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CLIFFORD SEMIRINGS AND GENERALIZED CLIFFORD SEMIRINGS

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Abstract. It is well known that a semigroup S is a Clifford semigroup if and only if S is a strong semilattice of groups. In this paper, we extend this important result from semigroups to semirings by showing that a semiring S is a Clifford semiring if and only if S is a strong distributive lattice of skew-rings. Also, as a further generalization, we prove that a semiring S is a genneralized Clifford semiring if and only if S is a strong b-lattice of skew-rings. Some results which have been recently obtained in the literature [2] are strengthened and extended.

1. Introduction

Recall that a semiring $(S, +, \cdot)$ is a type $(2, 2)$ algebra whose semigroup reducts $(S, +)$ and $(S, \cdot)$ are connected by distributivity, that is, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$. We call a semiring $(S, +, \cdot)$ additive regular if for every element $a \in S$ there exists an element $x \in S$ such that $a + x + a = a$. Additive regular semirings were first studied by J. Zeleznekow [11] in 1981. We call a semiring $(S, +, \cdot)$ an additive inverse semiring if $(S, +)$ is an additive inverse semigroup. Additive inverse semirings were first studied by Karvellas [6] in 1974.

In our paper [9], we call an element a of a semiring $(S, +, \cdot)$ completely regular if there exists an element $x \in S$ such that (i) $a + x + a = a$, (ii) $a + x = x + a$ and (iii) $a(a + x) = a + x$.

In fact, conditions (i) and (ii) follow immediately from the definition of complete regularity when the additive reduct $(S, +)$ of the semiring $(S, +, \cdot)$ is a completely regular semigroup, however condition (iii) above is an extra condition which makes the element a in the semiring $(S, +, \cdot)$ to be completely regular. Naturally, we call a smiring $(S, +, \cdot)$ completely regular if every element a of S is completely

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regular. We notice that the condition (iii) can be replaced by the condition $(iii)'$ $(a+x)a = a+x$.

In fact, we have obtained the following theorem in [9].

Theorem 1.1. *A semiring $(S, +, \cdot)$ is a completely regular semiring if and only if for all $a \in S$, there exists an element $x \in S$ such that the following conditions are satisfied:*

$$(i) \quad a + x + a = a$$
$$(ii) \quad a + x = x + a$$
$$\text{and} \quad (iii)' \quad (a + x)a = a + x$$

The following useful concept is due to M. P. Grillet [4].

A semiring $(S, +, \cdot)$ is called a skew-ring if its additive reduct $(S, +)$ is a group, not necessarily an abelian group.

We have also obtained in [9] the following characterization theorem for completely regular elements in semirings.

Theorem 1.2. *The following statements on a semiring $(S, +, \cdot)$ are equivalent:*

(i) *a is a completely regular element of S.*

(ii) *There exists a unique element $y \in V^+(a)$ such that $a(a+y) = a+y$, $a(y+a) = y+a$, $a+(a+y)a = a$, $a(y+a)+a = a$, $a(a+y) = (a+y)a$.*

(iii) *There exists a unique element $y \in V^+(a)$ such that $a+y = y+a, a(a+y) = a+y$.*

(iv) *H_a^+ is a skew-ring, where H_a^+ is the $\mathcal{H}$ -class on the semigroup $(S, +)$ containing $a \in S$.*

We denote the unique element in a completely regular semiring satisfying the condition (iii) of the Theorem 1.2 by a' .

Let us call a semiring $(S, +, \cdot)$ a b-lattice if its additive reduct $(S, +)$ is a semilattice and its multiplicative reduct $(S, \cdot)$ is a band. Also, a completely regular semiring S is called a completely simple semiring if any two elements of S are $\mathcal{J}^+$-related.

Definition 1.3. A congruence ρ on a semiring S is called a b-lattice congruence if S/ρ is a b-lattice. A semiring S is called a b-lattice Y of semirings $S_\alpha (\alpha \in Y)$ if S admits a b-lattice congruence ρ on S such that $Y = S/\rho$ and each S_α is a ρ-class.

By using the concept of b-lattice, we obtained the following characterization theorem for completely regular semirings in [9].

Theorem 1.4. *The following conditions on a semiring $(S, +, \cdot)$ are equivalent:*

(A) *S is completely regular semiring.*

(B) *Every $\mathcal{H}^+$ - class is a skew-ring.*

(C) *S is a union of skew-rings.*

(D) *S is a b-lattice of completely simple semirings.*

As a special case of completely regular semigroup, we recall that a semigroup S is Clifford semigroup if for each $a \in S$, there exists an element $x \in S$ such that $axa = a$ and $ae = ea$, for all idempotents e of S.

Clearly, a semigroup S is a Clifford semigroup if S is completely regular and its idempotents commmute with all elements of S. Similar to the result of Clifford semigroups, Bandelt and Petrich [1] have shown that a semiring S whose additive reduct $(S, +)$ is a regular semigroup can be expressed as a subdirect product of a distributive lattice and a ring if and only if $(S, +)$ is commutative and the following conditions hold

(i) $(a + a')b = b(a + a')$

(ii) $a(a + a') = a + a'$

(iii) $a + (a + a')b = a,$ for all $a, b \in S$

and (iv) If $a \in S$ and $b + a = b$ for some $b \in S$, then $a + a = a$.

Recall that an ideal I of a semiring S is a k-ideal of S if $a \in I$ and either $a + x \in I$ or $x + a \in I$ for some $x \in S$ implies $x \in I$.

In view of above result, Ghosh [2] has further given a characterization for semirings whose additive reduct $(S, +)$ is commutative and he has consequently defined Clifford semirings, by assuming that the additive reduct is commutative. According to Ghosh [2], a Clifford semiring S is an additively commutative inverse semiring such that $E^+(S)$ is a distributive sublattice as well as a k-ideal of S. Later on, Mukhopadhyay, P. [10] has verified that an additive commutative inverse semiring S satisfies the above conditions (i), (ii) and (iii) if and only if $E^+(S)$ is a distributive lattice of S and the semiring S satisfies the condition (iv) if and only if $E^+(S)$ is a k-ideal of S. Thus, we can see that $E^+(S)$ of a semiring S plays an important role in studying the structure of semirings.

In this paper, we consider the Clifford semiring without assuming that its additive reduct is commutative.

If S is a completely regular semiring as well as an additive inverse semiring then $E^+(S)$ is an ideal of S but $E^+(S)$ may not be a k-ideal of S, for instance, if we let $S = \{0, a, b\}$ be a semiring with the following Cayley tables:

+	0	a	b
0	0	a	b
a	a	0	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

Then we can easily see that the additive reduct $(S, +)$ is an additive inverse semigroup. It is also easy to see that $(S, +, \cdot)$ is a completely regular semiring because $a(a + a) = a0 = 0 = a + a$ and $b(b + b) = bb = b = b + b$ hold. In this example, $E^+(S) = \{0, b\}$ is clearly an ideal of S but since $a + b = b \in E^+(S)$, and $a \notin E^+(S)$, $E^+(S)$ is not a k-ideal of S.

In view of the above example, we now call a completely regular semiring S a generalized Clifford semiring if S is an additive inverse semiring whose $E^+(S)$ is a k-ideal of S. Also, we call a completely regular semiring S a Clifford semiring if S is an additive inverse semiring such that $E^+(S)$ is a distributive sublattice of S as well as a k-ideal of S.

In this paper, we will show that a semiring S is a Clifford semiring if and only if S is a strong distributive lattice of skew-rings. As an extension of this result, we further prove that a semiring S is a generalized Clifford semiring if and only if S is a strong b-lattice of skew-rings.

2. Generalized Clifford Semirings

In this section, we let $(S, +, \cdot)$ be a completely regular semiring. If $(S, +)$ is an inverse semigroup and $E^+(S)$ is a k-ideal of the semiring $(S, +, \cdot)$, then we call $(S, +, \cdot)$ a generalized Clifford semiring. We also call a semiring $(S, +, \cdot)$ an AR-semiring if its additive reduct $(S, +)$ is a regular semigroup and in particular we call a semiring $(S, +, \cdot)$ an AI-semiring if its additive reduct $(S, +)$ is an inverse semigroup. For the sake of brevity, we sometimes just denote the semiring $(S, +, \cdot)$ by S.

Generalized Clifford semiring as special completely regular semiring can be characterized by some of the conditions given by Bandelt and Petrich for AR-semirings in [1]. The following is a characterization for generalized Clifford semirings.

Theorem 2.1. *An AI-semiring $(S, +, \cdot)$ is a generalized Clifford semiring if and only if the following conditions are satisfied*:

(*i*) $a + a' = a' + a$

(*ii*) $a(a + a') = a + a'$

(*iii*) *If $a \in S$ and $a + b = b$ for some $b \in S$, then $a + a = a$.*

Proof. We first suppose that the AI-semiring $(S,+,\cdot)$ is a generalized Clifford semiring. Then $(S,+,\cdot)$ is a completely regular semiring and $E^+(S)$ is a k-ideal of S. Hence $a + a' = a' + a$ and $a(a + a') = a + a'$. Let $a \in S$ and $a + b = b$ for some $b \in S$. Then $a + b + b' = b + b'$. Since $E^+(S)$ is a k-ideal of S and $b + b' \in E^+(S)$, we have $a \in E^+(S)$, i.e., $a+a=a$. This shows that all the conditions of Theorem 2.1. are satisfied.

Conversely, suppose that an AI-semiring $(S,+,\cdot)$ satisfies the given conditions (i), (ii) and (iii) of Theorem 2.1. Now by conditions (i) and (ii), we immediately see that S is a completely regular semiring. Since S is an AI-semiring, it follows that $E^+(S)$ is an ideal of S. To show $E^+(S)$ a k-ideal, let $e, f+e \in E^+(S)$. Then, we have $f+e+f+e = f+e$, i.e., $f+(f+e)+e = f+e$. Hence, we obtain $f+(f+e) = f+e$. Now, by the given condition $f+f=f$, we see that $f \in E^+(S)$. Similarly from $e, e+f \in E^+(S)$, we can still show that $f \in E^+(S)$. Hence $E^+(S)$ is a k-ideal of S. The proof is completed.

By using Theorem 2.1., we construct an example of generalized Clifford semiring.

Example 2.2. Let T be a b-lattice and R a skew-ring . Construct the direct product of T and R and denote it by S. Then, we can check that $E^+(S) = T \times \{0_R\}$, where 0_R is the zero element of the skew-ring R. We can also check that $(S,+,\cdot)$ is a semiring whose additive reduct $(S,+)$ is clearly an inverse semigroup. Now let $(a,u) \in S = T \times R$. Then, we have $(a,u)' = (a,-u)$. Now $(a,u)+(a,u)' = (a,u)+(a,-u) = (a,0) = (a,u)'+(a,u)$ and $(a,u)((a,u)+(a,u)') = (a,u)(a,0) = (a,0) = (a,u)+(a,u)'$. Suppose, that $(a,u)+(b,v) = (b,v)$ for some $(b,v) \in S$ where $(a,u) \in S$. Then $a+b=b$ and $u+v=v$. This leads to $u=0$, and consequently $(a,u)+(a,u) = (a,u)$.

In order to construct a generalized Clifford semiring, we present a construction method which is analogous to the construction method of strong semilattice of semigroups. However, instead of using semilattice as a frame, we use a b-lattice T and instead of using semigroups as an ingredient, we use semirings. We also note that for all $a,b \in T$, we always have $a+b+ab = a+b$. We now give the following definition.

Definition 2.3. Let T be a b-lattice and $\{S_\alpha : \alpha \in T\}$ be a family of pairwise disjoint semirings which are indexed by the elements of T. For each $\alpha \leq \beta$ in T, we now embed S_α in S_β via a semiring monomorphism $\phi_{\alpha,\beta}$ satisfying the following conditions

(1.1) $$\phi_{\alpha,\alpha} = I_{S_\alpha}, \text{ the identity mapping on } S_\alpha$$

(1.2) $$\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma} \qquad \text{if } \alpha \leq \beta \leq \gamma$$

(1.3) $$S_\alpha\phi_{\alpha,\gamma}S_\beta\phi_{\beta,\gamma} \subseteq S_{\alpha\beta}\phi_{\alpha\beta,\gamma} \text{ if } \alpha+\beta \leq \gamma, i.e., \alpha+\beta+\alpha\beta \leq \gamma$$

On $S = \bigcup_{\alpha \in T} S_\alpha$ we define addition + and multiplication . for $a \in S_\alpha, b \in S_\beta$, as follows:

(1.4) $$a + b = a\phi_{\alpha,\alpha+\beta} + b\phi_{\beta,\alpha+\beta}$$

and

(1.5) $$a.b = c \in S_{\alpha\beta} \text{ such that } c\phi_{\alpha\beta,\alpha+\beta} = a\phi_{\alpha,\alpha+\beta}.b\phi_{\beta,\alpha+\beta}$$

Same as the notation of strong semilattice of semigroups, we denote the above system by $S =< T, S_\alpha, \phi_{\alpha,\beta} >$ and call it the strong b-lattice T of the semirings $S_\alpha, \alpha \in T$.

Theorem 2.4. *With the above notation in Definition 2.3., the system $S =< T, S_\alpha, \phi_{\alpha,\beta} >$ is a semiring.*

Proof. We first show that the operation of multiplication $'\cdot'$ defined above is well defined. For this purpose, we let $a \in S_\alpha$ and $b \in S_\beta$, with $\alpha, \beta \in T$. Then, by (1.3), there exists an element $c \in S_{\alpha\beta}$ satisfying (1.5) and the uniqueness of the element follows directly from the injectivity of the mapping $\phi_{\alpha\beta,\alpha+\beta}$. The associativity of the addition is clear. We only need to prove the associativity of the multiplication. For this purpose, we let $a \in S_\alpha, b \in S_\beta$ and $c \in S_\gamma$, with $\alpha, \beta, \gamma \in T$. Let $x = a.b$ and $d = x.c = (a.b).c$. Then by definition, we have $x\phi_{\alpha\beta,\alpha+\beta} = a\phi_{\alpha,\alpha+\beta} b\phi_{\beta,\alpha+\beta}$ and $d\phi_{\alpha\beta\gamma,\alpha\beta+\gamma} = x\phi_{\alpha\beta,\alpha\beta+\gamma} c\phi_{\gamma,\alpha\beta+\gamma}$.
Applying $\phi_{\alpha\beta+\gamma,\alpha+\beta+\gamma}$ to both sides of the second equation, we get

$$d\phi_{\alpha\beta\gamma,\alpha+\beta+\gamma} = x\phi_{\alpha\beta,\alpha+\beta+\gamma} c\phi_{\gamma,\alpha+\beta+\gamma}$$

Applying $\phi_{\alpha+\beta,\alpha+\beta+\gamma}$ to both sides of the first equation, we get

$$x\phi_{\alpha\beta,\alpha+\beta+\gamma} = a\phi_{\alpha,\alpha+\beta+\gamma} b\phi_{\beta,\alpha+\beta+\gamma}.$$

Thus, we obtain $d\phi_{\alpha\beta\gamma,\alpha+\beta+\gamma} = a\phi_{\alpha,\alpha+\beta+\gamma} b\phi_{\beta,\alpha+\beta+\gamma} c\phi_{\gamma,\alpha+\beta+\gamma}$. Similarly, we can show that $e\phi_{\alpha\beta\gamma,\alpha+\beta+\gamma} = a\phi_{\alpha,\alpha+\beta+\gamma} b\phi_{\beta,\alpha+\beta+\gamma} c\phi_{\gamma,\alpha+\beta+\gamma}$, where $e = a.(b.c)$. Since the mapping $\phi_{\alpha\beta\gamma,\alpha+\beta+\gamma}$ is injective, we have $d = e$. Hence, we have $(a.b).c = a.(b.c)$. Finally we prove the distributivity of the semiring $S =< T, S_\alpha, \phi_{\alpha,\beta} >$. Let $a \in S_\alpha, b \in S_\beta, c \in S_\gamma$ with $\alpha, \beta, \gamma \in T$. Let $d = a.(b + c) = a.(b\phi_{\beta,\beta+\gamma} + c\phi_{\gamma,\beta+\gamma})$. Then $d\phi_{\alpha(\beta+\gamma),\alpha+\beta+\gamma} = a\phi_{\alpha,\alpha+\beta+\gamma}(b\phi_{\beta,\beta+\gamma} + c\phi_{\gamma,\beta+\gamma})\phi_{\beta+\gamma,\alpha+\beta+\gamma} = a\phi_{\alpha,\alpha+\beta+\gamma}(b\phi_{\beta,\alpha+\beta+\gamma} + c\phi_{\gamma,\alpha+\beta+\gamma}) = a\phi_{\alpha,\alpha+\beta+\gamma}.b\phi_{\beta,\alpha+\beta+\gamma} + a\phi_{\alpha,\alpha+\beta+\gamma} c\phi_{\gamma,\alpha+\beta+\gamma}$. Let $e = a.b$ and $f = a.c$. Then, we have $e\phi_{\alpha\beta,\alpha+\beta} = a\phi_{\alpha,\alpha+\beta}.b\phi_{\beta,\alpha+\beta}$ and $f\phi_{\alpha\gamma,\alpha+\gamma} = a\phi_{\alpha,\alpha+\gamma}.c\phi_{\gamma,\alpha+\gamma}$. Then, $(e+f)\phi_{\alpha(\beta+\gamma),\alpha+\beta+\gamma} = a\phi_{\alpha,\alpha+\beta+\gamma}.b\phi_{\beta,\alpha+\beta+\gamma} + a\phi_{\alpha,\alpha+\beta+\gamma}.c\phi_{\gamma,\alpha+\beta+\gamma}$. Since $\phi_{\alpha(\beta+\gamma),\alpha+\beta+\gamma}$ is injective, we have $d = e + f$ i.e., $a.(b + c) = a.c + b.c$. The proof of the other distributive law is similar. Thus, S is indeed a semiring.

Theorem 2.5. *A semiring S is a generalized Clifford semiring if and only if S is a strong b-lattice of skew-rings.*

Proof. First we suppose that S is a generalized Clifford semiring. Then S is a completely regular semiring. Then by Theorem 1.4, S can be regarded as a b-lattice T of completely simple semirings $R_\alpha(\alpha \in T)$, where $T = S/\mathcal{J}^+$ and R_α is a $\mathcal{J}^+$-class in S containing a. Let $a \in S = \bigcup_{\alpha \in T} S_\alpha$. Then $a \in R_\alpha$, for some $\alpha \in T$. Also, we have $a + a' \in R_\alpha$. Thus R_α contains some additive idempotents. Let e and f be two additive idempotents in R_α. Then $e\mathcal{J}^+ f$. Since R_α is completely simple semiring, $(R_\alpha, +)$ is a completely simple semigroup and so $e\mathcal{D}^+ f$. Also, since S is an AI-semiring as well as a completely regular semiring, we have $e = f$. This shows that each R_α contains a single additive idempotent, so that $(R_\alpha, +)$ is a group and hence $(R_\alpha, +, \cdot)$ is a skew-ring. In other words, we have shown that S is a b-lattice T of skew-rings R_α.

Let $\alpha, \beta \in T$ be such that $\alpha \leq \beta$. Then, we define $\phi_{\alpha,\beta} : R_\alpha \longrightarrow R_\beta$ by

$$a\phi_{\alpha,\beta} = a + 0_\beta \qquad a \in R_\alpha,$$

where 0_β is the zero element of the skew-ring R_β.

We first show that $\phi_{\alpha,\beta}$ is injective. For this purpose, let $a, b \in R_\alpha$ be such that $a\phi_{\alpha,\beta} = b\phi_{\alpha,\beta}$ i.e., $a + 0_\beta = b + 0_\beta$. then, we have $b' + a + 0_\beta = b' + b + 0_\beta$. However, this leads to $b' + a \in E^+(S)$, as $E^+(S)$ is a k-ideal of S. Also, $b' + a \in R_\alpha$. Hence $b' + a = 0_\alpha = a + a' = b + b'$ i.e., $b' + a = b' + b$. This leads to $b + b' + a = b + b + b'$ i.e., $a = a + a' + a = b$.
Consequently, we obtain $a = b$, and this shows that $\phi_{\alpha,\beta}$ is injective. To show that $\phi_{\alpha,\beta}$ is a monomorphism, we let $a \in R_\alpha$. Then, $a\mathcal{J}^+\alpha$. Also by $0_\beta\mathcal{J}^+\beta$, we have $a0_\beta\mathcal{J}^+\alpha\beta$, i.e., $a.0_\beta = 0_{\alpha\beta}$. Similarly, we have $0_\beta.a = 0_{\beta\alpha}$. Also, we can easily see that $0_\alpha + 0_\beta = 0_{(\alpha+\beta)}$. Now, let $a, b \in R_\alpha$. Then, $a\phi_{\alpha,\beta} + b\phi_{\alpha,\beta} = a + 0_\beta + b + 0_\beta = a + b + 0_\beta = (a + b)\phi_{\alpha,\beta}$.

Also, $a\phi_{\alpha,\beta}b\phi_{\alpha,\beta} = (a+0_\beta)(b+0_\beta) = ab+a0_\beta+0_\beta b+0_\beta = ab+0_{\alpha\beta}+0_{\beta\alpha}+0_\beta$ = $ab + 0_\beta = (ab)\phi_{\alpha,\beta}$.

Thus, we have proved that, $\phi_{\alpha,\beta}$ is a monomorphism.
Clearly, $\phi_{\alpha,\alpha} = I_{R_\alpha}$ and $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \leq \beta \leq \gamma$. For $\alpha, \beta, \gamma \in T$ with $\alpha + \beta \leq \gamma$, let $a \in R_\alpha$ and $b \in R_\beta$. Note that in T , we always have $\alpha + \beta = \alpha + \beta + \alpha\beta$. Then $a\mathcal{J}^+\alpha$ and $b\mathcal{J}^+\beta$, and thereby, we have $ab\mathcal{J}^+\alpha\beta$ and $(a + b)\mathcal{J}^+(\alpha + \beta)$. These implies that $ab \in R_{\alpha\beta}$ and $a + b \in R_{\alpha+\beta}$. Now, $a\phi_{\alpha,\gamma}b\phi_{\beta,\gamma} = (a+0_\gamma)(b+0_\gamma) = ab+a0_\gamma+0_\gamma b+0_\gamma = ab+0_{\alpha\gamma}+0_{\gamma\beta}+0_\gamma = ab+0_\gamma$ (as $\alpha\gamma \leq \gamma$ and $\gamma\beta \leq \gamma$) = $(ab)\phi_{\alpha\beta,\gamma}$, hence we get $R_\alpha\phi_{\alpha,\gamma}R_\beta\phi_{\beta,\gamma} \subseteq R_{\alpha\beta}\phi_{\alpha\beta,\gamma}$ if $\alpha + \beta \leq \gamma$. Also, we can derive that $a\phi_{\alpha,\alpha+\beta} + b\phi_{\beta,\alpha+\beta} = a + 0_{\alpha+\beta} + b + 0_{\alpha+\beta} = a + b + 0_{\alpha+\beta} = a + b$ and $a\phi_{\alpha,\alpha+\beta}b\phi_{\beta,\alpha+\beta} = (a + 0_{\alpha+\beta})(b + 0_{\alpha+\beta}) = ab + a0_{\alpha+\beta} + 0_{\alpha+\beta} + 0_{\alpha+\beta} = ab + 0_{\alpha+\beta} = (ab)\phi_{\alpha\beta,\alpha+\beta}$. Thus, we have proved that S is a strong b-lattice of skew-rings.

Conversely, let $S =< T, R_\alpha, \phi_{\alpha,\beta} >$ be strong b-lattice T of skew-rings $R_\alpha(\alpha \in T)$. Then, S is clearly an AI-semiring and of course , a completely regular semiring.

It remains to show that $E^+(S)$ is a k-ideal of S. But this follows from the fact that the semigroup $(S,+)$ is a strong semilattice of groups $(R_\alpha,+)$ on the semilattice $Y=(T,+)$, where all the structure mappings $\phi_{_{\alpha,\beta}}$ are one-to-one and hence $(S,+)$ is E-unitary which implies $E^+(S)$ is a k-ideal of S. Thus, our proof is completed.

We now recall that a subdirect product algebra T is a subalgebra of a direct product of algebras such that the projection mapping from the algebra T to each of its components is surjective.

Lemma 2.6. *Let $S=<T, S_\alpha, \phi_{_{\alpha,\beta}}>$ be a strong b-lattice T of semirings $S_{\,\alpha}$, $\alpha\in T$ and θ a binary relation on S defined by $a\theta b$ if and only if $a\phi_{\,\alpha,\alpha+\beta}=b\phi_{\beta,\alpha+\beta}$ ($a\in S_\alpha, b\in S_\beta$). Then θ is a congruence on S and S is a subdirect product of T and S/θ.*

Proof. Clearly, the relation θ defined in Lemma 2.6. is reflexive and symmetric. To show that θ is transitive, we let $a\in S_\alpha, b\in S_\beta$ and $c\in S_\gamma$, where $\alpha,\beta,\gamma\in T$. Also, we assume that $a\theta b$ and $b\theta c$. Then we have $a\phi_{\alpha,\alpha+\beta+\gamma}= b\phi_{\beta,\alpha+\beta+\gamma}=c\phi_{\gamma,\alpha+\beta+\gamma}$. Hence, it follows that $a\phi_{\alpha,\alpha+\gamma}=c\phi_{\gamma,\alpha+\gamma}$, since the mapping $\phi_{\alpha+\gamma,\alpha+\beta+\gamma}$ is injective. Thus $a\theta c$ and so θ is transitive. Now, assume that $a\theta b$. Then, we have $a\phi_{\alpha,\alpha+\beta}=b\phi_{\beta,\alpha+\beta}$. This leads to $a\phi_{\alpha,\alpha+\beta+\gamma}+c\phi_{\gamma,\alpha+\beta+\gamma}= b\phi_{\beta,\alpha+\beta+\gamma}+c\phi_{\gamma,\alpha+\beta+\gamma}$ and so $(a+c)\phi_{\alpha+\gamma,(\alpha+\gamma)+(\beta+\gamma)}=(b+c)\phi_{\beta+\gamma,(\alpha+\gamma)+(\beta+\gamma)}$, i.e., $(a+c)\theta(b+c)$. By using a symmetric argument, we also have $(c+a)\theta(c+b)$. Again let $x=ac$ and $y=bc$. Then, we have $x\phi_{_{\alpha\gamma,\alpha+\gamma}}=a\phi_{_{\alpha,\alpha+\gamma}}c\phi_{_{\gamma,\alpha+\gamma}}$, i.e., $x\phi_{_{\alpha\gamma,\alpha+\beta+\gamma}}=a\phi_{_{\alpha,\alpha+\beta+\gamma}}c\phi_{_{\gamma,\alpha+\beta+\gamma}}=b\phi_{_{\beta,\alpha+\beta+\gamma}}c\phi_{_{\gamma,\alpha+\beta+\gamma}}=y\phi_{_{\beta\gamma,\alpha+\beta+\gamma}}$.
Consequently, we have $x\phi_{_{\alpha\gamma,\alpha\gamma+\beta\gamma}}=y\phi_{_{\beta\gamma,\alpha\gamma+\beta\gamma}}$ and so $x\theta y$. This shows that $(ac)\theta(bc)$.
Similarly, we can prove that $(ca)\theta(cb)$. Thus, θ is indeed a congruence on the semiring S.

Finally, we define a mapping $\Psi: S\longrightarrow T\times S/\theta$ by $a\Psi=(\alpha, a\theta)$, where $a\in S_\alpha$.
Clearly Ψ is a homomorphism. Also Ψ is injective and the projection homomorphisms map $S\Psi$ onto T and S/θ respectively. Therefore, S is a subdirect product of T and S/θ.

Theorem 2.7. *A semiring S is an AI-semiring and is a subdirect product of a b-lattice and a skew-ring if and only if $S=<T, R_{\,\alpha}, \phi_{_{\alpha,\beta}}>$, where the latter is a generalized Clifford semiring.*

Proof. First, we suppose that the semiring S is an AI-semiring and is a subdirect product of a b-lattice T and a skew-ring R. Then we may consider $S\subseteq T\times R$. For each $\alpha\in T$, let $R_\alpha=(\{\alpha\}\times R)\cap S$. Then R_α is a skew-ring for each $\alpha\in T$ and $S=\bigcup\limits_{\alpha\in T}R_\alpha$. Now for each pair $\alpha,\beta\in T$ with $\alpha\leq\beta$, we define

$\phi_{\alpha,\beta} : R_\alpha \longrightarrow R_\beta$ by $(\alpha, r)\phi_{\alpha,\beta} = (\beta, r)$.
Then clearly $\phi_{\alpha,\beta}$ is a monomorphism satisfying the conditions $\phi_{\alpha,\alpha} = I_{R_\alpha}$ and $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \leq \beta \leq \gamma$.
Let $\alpha, \beta, \gamma \in T$ such that $\alpha + \beta \leq \gamma$. Let $a = (\alpha, r) \in R_\alpha$ and $b = (\beta, r') \in R_\beta$. Then, we have $a + b = (\alpha, r) + (\beta, r') = (\alpha + \beta, r + r') \in R_{\alpha+\beta}$ and $ab = (\alpha, r)(\beta, r') = (\alpha\beta, rr') \in R_{\alpha\beta}$. Now $(a\phi_{\alpha,\gamma})(b\phi_{\beta,\gamma}) = (\gamma, r)(\gamma, r') = (\gamma, rr') = (\alpha\beta, rr')\phi_{\alpha\beta,\gamma} = (ab)\phi_{\alpha\beta,\gamma}$. Therefore, $R_\alpha\phi_{\alpha,\gamma}R_\beta\phi_{\beta,\gamma} \subseteq R_{\alpha\beta}\phi_{\alpha\beta,\gamma}$ if $\alpha + \beta \leq \gamma$. Also, $a + b = (\alpha, r) + (\beta, r') = (\alpha + \beta, r + r') = (\alpha + \beta, r) + (\alpha + \beta, r') = a\phi_{\alpha,\alpha+\beta} + b\phi_{\beta,\alpha+\beta}$ and $(a\phi_{\alpha,\alpha+\beta})(b\phi_{\alpha,\alpha+\beta}) = (\alpha + \beta, r)(\alpha + \beta, r') = (\alpha + \beta, rr') = (ab)\phi_{\alpha\beta,\alpha+\beta}$. Therefore, S is a strong b-lattice of skew-rings R_α i.e., $S =< T, R_\alpha, \phi_{\alpha,\beta} >$.

Conversely, let $S =< T, R_\alpha, \phi_{\alpha,\beta} >$. Then, by Lemma 2.6., S is a subdirect product of T and S/θ. Now S/θ, being a homomorphic image of a completely regular semiring is a completely regular semiring. If $e, f \in E^+(S)$, say $e \in R_\alpha, f \in R_\beta$. Then, we have $e\,\theta\,(e\phi_{\alpha,\alpha+\beta}) = (f\phi_{\beta,\alpha+\beta})\,\theta\, f$, i.e., $e\theta = f\theta$. This shows that S/θ has just one additive idempotent so that $(S/\theta, +)$ is a group and hence $(S/\theta, +, \cdot)$ is a skew-ring. In other words, S is a subdirect product of a b-lattice T and a skew-ring S/θ.

3. Clifford Semirings and Characterizations

In this section, we will study completely regular semiring S which is AI-semiring in which $E^+(S)$ is a distributive lattice as well as a k-ideal of S. We first give the following definition of Clifford semirings.

Definition 3.1. Let S be a completely regular semiring. Then S is called a Clifford semiring if S is an AI-semiring and $E^+(S)$ is a distributive sublattice of S as well as a k-ideal of S.

One can easily see that every Clifford semiring is a generalized Clifford semiring, however, the converse is not necessarily true. This is evident if we let $(S, +, \cdot)$ be a semiring such that $(S, +)$ is a semilattice and $(S, \cdot)$ is a left zero semigroup, then $(S, +, \cdot)$ is clearly a generalized Clifford semiring, but according to our definition, S is not a Clifford semiring.

We now classify the Clifford semirings.

Theorem 3.2. *An AI-semiring S is a Clifford semiring if and only if the following conditions hold*:

(*i*) $a + a' = a' + a$

(*ii*) $a(a + a') = a + a'$

(*iii*) $(a + a')b = b(a + a')$

(*iv*) $a + (a + a')b = a$, *for all* $a, b \in S$

and (v) If $a \in S$ *and* $a + b = b$ *for some* $b \in S$ *then* $a + a = a$.

Proof. First, we suppose that S is a Clifford semiring. Then by Theorem 2.1., we see that the conditions (i), (ii) and (v) are satisfied. To prove that the conditions (iii) and (iv) also hold, we let $a, b \in S$. Since $E^+(S)$ is a distributive lattice of S, we have $(a + a')(b + b') = (b + b')(a + a')$ so that $(a + a')b + (a + a')b' = b(a+a')+b'(a+a')$. This is equivalent to $(a+a')b+(a+a')b = b(a+a')+b(a+a')$ i.e., $(a+a')b = b(a+a')$. Also, we have $(a+a')+(a+a')(b+b') = (a+a')$ i.e., $(a + a') + (a + a')b = a + a'$. This leads to $a + (a + a')b = a$. Thus, conditions (iii) and (iv) are satisfied.

Conversely, suppose that all the above conditions (i) - (v) hold. Then by Theorem 2.1., we see that S is a generalized Clifford semiring. To see that S is a Clifford semiring, it remains to show that $E^+(S)$ is a distributive lattice of S.

Clearly $e^2 = e$ and $e + f = f + e$ for all $e, f \in E^+(S)$. Let $e, f \in E^+(S)$. Then we have $e = a + a'$ and $f = b + b'$ for some $a, b \in S$. Now, by $(a + a')b = b(a + a')$, we deduce that $(a + a')b + (a + a')b' = b(a + a') + b'(a + a')$, and so $(a + a')(b + b') = (b + b')(a + a')$ i.e., $ef = fe$. Again, by $a + (a + a')b = a$, we have $a' + a + (a + a')b = a' + a$, and so $a + a' + (a + a')b + (a + a')b' = a + a'$, or equivalently, $(a + a') + (a + a')(b + b') = (a + a')$ i.e., $e + ef = e$. This proves that $E^+(S)$ is a distributive lattice of S. Hence, S is a Clifford semiring. Thus the proof is completed.

Finally, we give the following interesting characterization theorem for Clifford semiring, in fact, this is the main result of our paper.

Theorem 3.3. *A semiring* S *is a Clifford semiring if and only if* S *is a strong distributive latice of skew-rings.*

Proof. We first suppose that S is a Clifford semiring. Then S is a generalized Clifford semiring. Hence, by Theorem 2.5., S is a strong b-lattice T of skew-rings $R_\alpha(\alpha \in T)$, where $T = S/\mathcal{J}^+$ and R_α is a $\mathcal{J}^+$-class of $(S, +)$ containing a. We now show that the $\mathcal{J}^+$-relation is a distributive lattice congruence on S. Let $a, b \in S$. Then we deduce the following equalities:
$ab = (a + a' + a)b = (a + a')b + ab = b(a' + a) + ab$ [by (i) and (iii) of Theorem 3.2.] $= ba' + ba + ab$ and $ba = b(a + a' + a) = b(a + a') + ba = (a' + a)b + ba$ [by (i) and (iii) of Theorem 3.2.] $= a'b + ab + ba$. This shows that $ab\mathcal{J}^+ba$.

Also, $a = a+(a+a')b = (a+a')+(a+ab)+a'b$ and $a+ab = (a+a')+a+ab$. Hence, $(a+ab)\mathcal{J}^+a$ as well. Consequently, the $\mathcal{J}^+$-relation is a distributive lattice congruence on S and hence $S/\mathcal{J}^+$ is a distributive lattice. This implies that S is a strong distributive lattice T of skew-rings $R_\alpha(\alpha \in T)$.

Conversely, let $S =< D, R_\alpha, \phi_{\alpha,\beta} >$ be a strong distributive lattice D of skew-rings $R_\alpha(\alpha \in D)$. Since every distributive lattice is also a b-lattice, it follows from

Theorem 2.5 that S is a generalized Clifford semiring. To complete our proof, it suffices to prove that $(a+a')b = b(a+a')$ and $a+(a+a')b = a$ for all $a, b \in S$.

Let $a, b \in S$. Suppose that $a \in R_\alpha$ and $b \in R_\beta$. Since S is a generalized Clifford semiring, we can let a' be the inverse element of a in the skew-ring R_α. Let $(a+a')b = c$ and $a+a' = 0_\alpha$ in R_α.Then, we have $c\phi_{\alpha\beta,\alpha+\beta} = 0_\alpha\phi_{\alpha,\alpha+\beta}b\phi_{\beta,\alpha+\beta} = 0_{\alpha+\beta}b\phi_{\beta,\alpha+\beta} = 0_{\alpha+\beta}$. Also, we let $b(a+a') = d$. Then, we have $d\phi_{\beta\alpha,\beta+\alpha} = b\phi_{\beta,\beta+\alpha}0_\alpha\phi_{\alpha,\beta+\alpha} = b\phi_{\beta,\beta+\alpha}0_{\beta+\alpha} = 0_{\beta+\alpha}$.
Since D is a distributive lattice, we have $c\phi_{\alpha\beta,\alpha+\beta} = d\phi_{\beta\alpha,\beta+\alpha} = d\phi_{\alpha\beta,\alpha+\beta}$. Again, from injectivity of $\phi_{\alpha\beta,\alpha+\beta}$, we have $b(a+a') = (a+a')b$. Similarly, we can also show that $a+(a+a')b = a$. Thus, S is a Clifford semiring.

By using the above theorem, we immediately obtain the following corollary.

Corollary 3.4. *Let $(S, +, \cdot)$ be an AI-semiring whose additive reduct $(S, +)$ is commutative. Then $(S, +, \cdot)$ is a Clifford semiring if and only if S is a strong distributive lattice of rings.*

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