TAIWANESE JOURNAL OF MATHEMATICS
Vol. 9, No. 3, pp. 411-420, September 2005
This paper is available online at http://www.math.nthu.edu.tw/tjm/

NOTES ON THE SCHUR-CONVEXITY OF THE EXTENDED MEAN VALUES

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Abstract. In this article, the Schur-convexities of the weighted arithmetic mean of function and the extended mean values are proved. Moreover, some inequalities involving the arithmetic mean, the harmonic mean, the logarithmic mean, and comparison between the extended mean values and the generalized weighted mean with two parameters and constant weight are obtained.

1. INTRODUCTION

It is well known [**9**, pp. 75-76] that a function f with n arguments defined on I^n is Schur-convex on I^n if $f(x) \leq f(y)$ for each two n-tuples $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in I^n such that $x \prec y$ holds, where I is an interval with nonempty interior and the relationship of majorization $x \prec y$ means that

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \qquad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \tag{1}$$

where $1 \leq k \leq n-1$, $x_{[i]}$ denotes the i-th largest component in x.

A function f is Schur-concave if and only if $-f$ is Schur-convex.

For a positive sequence $a = (a_1, \cdots, a_n)$ with $a_i > 0$ and a positive weight $w = (w_1, \cdots, w_n)$ with $w_i > 0$ for $1 \leq i \leq n$, the generalized weighted mean of

Received April 18, 2002; Accepted December 22, 2003.
Communicated by H. M. Srivastava.
2000 *Mathematics Subject Classification*: Primary 26B25, Secondary 26D07, 26D20.
Key words and phrases: Extended mean values, Schur-convexity, Inequality, Generalized weighted mean values, Weighted arithmetic mean of function.
The first author was supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), SF for Pure Research of Natural Science of the Education Department of Henan Province (#1999110004), Doctor Fund of Jiaozuo Institute of Technology, China.

positive sequence a with two parameters r and s is defined in [11] as

$$M_n(w;a;r,s)=\begin{cases}\left(\dfrac{\sum_{i=1}^n w_i a_i^r}{\sum_{i=1}^n w_i a_i^s}\right)^{1/(r-s)}, & r-s\neq 0;\\ \exp\left(\dfrac{\sum_{i=1}^n w_i a_i^r \ln a_i}{\sum_{i=1}^n w_i a_i^r}\right), & r-s=0.\end{cases} \tag{2}$$

For $x,y>0$ and $t\in\mathbb{R}$, let us define a function g by

$$g(t)\triangleq g(t;x,y)\triangleq\begin{cases}\dfrac{(y^t-x^t)}{t}, & t\neq 0;\\ \ln y-\ln x, & t=0.\end{cases} \tag{3}$$

It is easy to see that for $n\in\mathbb{N}$

$$g^{(n)}(t)=\int_x^y (\ln u)^n u^{t-1}\,\mathrm{d}u. \tag{4}$$

Therefore, the extended mean values $E(r,s;x,y)$ defined firstly in [29] can be expresssed in terms of g by

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{g(s;x,y)}{g(r;x,y)}\right)^{1/(s-r)}, & (r-s)(x-y)\neq 0;\\ \exp\left(\dfrac{\partial g(r;x,y)/\partial r}{g(r;x,y)}\right), & r=s,\ x-y\neq 0.\end{cases} \tag{5}$$

Now there is a rich literature about $E(r,s;x,y)$ (see [**3, 5, 6, 8, 10, 13, 15, 16, 19, 22, 24, 25, 29**] and references therein) and other mean values (see [**1, 2, 11, 12, 14, 16, 20, 21, 23, 26, 27, 28, 30, 31**] and references therein) and their applications (see [16, 17, 18] and references therein).

In this article, as a subsequent paper of [15], our main purpose is to prove the Schur-convexities of the weighted arithmetic mean of function and the extended mean values $E(r,s;x,y)$ with respect to (x,y) for fixed (r,s), and then we obtain the following

Theorem 1.1. *Let f be a continuous function on I, let p be a positive continuous weight on I. Then the weighted arithmetic mean of function f with weight p defined by*

$$F(x,y)=\begin{cases}\dfrac{\int_x^y p(t)f(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t}, & x\neq y,\\ f(x), & x=y\end{cases} \tag{6}$$

is Schur-convex (Schur-concave) on I^2 if and only if inequality

$$\frac{\int_x^y p(t)f(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t} \leq \frac{p(x)f(x)+p(y)f(y)}{p(x)+p(y)} \tag{7}$$

holds (reverses) for all $x, y \in I$.

Theorem 1.2. *Let $x > 0$ and $y > 0$ be positive real numbers and $r \in \mathbb{R}$, further let $A(x,y)$, $G(x,y)$, $H(x,y)$ and $L(a,y)$ denote the arithmetic, geometric, harmonic and logarithmic mean values.*

(i) *If $r \leq 0$, then*

$$L(x^r, y^r) \geq [G(x,y)]^r \geq A(x,y)H(x^{r-1}, y^{r-1}), \tag{8}$$

the equalities in (8) hold only if $x = y$ or $r = 0$.

(ii) *If $r \geq \frac{3}{2}$, we have*

$$L(x^r, y^r) \geq A(x,y)H(x^{r-1}, y^{r-1}), \tag{9}$$

the equality in (9) holds only if $x = y$.

(iii) *If $r \in (0,1]$, inequality (9) reverses without equality unless $x = y$.*

(vi) *Otherwise, the validity of inequality (9) may not be certain.*

Theorem 1.3. *For fixed point (r,s) such that $r, s \notin (0, \frac{3}{2})$ (or $r, s \in (0,1]$, resp.), the extended mean values $E(r,s;x,y)$ is Schur-concave (or Schur-convex, resp.) with (x,y) on the domain $(0,\infty) \times (0,\infty)$.*

2. Proofs of Theorems

Proof of Theorem 1.1. The function F is obviously symmetric.
Straightforward computation gives us

$$\left[\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right](y-x) = \left[\frac{p(y)f(y)+p(x)f(x)}{p(x)+p(y)} - \frac{\int_x^y p(t)f(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t}\right]\frac{p(x)+p(y)}{\int_x^y p(t)\mathrm{d}t}. \tag{10}$$

The proof follows from [**9**, 12.25. Theorem in p. 333] which can also be found in [**4**] and [**7**, p. 57]. ∎

Proof of Theorem 1.2. For $r = 0$, it is easy to see that equality in (9) holds for all $x, y > 0$.

Case 1. For $r < 0$, set $s = -r > 0$, then inequality (9) can be rewritten as

$$L\Big(\frac{1}{x^s}, \frac{1}{y^s}\Big) \ge \frac{x+y}{2} H\Big(\frac{1}{x^{s+1}}, \frac{1}{y^{s+1}}\Big), \tag{11}$$

which is equivalent to

$$\frac{y^s - x^s}{s(\ln y - \ln x)x^s y^s} \ge \frac{x+y}{x^{s+1} + y^{s+1}}. \tag{12}$$

From the logarithmic mean inequality $L(a,b) \ge \sqrt{ab}$ for $a, b > 0$ (see [**24**]), we have

$$\frac{y^s - x^s}{s(\ln y - \ln x)} \ge \sqrt{x^s y^s}. \tag{13}$$

Since the function $u(t) = t^{s+1}$ is convex on $(0, \infty)$ for $s > 0$, from definition of convex function it follows that

$$\frac{x^{s+1} + y^{s+1}}{2} \ge \Big(\frac{x+y}{2}\Big)^{s+1} \tag{14}$$

for $s > 0$. Combining (14) with the arithmetic-geometric mean inequality yields that

$$x^{s+1} + y^{s+1} \ge (x+y)\Big(\frac{x+y}{2}\Big)^s \ge (x+y)(\sqrt{xy})^s, \tag{15}$$

then we have

$$\frac{1}{\sqrt{x^s y^s}} \ge \frac{x+y}{x^{s+1} + y^{s+1}}. \tag{16}$$

Therefore, from (12), (13) and (16), it follows that

$$\frac{y^s - x^s}{s(\ln y - \ln x)x^s y^s} \ge \frac{1}{\sqrt{x^s y^s}} \ge \frac{x+y}{x^{s+1} + y^{s+1}}, \tag{17}$$

which implies inequality (8) for $r < 0$.

Case 2. If $r > 0$, without loss of generality, assume $y > x > 0$, then inequality (9) becomes

$$(y^r - x^r)(y^{r-1} + x^{r-1}) \le r(x+y)x^{r-1}y^{r-1} \ln \frac{y}{x}. \tag{18}$$

Dividing on both sides of (18) by x^{2r-1} produces

$$\left(\frac{y^r}{x^r} - 1\right)\left(\frac{y^{r-1}}{x^{r-1}} + 1\right) \le r\left(1 + \frac{y}{x}\right)\frac{y^{r-1}}{x^{r-1}} \ln \frac{y}{x}. \tag{19}$$

Let $\frac{y}{x} = t > 1$ and define a function $p(t)$ on $(1, \infty)$ such that

$$p(t) = (1 - t^r)(1 + t^{r-1}) + r(1 + t)t^{r-1} \ln t. \tag{20}$$

Direct and standard calculating leads to

$$\begin{aligned} p'(t) &= t^{r-2}[(2r-1)(1-t^r) + r(r-1+rt)\ln t] \triangleq t^{r-2} g(t), \\ g'(t) &= \frac{r(r-1) + r^2 t + r(1-2r)t^r + r^2 t \ln t}{t} \triangleq \frac{h(t)}{t}, \\ h'(t) &= r^2[2 + \ln t + (1-2r)t^{r-1}], \\ h''(t) &= \frac{r^2[1 + (1-2r)(r-1)t^{r-1}]}{t} \triangleq \frac{r^2 w(t)}{t}. \end{aligned} \tag{21}$$

Case 2.1. For $r \in [\frac{1}{2}, 1]$ the function $w(t) > 0$ and $h''(t) > 0$, then $h'(t)$ increases. Since $h'(1) = r^2(3-2r) > 0$, we have $h'(t) > 0$, and then $h(t)$ increases. Since $h(1) = 0$, thus $h(t) > 0$, and $g'(t) > 0$, and then $g(t)$ is increasing. From $g(1) = 0$ it follows that $g(t) > 0$, which means that $p'(t) > 0$ and $p(t)$ increases. Further, since $p(1) = 0$, we obtain $p(t) > 0$ for $r \in [\frac{1}{2}, 1]$ and $t \in (1, \infty)$. This implies that inequality (9) is reversed for $r \in [\frac{1}{2}, 1]$.

Case 2.2. For $r \geq \frac{3}{2}$, the function $w(t)$ decreases and $w(1) = r(3-2r) \leq 0$, and then $w(t) \leq 0$, and $h''(t) \leq 0$ and $h'(t)$ decreases. Since $h'(1) \leq 0$, we have $h'(t) \leq 0$, and $h(t)$ is decreasing. From $h(1) = 0$ it follows that $h(t) \leq 0$, and $g'(t) \leq 0$, and then $g(t)$ is dereasing. The fact that $g(1) = 0$ yields $g(t) \leq 0$, and $p'(t) \leq 0$, and then $p(t)$ is decreasing. The fact that $p(1) = 0$ results in $p(t) \leq 0$. This means that inequality (9) holds for $r \geq \frac{3}{2}$.

Case 2.3. For $0 < r < \frac{1}{2}$, it is easy to see that the function $w(t)$ is increasing. Since $w(1) = r(3-2r) > 0$, we obtain $w(t) > 0$, and $h''(t) > 0$, and then $h'(t)$ increases strictly. The fact that $h'(1) = r^2(3-2r) > 0$ leads to $h'(t) > 0$, and $h(t)$ increases. Meanwhile, $h(1) = 0$ produces $h(t) > 0$, and $g'(t) > 0$, and then $g(t)$ is increasing. since $g(1) = 0$, thus $g(t) > 0$ and $p'(t) > 0$, and then $p(t)$ is increasing. From $p(1) = 0$, it follows that $p(t) > 0$, that is, inequality (9) reverses for $r \in (0, \frac{1}{2})$.

Case 2.4. For $r \in (1, \frac{3}{2})$, the function $w(t)$ has a zero $t_0 = \frac{1}{[(r-1)(2r-1)]^{1/(r-1)}}$. Rearranging equality $w(1) = (1-2r)(r-1) + 1 = r(3-2r) > 0$ yields that $0 < (r-1)(2r-1) = 1 - w(1) < 1$, hence we have $t_0 > 1$.

In the case of $t \in (1, t_0)$, we have $w(t) > 0$ and $h''(t) > 0$, since $w(t)$ is decreasing for all $t > 1$ and $r \in (1, \frac{3}{2})$. By the same arguments as in Case 2.1, we obtain that inequality (9) is reversed when $\frac{y}{x} \in \big(1, 1/[(r-1)(2r-1)]^{1/(r-1)}\big)$, where $r \in (1, \frac{3}{2})$.

In the case of $t \in (t_0, \infty)$, we have $w(t) < 0$ and $h''(t) < 0$, and then $h'(t)$ decreases. It is easy to see that $\lim_{t\to\infty} h'(t) = -\infty$. Therefore, there exists a point t_1 such that $t_1 \geq t_0$ and $h'(t) < 0$ for $t \in (t_1, \infty)$. On the interval (t_1, ∞), the function $h(t)$ decreases and $\lim_{t\to\infty} h(t) = -\infty$. Similarly, there exists a number t_2 such that $t_2 \geq t_1$ and $h(t) < 0$ and $g'(t) < 0$ for $t \in (t_2, \infty)$. On the interval (t_2, ∞), the function $g(t)$ decreases and $\lim_{t\to\infty} g(t) = -\infty$. Then there exists another number $t_3 \geq t_2$ such that $g(t) < 0$ and $p'(t) < 0$, and then $p(t)$ is decreasing on the interval (t_3, ∞). Since $\lim_{t\to\infty} p(t) = -\infty$, then there exists a number $t_4 \geq t_3$ such that $p(t)$ is negative on the interval (t_4, ∞). This means that, for $\frac{y}{x} \in (t_4, \infty)$ and $r \in (1, \frac{3}{2})$, inequality (9) holds. Note that the numbers t_i, $0 \leq i \leq 4$, are all dependent on r undoubtedly.

Thus, for $r \in (1, \frac{3}{2})$, the validity of inequality (9) depends on values of the ratio $\frac{y}{x}$, that is, inequality (9) cannot hold for all $x, y > 0$. The proof is complete. ■

Proof of Theorem 1.3. To prove the Schur-convexity of the extended mean values, from Theorem 1.1, it suffices to prove the following inequality

$$\frac{g(r;x,y)}{g(s;x,y)} = \frac{\int_x^y t^{r-1} dt}{\int_x^y t^{s-1} dt} = \frac{s(y^r - x^r)}{r\,(y^s - x^s)} < \frac{x^{r-1} + y^{r-1}}{x^{s-1} + y^{s-1}}, \tag{22}$$

which is equivalent to the monotonicity with t of function $\frac{g(t;x,y)}{x^{t-1}+y^{t-1}}$, this is further reduced to the reversed inequality of (9), since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{g(t;x,y)}{(x^{t-1}+y^{t-1})}\right] = \frac{[\ln y - \ln x]\left[A(x,y)H(x^{t-1},y^{t-1}) - L(x^t,y^t)\right]}{t(x^{t-1}+y^{t-1})}. \tag{23}$$

Therefore, the proof of Theorem 1.3 follows. ■

3. A Corollary and an Open Problem

In this section, we shall deduce a corollary and pose an open problem.

Corollary 3.1. *Let $x, y > 0$. Then*

(i) *if $r, s \in (0, 1]$, we have*

$$E(r,s;x,y) \leq M_2((1,1);(x,y);r-1,s-1), \tag{24}$$

where $M_2((1,1);(x,y);r-1,s-1)$ denotes the generalized weighted mean of positive sequence (x,y) with two parameters $r-1$ and $s-1$ and constant weight $(1,1)$ defined by (2);

(ii) *if* $r, s \notin (0, \frac{3}{2})$, *inequality* (24) *reverses*;

(iii) *otherwise*, *the validity of inequality* (24) *may not be certain.*

Proof of Corollary 3.1. This follows from standard argument by combining (22) and Theorem 1.2 with (2) and definition of the extended mean values.

In fact, the inequality $E(r, s; x, y) \le M_2((1, 1); (x, y); r - 1, s - 1)$ can be rewritten as

$$\left[\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r}\right]^{1/(s-r)} \le \left[\frac{x^{r-1} + y^{r-1}}{x^{s-1} + y^{s-1}}\right]^{1/(r-s)},$$

which is equivalent to inequality (22). This follows from Theorem 1.2. ∎

At last, we propose the following open problem.

Open Problem 3.1. Under what conditions do the following inequalities

$$f\left(\frac{xp(x) + yp(y)}{p(x) + p(y)}\right) \le \frac{\int_x^y p(t) f(t) \mathrm{d}t}{\int_x^y p(t) \mathrm{d}t} \le \frac{p(x) f(x) + p(y) f(y)}{p(x) + p(y)} \tag{25}$$

hold for all $x, y \in I$? Here I denotes an interval on $\mathbb{R}$ and $p(x)$ is positive.

Acknowledgements

The authors would like to thank the anonymous referee for this valuable comments on improving this paper. This paper was finalized during the first author's visit to the RGMIA between November 1, 2001 and January 31, 2002 as a Visiting Professor with grants from the Victoria University of Technology and Jiaozuo Institute of Technology.

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