < \infty} a_j |w_j| < \infty\}$. In this paper, we study the following problem : Find a necessary and sufficient condition on (z_j) so that $\{(f(z_j)); f \in H^p\} \supset \ell^s(a)$ where $1 \leq p \leq \infty$ and $1 \leq s \leq \infty$.

Suppose $a_j = 1$ for all $j \geq 1$. When $p = s = \infty$, this was solved by L. Carleson [1]. That is, $\{(f(z_j)); f \in H^\infty\} \supset \ell^\infty$ if and only if $\inf_j \rho_j > 0$. (z_j) is called a uniformly separated sequence when $\inf_j \rho_j > 0$. When $p = \infty$ and $1 \leq s < \infty$, A. K. Snyder [13] (cf. [7],[11]) proved that $\{(f(z_j)); f \in H^\infty\} \supset \ell^s$ if and only if $\inf_j \rho_j > 0$. A. K. Snyder [13] and P. L. Duren and H. S. Shapiro [3] showed that there exists a sequence (z_j) which is not uniformly separated, that is, $\inf_j \rho_j = 0$ and has the property : $\{(f(z_j)); f \in H^p\} \supset \ell^\infty$ when $p \neq \infty$. B. A. Taylor and D. L. Williams [14] showed that for $1 \leq p \leq \infty$ $\{(f(z_j)); f \in H^p\} \supset \ell^\infty$ if and only if there exists a positive finite constant γ such that $\sum_{j=1}^{\infty} \frac{1}{\rho_j} (1 - |z_j|^2) |g(z_j)| \leq \gamma \|g\|_q$ for all g in H^q and $1/p + 1/q = 1$.

Suppose $1 \leq p = s \leq \infty$. When $a_j = (1 - |z_j|^2)^{1/p}$ for all $j \geq 1$, this was solved by H. S. Shapiro and A. L. Shields [11]. That is, $\{(f(z_j)); f \in H^p\} \supset \ell^p(a)$ if and only if $\inf_j \rho_j > 0$. When $a_j = \rho_j^2$ for all $j \geq 1$, J. P. Earl [4] showed that $\{(f(z_j)); f \in H^\infty\}$ contains $\ell^\infty(a)$ always. This was pointed out by A. M. Gleason (see [6]). On the other hand, when $a_j = \rho_j$ for all $j \geq 1$, T. Nakazi [10] showed that $\{(f(z_j)); f \in H^\infty\} \supset \ell^\infty(a)$ if and only if (z_j) is the union of a finite number of uniformly separated sequences. For a general weight $a = (a_j)$, J. D. McPhail [9] gave a necessary and sufficient condition about (z_j) that $\{(f(z_j)); f \in H^p\} \supset \ell^p(a)$. In fact, he studied such a problem in weighted Hardy spaces.

In §2, we give a necessary and sufficient condition about (z_j) for that $\{(f(z_j)); f \in H^p\} \supset \ell^s(a)$ where $1 \leq p \leq \infty$, $1 \leq s \leq \infty$ and $a = (a_j)$ is arbitrary weight. As a result, we show that $\{(f(z_j)); f \in H^1\} \supset \ell^s(a)$ if and only if $\sum_{j=1}^{\infty} (a_j \rho_j)^{-t} (1 - |z_j|^2)^t < \infty$ where $1/s + 1/t = 1$. Moreover, when $1 < p \leq \infty$ and $a = (\rho_j^{-1})$, we show that $\{(f(z_j)); f \in H^p\} \supset \ell^p(a)$ if and only if (z_j) is a finite sum of uniformly separated sequences. This is a generalization of a result in [10] for $p = \infty$.

In §3, when $1 \leq p \leq \infty$, we show that $\{(f(z_j)); f \in H^p\} \supset \ell^1(a)$ if and only if $\sup_j (a_j \rho_j)^{-1} (1 - |z_j|^2)^{1/p} < \infty$. As a result, a theorem of A. K. Snyder [13] follows, that is, $\{(f(z_j)); f \in H^\infty\} \supset \ell^s$ if and only if $\inf_j \rho_j > 0$.

In §4, we give a necessary and sufficient condition about (z_j) for that $\{(f(z_j)); f \in H^p\} \supset \ell^\infty(a)$. Put $\mu = \sum_{j=1}^{\infty} (a_j \rho_j)^{-1} (1 - |z_j|^2) \delta_{z_j}$. Then $\{(f(z_j)); f \in H^1\} \supset \ell^\infty(a)$ if and only if μ is a finite measure on D , and $\{(f(z_j)); f \in H^\infty\} \supset \ell^\infty(a)$ if and only if μ is a Carleson measure on D .

In §5, we give a necessary and sufficient condition about (z_j) for that $\{(s(z_j)f(z_j)); f \in H^p(W)\} \supset \ell^p$, where $H^p(W)$ is a weighted Hardy space and $s(z_j) = \inf\{\int |f|^p W d\theta/2\pi; f(z_j) = 1\}$. We assume only that $\log W$ is in L^1 . J. D. McPhail [9] studied such a problem when W satisfies the (A_p) -condition of Muckenhoupt.

Our interests in this paper are in the differences between interpolations for $\ell^1(a)$ and $\ell^\infty(a)$ and in the interpolation problems for weighted Hardy spaces. For example, it is very easy to prove that $\{(f(z_j)); f \in H^\infty\} \supset \ell^1$ if and only if $\{z_j\}$ is uniformly separated.

2. GENERAL RESULTS

In this section, we obtain a general result for interpolation problems for $\ell^s(a)$ ($1 \leq s \leq \infty$) by H^p ($1 \leq p \leq \infty$). For $1 \leq j \leq n$, let

$$B_n(z) = \prod_{j=1}^n \frac{z - z_j}{1 - \bar{z}_j z} \text{ and } B_{nj}(z) = B_n(z) \frac{1 - \bar{z}_j z}{z - z_j}.$$

If we put $b_{nj} = B_{nj}(z_j)$, then

$$\rho_{j,n} = |b_{nj}| \quad (1 \leq j \leq n).$$

Suppose for $n = 1, 2, \dots$

$$f_n(z) = \sum_{j=1}^n b_{nj}^{-1} w_j B_{nj}(z).$$

Then f_n is in H^∞ and $f_n(z_j) = w_j$ ($1 \leq j \leq n$). Lemma 1 is essentially known.

Lemma 1. *Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. Suppose w_j is a complex number for $j = 1, 2, \dots$. There exists a function f in H^p such that $f(z_j) = w_j$ for $j = 1, 2, \dots$ if and only if there exists a positive finite constant γ such that for any $n \geq 1$ and for all g in H^q ,*

$$\left| \sum_{j=1}^n \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \leq \gamma \|g\|_q.$$

Proof. Put for $n \geq 1$

$$m_{p,n}(w) = \inf\{\|f_n + B_n h\|_p; h \in H^p\}.$$

Then by [2, p. 142],

$$m_{p,n}(w) = \sup \left\{ \left| \sum_{j=1}^n \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| ; g \in H^q \text{ and } \|g\|_q \leq 1 \right\}.$$

There exists a function f in H^p such that $f(z_j) = w_j$ for $j = 1, 2, \dots$ if and only if $\sup_n m_{p,n}(w) < \infty$ because the unit ball of H^p is compact in the weak topology or the weak $*$ topology. This implies the lemma.

Theorem 1. *Let $1 \leq p \leq \infty$ and $1 \leq s \leq \infty$. $\{(f(z_n)); f \in H^p\} \supset \ell^s(a)$ if and only if there exists a finite positive constant γ such that*

$$\left\{ \sum_{n=1}^{\infty} (a_n \rho_n)^{-t} (1 - |z_n|^2)^t |f(z_n)|^t \right\}^{1/t} \leq \gamma \|f\|_q$$

for f in H^q , where $1/p + 1/q = 1$ and $1/s + 1/t = 1$.

Proof. For the ‘only if’ part, since $\{(f(z_j)); f \in H^p\} \supset \ell^s(a)$, by Lemma 1 there exists a positive finite constant γ such that for any $n \geq 1$

$$\sup_{\substack{w \in \ell^s(a) \\ \|w\| \leq 1}} \left| \sum_{j=1}^n \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \leq \gamma \|g\|_q \quad (g \in H^q)$$

where $w = (w_j)$ and $\|w\| = \left(\sum_{j=1}^{\infty} |w_j a_j|^s \right)^{1/s}$. Hence for any $n \geq 1$

$$\left\{ \sum_{j=1}^{\infty} (a_j \rho_{n,j})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \right\}^{1/t} \leq \gamma \|g\|_q \quad (g \in H^q).$$

Assuming $\|g\|_q = 1$,

$$\sum_{j=1}^{\infty} (a_j \rho_{n,j})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \leq \gamma^t.$$

For any $\varepsilon > 0$, there exists a positive integer n_j for each j such that for all $n \geq n_j$

$$(a_j \rho_{n,j})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t - \frac{\varepsilon}{2^j} \leq (a_j \rho_{n,j})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t$$

because $\rho_{j,n} \geq \rho_{j,n+1}$ and $\lim_{n \rightarrow \infty} \rho_{j,n} = \rho_j$. Thus, $\{(f(z_j)); f \in H^p\} \supset \ell^s(a)$ if and only if for any $\varepsilon > 0$ and any $n \geq \max(n_1, \dots, n_n)$

$$\sum_{j=1}^n (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t - \varepsilon \leq \sum_{j=1}^n (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \leq \gamma^t$$

This implies the ‘only if’ part.

For the ‘if’ part, by Lemma 1 it is sufficient to show that there exists a finite positive constant γ such that for all $n \geq 1$

$$\sup_{\substack{w \in \ell^s(a) \\ \|w\| \leq 1}} \sup_{\|g\|_q \leq 1} \left| \sum_{j=1}^n \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \leq \gamma < \infty.$$

In fact, for all $n \geq 1$

$$\begin{aligned} & \sup_{\substack{w \in \ell^s(a) \\ \|w\| \leq 1}} \sup_{\|g\|_q \leq 1} \left| \sum_{j=1}^n \frac{w_j}{b_{nj}} (1 - |z_j|^2) g(z_j) \right| \\ & \leq \sup_{\|g\|_q \leq 1} \left\{ \sum_{j=1}^n (a_j \rho_{j,n})^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \right\}^{1/t} \\ & \leq \sup_{\|g\|_q \leq 1} \left\{ \sum_{j=1}^{\infty} (a_j \rho_j)^{-t} (1 - |z_j|^2)^t |g(z_j)|^t \right\}^{1/t} < \infty \end{aligned}$$

Corollary 1. Let $1 \leq s \leq \infty$. $\{(f(z_n)); f \in H^1\} \supset \ell^s(a)$ if and only if

$$\sum_{n=1}^{\infty} (a_n \rho_n)^{-t} (1 - |z_n|^2)^t < \infty$$

where $1/s + 1/t = 1$. Hence, when $a = (a_n) = (\rho_n^{-1})$ it is always true that $\{(f(z_n)); f \in H^1\} \supset \ell^s(a)$.

Proof. The first part is clear by Theorem 1. When $a = (\rho_n^{-1})$, $\{(f(z_n)); f \in H^1\} \supset \ell^s(a)$ if and only if $\sum_{n=1}^{\infty} (1 - |z_n|^2)^t < \infty$. This implies the second part.

Corollary 2. Let $1 \leq p \leq \infty$, $1 \leq s \leq \infty$ and $a = (\rho_n^{-1})$. $\{(f(z_n)); f \in H^p\} \supset \ell^s(a)$ if and only if there exists a finite positive constant γ such that

$$\left\{ \sum_{n=1}^{\infty} (1 - |z_n|^2)^t |f(z_n)|^t \right\}^{1/t} \leq \gamma \|f\|_q$$

for f in H^q , where $1/p + 1/q = 1$ and $1/s + 1/t = 1$. When $1 < p \leq \infty$, $\{(f(z_n)); f \in H^p\} \supset \ell^p(a)$ if and only if (z_n) is a finite sum of uniformly separated sequences.

Proof. The first part is clear by Theorem 1. The second part follows from the first one and [8].

In Corollary 2, when $1 < p \leq \infty$ and $1 < s \leq \infty$ and $s > p$, if $\{(f(z_n)); f \in H^p\} \supset \ell^s(a)$ then (z_n) is a finite sum of uniformly separated sequences but the converse is not true. When $s < p$, if (z_n) is a finite sum of uniformly separated sequences then $\{(f(z_n)); f \in H^p\} \supset \ell^s(a)$ but the converse is not true.

3. INTERPOLATIONS FOR $\ell^1(a)$

$\ell^1(a)$ is the smallest sequence space among $\ell^p(a)$ ($1 \leq p \leq \infty$) for the same $a = \{a_j\}$. Then the interpolations for $\ell^1(a)$ are very special as the following shows.

The case of $p = \infty$ in Corollary 3 was proved by A. Snyder [13] (see [7], [11]). Corollary 4 is due to O. Hatori [7].

Theorem 2. Let $1 \leq p \leq \infty$. $\{(f(z_n)); f \in H^p\} \supset \ell^1(a)$ if and only if

$$\sup_n (a_n \rho_n)^{-1} (1 - |z_n|^2)^{1/p} < \infty.$$

Proof. By Theorem 1, $\{(f(z_n)); f \in H^p\} \supset \ell^1(a)$ if and only if there exists a finite positive constant γ such that

$$\sup_n (a_n \rho_n)^{-1} (1 - |z_n|^2) |f(z_n)| \leq \gamma \|f\|_q$$

for all f in H^q . For each n , $\sup_{\|f\|_q=1} |f(z_n)| = (1 - |z_n|^2)^{-1/q}$ by [2, p144] and so the theorem follows.

Corollary 3. Let $1 \leq p \leq \infty$. $\{(f(z_n)); f \in H^p\} \supset \ell^1$ if and only if $\sup_n \frac{1}{\rho_n} (1 - |z_n|^2)^{1/p} < \infty$. Hence if $p = \infty$, $\{(f(z_n)); f \in H^\infty\} \supset \ell^1$ if and only if $\inf_n \rho_n > 0$.

Corollary 4. Let $1 \leq p \leq \infty$. $\{((1 - |z_n|^2)^{1/p} f(z_n)); f \in H^p\} \supset \ell^1$ if and only if $\inf_n \rho_n > 0$.

Proof. Note that $\{((1 - |z_n|^2)^{1/p} f(z_n)); f \in H^p\} \supset \ell^1$ if and only if $\{(f(z_n)); f \in H^p\} \supset \ell^1(a)$ and $a = ((1 - |z_n|^2)^{1/p})$.

Corollary 5. Let $1 \leq p \leq \infty$. For any (z_n) , $\{(f(z_n)); f \in H^p\} \supset \ell^1(a)$ where $a = (\rho_n^{-1})$.

Let (b_j) be a uniformly separated sequence in D such that $0 < \operatorname{Re} b_j \nearrow 1$ and $\operatorname{Im} b_j \searrow 0$. For $j \geq 1$, put $z_{2j-1} = b_j$ and $z_{2j} = \bar{b}_j$. Let B be the Blaschke product associated with $\{z_n\}$. Then for each j

$$B = \frac{z - b_j}{1 - \bar{b}_j z} \frac{z - \bar{b}_j}{1 - b_j z} B_{1j} B_{2j}$$

where B_{1j} (or B_{2j}) is a Blaschke product with zeros $\{b_\ell\}_{\ell \neq j}$ (or $\{\bar{b}_\ell\}_{\ell \neq j}$). Then

$$\rho_{2j-1} = \left| \frac{b_j - \bar{b}_j}{1 - \bar{b}_j b_j} \right| \prod_{\ell \neq j} \left| \frac{b_j - b_\ell}{1 - \bar{b}_\ell b_j} \right| \prod_{\ell \neq j} \left| \frac{b_j - \bar{b}_\ell}{1 - b_\ell b_j} \right|$$

and

$$\rho_{2j} = \left| \frac{\bar{b}_j - b_j}{1 - \bar{b}_j b_j} \right| \prod_{\ell \neq j} \left| \frac{\bar{b}_j - b_\ell}{1 - \bar{b}_\ell \bar{b}_j} \right| \prod_{\ell \neq j} \left| \frac{\bar{b}_j - \bar{b}_\ell}{1 - b_\ell \bar{b}_j} \right|.$$

Hence $\rho_{2j-1} = \rho_{2j}$ for $j \geq 1$ and

$$\delta^2 \frac{|\bar{b}_j - b_j|}{1 - |b_j|^2} \leq \rho_{2j} = \rho_{2j-1} \leq \frac{|\bar{b}_j - b_j|}{1 - |b_j|^2} \quad (j \geq 1)$$

where

$$0 < \delta = \min \left\{ \inf_j \prod_{\ell \neq j} \left| \frac{b_j - b_\ell}{1 - \bar{b}_\ell b_j} \right|, \inf_j \prod_{\ell \neq j} \left| \frac{b_j - \bar{b}_\ell}{1 - b_\ell b_j} \right| \right\}.$$

Hence

$$\frac{(1 - |z_n|^2)^{1+1/p}}{|z_n - \bar{z}_n|} \leq \frac{(1 - |z_n|^2)^{1/p}}{\rho_n} \leq \delta^{-2} \frac{(1 - |z_n|^2)^{1+1/p}}{|z_n - \bar{z}_n|}.$$

Thus $\{(f(z_n)); f \in H^p\} \supset \ell^1$ if and only if $\sup_n (1 - |z_n|^2)^{1+1/p} / |z_n - \bar{z}_n| < \infty$.

4. INTERPOLATIONS FOR $\ell^\infty(a)$

$\ell^\infty(a)$ is the largest sequence space among $\ell^p(a)$ ($1 \leq p \leq \infty$) for the same $a = (a_j)$. Then the interpolations for $\ell^\infty(a)$ are special as the following shows. The case of $p = \infty$ of Corollary 6 is known in [10].

Theorem 3. *Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$, $\{(f(z_n)); f \in H^p\} \supset \ell^\infty(a)$ if and only if there exists a finite positive constant γ such that*

$$\sum_n (a_n \rho_n)^{-1} (1 - |z_n|^2) |f(z_n)| \leq \gamma \|f\|_q$$

for all f in H^q . When $p = 1$, $\{(f(z_n)); f \in H^1\} \supset \ell^\infty(a)$ if and only if $\mu = \sum_n (a_n \rho_n)^{-1} (1 - |z_n|^2) \delta_{z_n}$ is a finite measure on D . When $p = \infty$, $\{(f(z_n)); f \in H^\infty\} \supset \ell^\infty(a)$ if and only if $\mu = \sum_n (a_n \rho_n)^{-1} (1 - |z_n|^2) \delta_{z_n}$ is a Carleson measure on D .

Corollary 6. *Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$ and $a = (\rho_n^{-1}) \cdot \{(f(z_n)); f \in H^p\} \supset \ell^\infty(a)$ if and only if there exists a finite positive constant γ such that*

$$\sum_n (1 - |z_n|^2) |f(z_n)| \leq \gamma \|f\|_q$$

for all f in H^q .

- (1) When $p = 1$, for any (z_n) , $\{(f(z_n)); f \in H^1\} \supset \ell^\infty(a)$.
- (2) When $p = \infty$, $\{(f(z_n)); f \in H^\infty\} \supset \ell^\infty(a)$ if and only if (z_n) is a finite union of uniformly separated sequences.
- (3) When $1 < p < \infty$, there exists a sequence (z_n) in D such that $\{(f(z_n)); f \in H^p\} \supset \ell^\infty(a)$ and (z_n) is not a union of finitely many uniformly separated sequences. If $\sum_{n=1}^\infty (1 - |z_n|^2)^{1/p} < \infty$, then $\{(f(z_n)); f \in H^p\} \supset \ell^\infty(a)$.

Suppose that (z_n) is the sequence in D which was used in the end of Section 3, and $1 \leq p < \infty$. If $0 < \gamma_1 \leq \frac{(1 - |z_n|^2)^{1+1/p-\varepsilon}}{|z_n - \bar{z}_n|} \leq \gamma_2 < \infty$ for some $0 < \varepsilon < 1/p$, then $\{(f(z_n)); f \in H^p\} \supset \ell^\infty$. This was proved by B. A. Taylor and D. L. Williams [14].

5. WEIGHTED HARDY SPACE

Let W be a nonnegative function in L^1 with $\log W \in L^1$ and $1 \leq p < \infty$. $H^p(W)$ denotes the closure of the set of all analytic polynomials in $L^p(W) = L^p(W d\theta/2\pi)$. $H^p(W)$ is called a weighted Hardy space. For $b \in D$, put

$$s(b) = s(b, p, W) = \inf \left\{ \int |f|^p W d\theta/2\pi; f(b) = 1 \right\}.$$

Let h be an outer function in H^p such that $|h|^p = W$.

Lemma 2. *For $1 \leq p < \infty$ and $b \in D$,*

$$\begin{aligned} s(b, p, W) &= (1 - |b|^2) \exp(\log W)^\sim(b) \\ &= (1 - |b|^2) |h(b)|^p, \end{aligned}$$

where $(\log W)^\sim(b)$ denotes the Poisson integral of $\log W$ at b .

Proof. It is well known (cf. [5, p136]) that $s(0, p, W) = \exp \int_0^{2\pi} \log W d\theta/2\pi$. It is easy to show the lemma using $f(b) = f \circ \phi_b(0)$, where $\phi_b(z) = (z+b)/(1+\bar{b}z)$.

Lemma 3. *Suppose (z_j) is a sequence of points in D . For $1 \leq p < \infty$ and $1 \leq s < \infty$, $\{(s(z_j, p, W)^{1/p}f(z_j)); f \in H^p(W)\} \supset \ell^s$ if and only if $\{(F(z_j)); F \in H^p\} \supset \ell^s(a)$, where $a = (a_j)$ and $a_j = s(z_j, p, W)^{1/p}/|h(z_j)|$.*

Proof. Since $H^p(W) = h^{-1}H^p$, $f \in H^p(W)$ if and only if $f = h^{-1}F$ and $F \in H^p$. For each j , $s(z_j)^{1/p}f(z_j) = w_j$ if and only if $F(z_j) = h(z_j)w_j/s(z_j)^{1/p}$ if and only if $F(z_j) = \zeta_j$, $w_j = a_j\zeta_j$. $(w_j) \in \ell^p$ if and only if $(\zeta_j) \in \ell^s(a)$. Now the lemma follows.

Theorem 4. *Let $1 \leq p < \infty$, $1 \leq s \leq \infty$, and $1/p + 1/q = 1/s + 1/t = 1$. Then, $\{(s(z_n, p, W)^{1/p}f(z_n)); f \in H^p(W)\} \supset \ell^s$ if and only if*

$$\left\{ \sum_{n=1}^{\infty} \frac{1}{\rho_n^t} s(z_n, p, W)^{t/q} |g(z_n)|^t \right\}^{1/t} \leq \gamma \|g\|_{H^q(W)}$$

for g in $H^q(W)$.

Proof. By Lemma 3, $\{(s(z_n)^{1/p}f(z_n)); f \in H^p(W)\} \supset \ell^s$ if and only if $\{(F(z_n)); F \in H^p\} \supset \ell^s(a)$, where $a_n = s(z_n)^{1/p}/|h(z_n)|$. By Theorem 1, this is equivalent to saying that there exists a finite positive constant γ such that

$$\left\{ \sum_{n=1}^{\infty} \frac{1}{\rho_n^t} \frac{1}{a_n^t} (1 - |z_n|^2)^t |G(z_n)|^t \right\}^{1/t} \leq \gamma \|G\|_q$$

for $G \in H^q$. Since $H^q(W) = h^{-p/q}H^q$, $g \in H^q(W)$ if and only if $g = h^{-p/q}G$ and $G \in H^q$. Hence $\|g\|_{H^q(W)} = \|G\|_{H^q}$ and for each $n \geq 1$

$$\begin{aligned} & a_n^{-t} (1 - |z_n|^2)^t |G(z_n)|^t \\ &= s(z_n)^{-(t/p)} |h(z_n)|^t (1 - |z_n|^2)^t |h(z_n)|^{pt/q} |g(z_n)|^t \\ &= s(z_n)^{-(t/p)} (1 - |z_n|^2)^t |h(z_n)|^{t(q+p)/q} |g(z_n)|^t \\ &= s(z_n)^{-(t/p)} s(z_n)^t |g(z_n)|^t \\ &= s(z_n)^{t/q} |g(z_n)|^t. \end{aligned}$$

This implies the theorem.

Corollary 7. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then $\{(s(z_n, p, W)^{1/p}f(z_n)); f \in H^p(W)\} \supset \ell^1$ if and only if $\sup_n \frac{1}{\rho_n} \exp \frac{p-2}{p} (\log W) \sim (z_n) < \infty$. When $p = 2$, this is equivalent to $\inf_n \rho_n > 0$.*

Proof. By Theorem 4, $\{(s(z_n, p, W)^{1/p}f(z_n)); f \in H^p(W)\} \supset \ell^1$ if and only if

$$\sup_n \frac{1}{\rho_n} s(z_n, p, W)^{1/q} s(z_n, q, W)^{-1/q} < \infty.$$

Now Lemma 2 implies the corollary.

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