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EXISTENCE AND STABILITY OF THREE-DIMENSIONAL BOUNDARY-INTERIOR LAYERS FOR THE ALLEN-CAHN EQUATION

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Abstract. A minimal surface intersecting the boundary of a smooth bounded domain $\subset \mathbb{R}^3$, when it is *non-degenerate*, gives rise to a family of transition layer solutions of the Allen-Cahn equation. The stability properties of the transition layer solution are determined by the eigenvalues of the Jacobi operator on the minimal surface with Robin type boundary conditions which encode the geometric information of the domain boundary.

1. Introduction

We are interested in transition-layer solutions of the following scalar reaction-diffusion equation

$$\text{(1.1)} \qquad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f(u) \quad (\text{in } \Omega, t > 0) \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 \quad (\text{on } \partial\Omega, t > 0) \end{cases}$$

with the homogeneous Neumann boundary conditions. This system, called the Allen-Cahn equation, has been studied extensively for bistable reaction kinetics. A typical example of the nonlinearity f is a cubic polynomial $f(u) = u - u^3$. In general, we assume that the nonlinearity f is obtained from a double-well potential $F(u)$ of equal depth, $f(u) = -F'(u)$. Namely, $F(u)$ with $F(u) \geq 0$ is smooth and attains its absolute minimum at exactly two ***non-degenerate*** critical points $u = \pm 1$, $F(\pm 1) = 0$. The non-degeneracy here means that $F''(\pm 1) > 0$. These conditions ensure the existence of a special function $Q(z)$ ($z \in \mathbb{R}$), called a standing wave, which satisfies

$$\text{(S-W)} \qquad \frac{d^2 Q}{dz^2} + f(Q) = 0, \quad z \in \mathbb{R}, \qquad \lim_{z \to \pm\infty} Q(z) = \pm 1, \quad Q(0) = 0.$$

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The function $Q(z)$, together with its derivatives, will play important roles in the following discussion.

Throughout this article, the domain $\Omega \subset \mathbb{R}^N$ is smooth and bounded, $\mathbf{n}$ stands for the unit *inward* normal vector on $\partial\Omega$, and the parameter $\varepsilon > 0$ is small.

Our main concern in this paper is to show the existence of internal transition layers which exhibit a sharp transition from $u \approx -1$ to $u \approx +1$ across such a hypersurface Γ that intersects the boundary of the domain; $\overline{\Gamma} \cap \partial\Omega \neq \emptyset$. We call this kind of internal transition layer a *boundary-interior layer*. We also characterize the stability property of boundary-interior layers in terms of geometric information of $\Gamma, \partial\Omega$ and $\partial\Gamma \subset \partial\Omega$.

When $\varepsilon > 0$ is small, the solutions of (1.1) for a class of initial functions are known to develop transition layers within a short time scale of $O(\varepsilon^2|\log\varepsilon|)$ [3]. This phenomenon is caused by the strong bistability of the ordinary differential equation $u_t = \frac{1}{\varepsilon^2}f(u)$ with $u = \pm 1$ being stable equilibria. According to the sign of the value of the initial function, the solution is quickly attracted to either $u = +1$ or $u = -1$, thus creating a sharp transition from $u \approx -1$ to $u \approx 1$ near the set, called an interface,

$$\Gamma(t) := \{x \in \Omega \mid u^\varepsilon(u,t) = 0\}.$$

The interface divides Ω into two sub-domains $\Omega^\pm(t)$ (cf. Fig. 1) defined by $\Omega^\pm(t) :$ $= \{x \in \Omega \mid \pm u^\varepsilon(x,t) > 0\}$. When $x \in \Omega^\pm(t)$, $u^\varepsilon(x,t) \to \pm 1$ as $\varepsilon \to 0$. Such solutions with sharp spatial transition are called *transition layer* solutions.

It is also well known (cf. [3], for instance) that, to the lowest order of approximation, the interface $\Gamma(t)$ evolves according to its mean curvature:

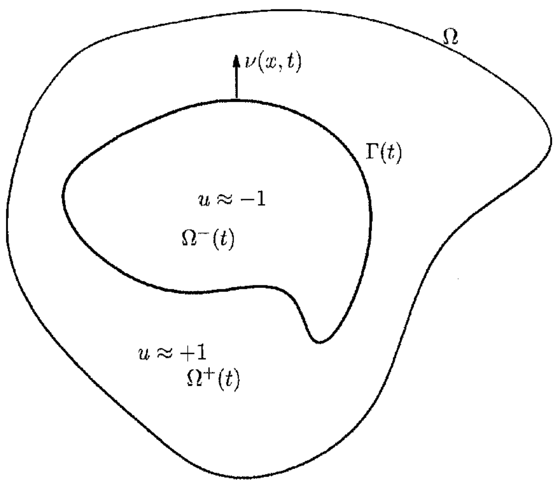


Fig. 1. The interface $\Gamma(t)$ and the normal vector $\nu(x,t)$.

$$\mathbf{V}_{\Gamma(t)}(x) = -\kappa\big(x;\Gamma(t)\big)\ (x \in \Gamma(t),\, t > 0\ ), \tag{1.2}$$

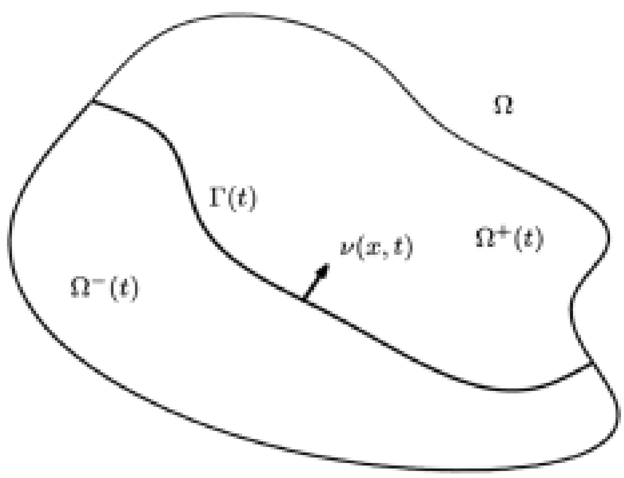
where $\mathbf{V}_{\Gamma(t)}(x)$ is the speed of the interface measured along the unit normal $\nu(x,t)$ of $\Gamma(t)$ at x (ν points to the $\Omega^+(t)$-side, cf. Fig. 1) and $\kappa(x;\Gamma)$ stands for the sum of the principal curvatures of Γ at $x \in \Gamma$. Hereafter, κ is simply called the mean curvature and the equation (1.2) is referred to as the mean curvature flow. To be precise about the sign of κ (which is the opposite to geometers' convention), let us extend the unit normal vector ν to a neighbourhood of Γ. Then our mean curvature is defined as the *divergence* of ν;

$$\kappa(x;\Gamma) = \mathrm{div}\ \nu(x), \quad x \in \Gamma.$$

When the interface $\Gamma(t)$ stays away from the boundary $\partial\Omega$, the dynamics of (1.2) has been studied rather extensively ([6, 8]). In such a case, the interface governed by the mean curvature flow (1.2) does not feel the presence of the boundary $\partial\Omega$. Therefore, the domain Ω does not play any role in the dynamics of (1.2).

Our concern in this paper, on the other hand, is the case in which the interface $\Gamma(t)$ intersects the boundary $\partial\Omega$ (cf. Fig. 2). The motion of $\Gamma(t)$ in such a situation is still described by the mean curvature flow (1.2) to the lowest order approximation. Main questions we raise in this article are:

> When (1.2) has an equilibrium interface which intersects the boundary of the domain Ω, does it give rise to an equilibrium boundary-interior layer for (1.1)? If the answer is affirmative, what is it that determines the stability of the layer?



. Fig. 2. The interface intersecting the boundary.

Since we have identified $\Gamma(t)$ as the 0-level set of the solution to (1.1), the homogeneous Neumann boundary conditions demand that $\Gamma(t)$ be perpendicular to $\partial\Omega$ at the intersection $\partial\Gamma(t) = \overline{\Gamma(t)} \cap \partial\Omega$. Therefore, the interface $\Gamma(t)$ immediately feels the presence of the boundary, and the geometry of $\partial\Omega$ certainly will influence the dynamics of (1.2).

The dynamics of interfaces intersecting the boundary of domain has been studied by several authors ([2, 14, 4, 5, 12, 15, 10]) from various viewpoints and by differing methods. The existence of energy-minimising solutions (of (1.1)) with interface intersecting the boundary was first rigorously established in [15] by a variational method. For competition-diffusion systems, stable internal layers intersecting the boundary was established in [12] for rotationally symmetric domains. Exponentially slow motions of flat interfaces are discussed in [2, 14], where interfaces intersect flat parallel parts of the boundary. Motions of interfaces with contact angle was treated in [4] for a generalized mean curvature flow. Dynamics of flat interfaces in a strip-like domain was discussed in [5], where the speed of the interface is of order $O(\epsilon^2)$ with respect to the time scale of (1.1). In [10], the existence and stability of equilibrium boundary-interior layers with flat interfaces were established. Recently, the same results as [10] have been obtained by [16] via a different method. In all of these works, the geometry of the boundary $\partial\Omega$ has essential effects on the dynamics of (1.1).

2. Main Results

The purpose of this article is to extend the results in [10] and [16] to 3-dimensional domains.

2.1. General Domains

The most difficult part of all to obtain results similar to the main theorems in [10] and [16] for general 3-dimensional domains is to find a minimal surface that intersects $\partial\Omega$ in the right angle. We therefore assume the existence of such a minimal surface. Later, we will exhibit some special situations in which the existence of such minimal surfaces are easily established.

> **(A1)**: There exists a minimal interface $\Gamma \subset \Omega$ that is smooth, embedded and intersects $\partial\Omega$ in the right angle along its boundary $\partial\Gamma = \overline{\Gamma} \cap \partial\Omega$.

As in [10, 16] the existence of minimal surfaces as in **(A1)** alone is not enough to ensure the existence of boundary-interior layers. We need some kind of non-degeneracy condition imposed on Γ. In order to state such a condition, let us consider an eigenvalue problem defined on Γ:

$$\text{(2.1)} \qquad \begin{cases} \Delta^\Gamma v + (\kappa_1^2 + \kappa_2^2)v = \lambda v & \text{in } \Gamma, \\ \partial v(y)/\partial \mathbf{n} - \overline{\kappa}(y) v(y) = 0 & \text{on } \partial\Gamma, \end{cases}$$

where Δ^Γ is the Laplace-Beltrami operator on Γ, κ_j ($j = 1, 2$) the principal curvatures of Γ, and

$$\text{(2.2)} \qquad \overline{\kappa}(y) = \left\langle \frac{\partial \mathbf{n}}{\partial \nu}, \nu \right\rangle, \quad y \in \partial\Gamma \subset \partial\Omega.$$

We recall again that $\mathbf{n}$ is the *inward* unit normal vector on $\partial\Omega$, and hence, it is the unit normal vector on $\partial\Gamma$ tangent to Γ because of the perpendicularity of $\overline{\Gamma}$ and $\partial\Omega$. Since a curve on the surface $\partial\Omega$ is a geodesic if and only if its normal vector is parallel to the normal vector $\mathbf{n}$ of $\partial\Omega$. Therefore, $\overline{\kappa}(y)$ is the curvature of the geodesic on $\partial\Omega$ passing through $y \in \partial\Gamma$ in the direction $\nu(y)$.

Let us denote by σ_Γ the set of *distinct* eigenvalues for (2.1);

$$\sigma_\Gamma = \{\lambda_j\}_{j=0}^\infty, \quad \lambda_0 > \lambda_1 > \ldots > \lambda_j > \ldots \to -\infty.$$

The multiplicity of λ_j is denoted by m_j.

The non-degeneracy condition for Γ is:

(A2): $\quad 0 \notin \sigma_\Gamma$.

Our main result is the following.

Theorem 2.1. (Existence and stability of boundary-interior layers) *Assume that conditions* **(A1)** *and* **(A2)** *are satisfied. Then there exist an $\varepsilon_* > 0$ and a family of equilibrium solutions $U^\varepsilon(x)$ of (1.1) defined for $\varepsilon \in (0, \varepsilon_*]$ with the following properties.*

(i) *For each $\delta > 0$,*

$$\lim_{\varepsilon \to 0} U^\varepsilon(x) = \begin{cases} 1 \\ -1 \end{cases} \text{uniformly in} \begin{cases} x \in \Omega^+ \backslash \Gamma^\delta, \\ x \in \Omega^- \backslash \Gamma^\delta, \end{cases}$$

where $\Gamma^\delta = \{x \in \Omega \mid \operatorname{dist}(x, \Gamma) < \delta\}$ is the δ-neighborhood of Γ in Ω.

(ii) *Near the interface Γ, the solution U^ε has the following behavior*

$$U^\varepsilon(x) \approx Q\left(\frac{\operatorname{dist}(x, \Gamma)}{\varepsilon}\right),$$

where $Q(z)$ is the standing wave (cf. (S-W) *in §1).*

(iii) *If $0 > \lambda_0$, then U^ε is asymptotically stable with respect to (1.1).*

(iv) *If there exits $j \geq 0$ satisfying $\lambda_j > 0 > \lambda_{j+1}$, then U^ε is unstable with Morse index equal to $\sum_{k=0}^j m_k$.*

It is illuminating to put the results of Theorem 2.1 in a variational formulation. Let us define the class of admissible interfaces;

$$\mathcal{A}_\Omega := \{\Gamma \mid \overline{\Gamma} \text{ is a } C^2 \text{ surface with } \overline{\Gamma} \cap \partial\Omega = \partial\Gamma \text{ and } \Gamma \subset \Omega\}.$$

Let $\mathcal{S} : \mathcal{A}_\Omega \to \mathbb{R}$ be the surface area functional. The problem (1.2) is nothing but the gradient flow with respect to the functional $\mathcal{S}(\Gamma)$;

$$\frac{\partial \Gamma}{\partial t} = -\frac{\delta \mathcal{S}(\Gamma)}{\delta \Gamma} = -\kappa(x;\Gamma),$$

where the interface Γ varies within the class $\mathcal{A}_\Omega$ of admissible surfaces. Critical points of $\mathcal{S}(\Gamma)$ are characterized as

$$\kappa(x;\Gamma) \equiv 0 \quad \text{and} \quad \Gamma \perp_{\partial\Gamma} \partial\Omega. \tag{2.3}$$

Moreover, (2.1) is an eigenvalue problem associated with the second variation of the functional $\mathcal{S}$ at the critical point $\Gamma \in \mathcal{A}_\Omega$ satisfying (2.3). Therefore we may restate Theorem 2.1 as follows (cf. Fig. 3):

> *A non-degenerate critical point $\Gamma \in \mathcal{A}_\Omega$ of $\mathcal{S}$ gives rise to an equilibrium boundary-interior layer of* (1.1). *The Morse index of the boundary-interior layer is the same as that of Γ with respect to the area functional $\mathcal{S}$.*

An interesting implication of Theorem 2.1 is that the boundary-interior layer with transition layers occurring near any *Plateau stable* minimal hypersurface $\Gamma \in \mathcal{A}_\Omega$, with $\Gamma \perp_{\partial\Gamma} \partial\Omega$, can be made stable by deforming the boundary $\partial\Omega$ near $\partial\Gamma$ so that

$$\inf_{y \in \partial\Gamma} \overline{\kappa}(y) =: \overline{\kappa}_0 \gg 1.$$

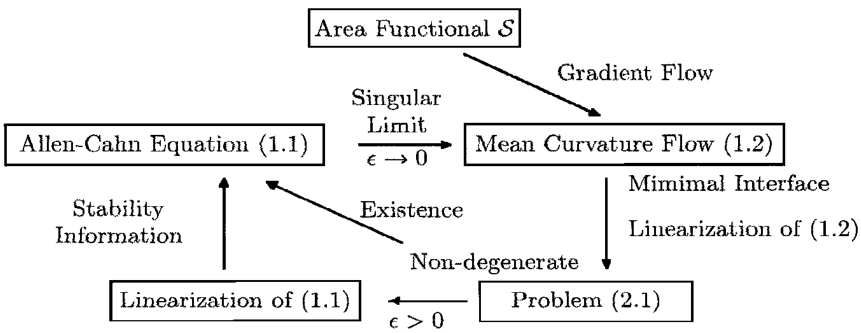


Fig. 3. Non-degenerate critical point of $\mathcal{S}$ give rise to boundary-interior layers.

A minimal surface is, by definition, *Plateau stable* if the principal eigenvalue λ_0^D of the associated Dirichlet eigenvalue problem

$$\begin{cases} \Delta^\Gamma \phi + (\kappa_1^2 + \kappa_2^2)\phi = \lambda\phi & \text{in } \Gamma, \\ \phi(y) = 0 & \text{on } \partial\Gamma \end{cases}$$

is *negative.* Let us denote by $\lambda_0(p)$, $p \in \mathbb{R}$, the principal eigenvalue of

$$\begin{cases} \Delta^\Gamma \phi + (\kappa_1^2 + \kappa_2^2)\phi = \lambda\phi & \text{in } \Gamma, \\ \partial\phi(y)/\partial\mathbf{n} - p\phi(y) = 0 & \text{on } \partial\Gamma. \end{cases}$$

One can readily verify that $\lambda_0(p)$ is monotone decreasing in $p \in \mathbb{R}$ and that $\lim_{p\to\infty} \lambda_0(p) = \lambda_0^D$. On the other hand, the principal eigenvalue λ_0 of (2.1) satisfies $\lambda_0 \leq \lambda_0(\overline{\kappa}_0)$. If Γ is Plateau stable, i.e., if $\lambda_0^D < 0$, the by choosing $\overline{\kappa}_0 > 0$ large, we obtain $\lambda_0 < 0$, showing the stability of U^ϵ thanks to Theorem 2.1. We summarize this as follows.

Corollary 2.1. *Let Γ be a minimal surface as in* **(A1)***.*

(i) *If Γ is Plateau stable, then one can deform the boundary $\partial\Omega$ of domain so that the corresponding boundary-interior layer is stable with respect to* (1.1).

(ii) *If Γ is not Plateau stable, then the associated boundary-interior layer can never be stable as an equilibrium solution of* (1.1), *no matter how one deforms the boundary $\partial\Omega$ of domain.*

2.2. Rotationary-symmetric Domains

We first apply Theorem 2.1 to a special class of domains; *rotationally symmetric domains*. Let the axis of rotation be in x-direction ($x \in \mathbb{R}$ here and below within §2.2), and consider a domain $\Omega \subset \mathbb{R}^3$ which (or, part of which) is obtained by rotating the graph of a *positive* function $\psi(x)$ around x-axis:

$$\Omega = \{(x, y) \in \mathbb{R}^3 \mid y \in \mathbb{R}^2, \ \ |y| < \psi(x)\}. \tag{2.4}$$

In this situation it is easy to find an equilibrium to (1.2).

Proposition 2.1. **(Existence of flat disk-type interfaces)** *Let $x_0 \in \mathbb{R}$ satisfy $\psi'(x_0) = 0$. Then the disk $\Gamma = \{x_0\} \times \Omega_{x_0} := \{(x_0, y) \mid |y| < \psi(x_0)\}$ is an equilibrium solution of* (1.2).

We have therefore a situation in which the condition **(A1)** is verified.

In order to see if the condition **(A2)** is satisfied, let us consider an eigenvalue problem:

$$(2.5)\qquad \begin{cases} \Delta_y \phi = \lambda\phi & \text{in } \Omega_{x_0} := \{y \in \mathbb{R}^2 \mid |y| < \psi(x_0)\}, \\ \partial\phi/\partial\mathbf{n}' - \psi''(x_0)\phi = 0 & \text{on } S_{x_0} := \{y \in \mathbb{R}^2 \mid |y| = \psi(x_0)\}, \end{cases}$$

where $\mathbf{n}'$ is the inward unit normal vector on S_{x_0}. Let us denote by σ_{x_0} the set of eigenvalues of (2.5). Applying to (2.5) the procedure of separation of variables, one finds that $\lambda \in \mathbb{R}$ belongs to σ_{x_0} if and only if the boundary value problem

$$(2.6)\qquad \begin{cases} \phi_{rr} + \frac{1}{r}\phi_r - \frac{k^2}{r^2}\phi = \lambda\phi, \quad r \in (0, \psi(x_0)), \\ -\phi_r(\psi(x_0)) - \psi''(x_0)\phi(\psi(x_0)) = 0, \\ \phi(r) \text{ is bounded on } [0, \psi(x_0)] \end{cases}$$

has a nontrivial solution for some $k = 0, 1, 2, \ldots$. For $\lambda = 0$, (2.6) has a nontrivial solution $\phi(r) = r^k$ if and only if

$$k + \psi(x_0)\psi''(x_0) = 0.$$

Namely, **(A2)**: $0 \notin \sigma_{x_0}$ is realized if and only if

$$(2.7)\qquad -\psi(x_0)\psi''(x_0) \notin \{0, 1, 2, 3, \ldots\}.$$

Therefore, we can apply Theorem 2.1 if (2.7) is fulfilled. In order to count the Morse index, however, it is more convenient to view (2.5) from a different angle.

Let us define the **Dirichlet-to-Neumann map** Π for the Laplacian:

$$\Pi : C^{2+\alpha}(S_{x_0}) \longrightarrow C^{1+\alpha}(S_{x_0}); \qquad \Pi\phi(y) := \frac{\partial v}{\partial \mathbf{n}}(y), \quad y \in S_{x_0},$$

where $v(y)$ is the unique solution of the boundary value problem:

$$\Delta_y v = 0, \quad y \in \Omega_{x_0}, \qquad v(y) = \phi(y), \quad y \in S_{x_0}.$$

Namely, to a given *Dirichlet data* $\phi \in C^{2+\alpha}(S_{x_0})$ on S_{x_0}, the map Π asigns the *Neumann data* $\partial v/\partial\mathbf{n}$ of the harmonic extension v of ϕ. It is known that the map Π is a first order *elliptic* operator on S_{x_0}. In the present case, the operator is exactly given by

$$\Pi = -\sqrt{-\Delta^{S_{x_0}}},$$

and extends to an unbounded operator on $L^2(S_{x_0})$. Let us denote by $\sigma(\Pi)$ the set of eigenvalues of Π:

$$(2.8)\qquad \sigma(\Pi) = \{\mu_j\}_{j=0}^{\infty}; \qquad 0 = \mu_0 > \mu_1 > \ldots > \mu_j > \ldots \to -\infty,$$

where we only listed *distinct* eigenvalues. We denote by m_j the *multiplicity* of μ_j. By using the separation of variables, one can easily compute these eigenvalues and their multiplicities;

$$\mu_j = -j/\psi(x_0) \ \ (j \geq 0), \quad m_0 = 1, \ \ m_j = 2 \ \ (j \geq 1).$$

To see the relationship between σ_{x_0} and $\sigma(\Pi)$, let us consider the following boundary value problem for a parameter $\mu \in \mathbb{R}$.

$$\begin{cases} \Delta_y \phi = \lambda \phi & \text{in } \Omega_{x_0}, \\ \partial\phi/\partial\mathbf{n}' - \mu\phi = 0 & \text{on } S_{x_0}, \end{cases} \tag{2.9}$$

We denote by $\{\lambda_j(\mu)\}_{j-0}^{\infty}$, $\lambda_0(\mu) > \lambda_1(\mu) > \ldots$, the distinct eigenvalues of (2.9), and by $\overline{m}_j(\mu)$ the multiplicity of $\lambda_j(\mu)$. The variational characterization of eigenvalues of (2.9) implies that $\lambda_j(\mu)$ is strictly monotone decreasing in μ for each j, and that $\lambda_j(\mu) > 0$ as $\mu \to -\infty$. On the other hand, the definition of the Dirichlet-to-Neumann map implies that $\lambda_j(\mu_j) = 0$ and $m_j = \overline{m}_j(\mu)$. Therefore, the number of positive eigenvalues (counted with multiplicity) of (2.9) is equal to

$$\sum_{\mu_j > \mu} \overline{m}_j(\mu) = \sum_{\mu_j > \mu} m_j.$$

Since we have

$$\sigma_{x_0} = \{\lambda_j(\psi''(x_0))\}_{j=0}^{\infty}, \qquad \lambda_0(0) = 0,$$

the number of positive eigenvalues of (2.5) equals

$$\sum_{j=0}^{k} m_j \quad \text{if} \quad \mu_k > \psi''(x_0) > \mu_{k+1}.$$

We are ready to state:

Theorem 2.2. (Existence and stability of flat layers) *Assume that x_0 is such that $\psi'(x_0) = 0$ and the following non-degeneracy condition is satisfied*

$$\text{(a):} \qquad \psi''(x_0) \notin \sigma(\Pi).$$

Then there exist an $\varepsilon_* > 0$ and a family of equilibrium solutions $U^{\varepsilon}(x, y)$ of (1.1) for $\varepsilon \in (0, \varepsilon_*]$, enjoying the following properties:

(i) *For each $\delta > 0$,*

$$\lim_{\varepsilon \to 0} U^{\varepsilon}(x, y) = \begin{cases} 1 \\ -1 \end{cases} \begin{cases} (x, y) \in \bar{\Omega}, \ x \leq x_0 - \delta, \\ (x, y) \in \bar{\Omega}, \ x \geq x_0 + \delta. \end{cases}$$

(ii) *Near $x = x_0$, the solution $U^{\varepsilon}(x, y)$ has the asymptotic characterization*:

$$U^{\varepsilon}(x, y) \approx Q\left(\frac{x - x_0}{\varepsilon}\right).$$

(iii) *As an equilibrium solution of* (1.1), $U^{\varepsilon}(x,y)$ *is*

(1) *stable if* $\psi''(x_0) > 0 = \mu_0$,
(2) *unstable if* $\mu_j > \psi''(x_0) > \mu_{j+1}$ *with the Morse index equal to* $\sum_{k=0}^{j} m_k$.

3. Proof of Theorem 2.1.

We give a proof to Theorem 2.1 in the case of disk type interfaces Γ by using the method employed in [10]. For other types of interfaces, the proof is essentially the same. Moreover, the proof to be presented below works equally well for higher $(N \geq 4)$ dimensional domains.

3.1. Coordinate System Near the Interface

Our method of proof consists of two steps: (1) to construct approximate solutions with desired properties, and (2) to find a solution near the approximation. For the first step, we need to work with a suitable coordinate system near the minimal interface.

Let $\gamma_0(\cdot) : \overline{\mathbb{D}} \to \Gamma \subset \Omega$ be a smooth parametrization of the interface Γ, where Γ is the miminal interface appeared in **(A1)** in Section 2.1 and $\mathbb{D} := \{y \in \mathbb{R}^2 \mid |y| < 1\}$ is the unit disk. We extend γ_0 smoothly to $\mathbb{D}_\delta = \{y \in \mathbb{R}^2 \mid |y| < 1+\delta\}$ for some fixed constant $\delta > 0$. The extension is still denoted by γ_0 and its image by Γ_δ. Let $\nu(y) \in \mathbb{R}^3$ be the unit normal vector of Γ_δ at $\gamma_0(y) \in \Gamma_\delta$. We now define a neighborhood $\Omega_\delta^{r_0}$ of Γ_δ by

$$\Omega_\delta^{r_0} := \{x \in \mathbb{R}^3 \mid x = \gamma_0(y) + r\nu(y),\ |r| < r_0,\ y \in \mathbb{D}_\delta\} \tag{3.1}$$

for some fixed constant $r_0 > 0$. When we deal with the portion of $\partial\Omega$ in $\Omega_\delta^{r_0}$, we use coordinate $(\theta, \rho) \in \partial\mathbb{D} \times [0, \delta)$ on $\mathbb{D}$ where $(\theta, \rho = 0)$ is sent to the boundary of interface $\partial\Gamma$ by γ_0 and

$$\left\langle \frac{\partial \gamma_0}{\partial \theta}, \frac{\partial \gamma_0}{\partial \rho} \right\rangle = 0 \quad \text{for} \ \ \rho = 0$$

is satisfied.

Lemma 3.1. *There exist constants* $\delta > 0, r_0 > 0$, *which depend only on* Γ *and* $\partial\Omega$, *and a smooth diffeomorphism*

$$\gamma(\cdot,\cdot) : (-r_0, r_0) \times \overline{\mathbb{D}} \to \overline{\Omega} \cap \Omega_\delta^{r_0}$$

such that

(i) $\gamma(0,y)=\gamma_0(y)$ *for* $y\in\overline{\mathbb{D}}$, $\gamma(r,y)\in\partial\Omega$ *for* $y\in\partial\mathbb{D}$, $-r_0<r<r_0$;

(ii) $\gamma_r(0,y)=\nu(y)$ *for* $y\in\overline{\mathbb{D}}$;

(iii) *as* $r\to 0$, $\gamma(r,y)$ *has the following expansion*

$$\gamma(r,y)=\gamma_0(y)+r\nu(y)+\frac{r^2}{2}p(y)+\frac{r^3}{6}q(y)+O(r^4),\qquad y\in\overline{\mathbb{D}}$$

where $p(y)$ *and* $q(y)$ *are vector functions orthogonal to* $\nu(y)$.

(iv) *If we write* γ *as* $\gamma(r,\theta,\rho)$ *in terms of the coordinates* (r,θ,ρ), *then the derivative along the inward unit normal vector* $\mathbf{n}$ *of* $\partial\Omega$ *is expressed as*

$$\frac{\partial}{\partial\mathbf{n}}=\frac{1}{\sqrt{\tilde g^{33}}}\left(\tilde g^{13}\frac{\partial}{\partial r}+\tilde g^{23}\frac{\partial}{\partial\theta}+\tilde g^{33}\frac{\partial}{\partial\rho}\right)$$

where at $(r,\theta,\rho=0)$

$$\tilde g^{13}(r,\theta)=r\left(-\left|\frac{\partial\gamma_0}{\partial\rho}\right|^{-2}\left\langle p,\frac{\partial\gamma_0}{\partial\rho}\right\rangle\right)+O(r^2),$$

$$\tilde g^{23}(r,\theta)=r\left(2\left|\frac{\partial\gamma_0}{\partial\theta}\right|^{-2}\left|\frac{\partial\gamma_0}{\partial\rho}\right|^{-2}\left\langle\frac{\partial\gamma_0}{\partial\theta},\frac{\partial\nu}{\partial\rho}\right\rangle\right)+O(r^2),$$

$$\tilde g^{33}(r,\theta)=\left|\frac{\partial\gamma_0}{\partial\rho}\right|^{-2}+r\left(2\left|\frac{\partial\gamma_0}{\partial\rho}\right|^{-4}\left\langle\frac{\partial\gamma_0}{\partial\rho},\frac{\partial\nu}{\partial\rho}\right\rangle\right)+O(r^2).$$

The proof will be given in §4.

3.2. Approximate Solutions

Let us construct approximate solutions to the boundary value problem

$$\begin{cases}\varepsilon^2\Delta u+f(u)=0 & \text{in }\ \Omega,\\[4pt] \dfrac{\partial u}{\partial\mathbf{n}}=0 & \text{on }\ \partial\Omega.\end{cases}\tag{3.2}$$

3.2.1. Outer expansion

Let $\Omega^{\pm}$ be two subdomains of Ω divided by the minimal interface Γ. For (3.2), our *outer* solution is very simple and is given by

$$u_{\text{out}}(x)=\begin{cases}-1, & x\in\Omega^-,\\ +1, & x\in\Omega^+.\end{cases}\tag{3.3}$$

We always agree that the unit normal ν on Γ is pointing into Ω^+-region.

3.2.2. Inner expansion

We now construct *inner* solutions which bridge the gap of the outer solution u_{out} along Γ. For this purpose, we use the coordinate system (r, y) introduced in §3.1. Since we need to deal with the jump of u_{out} from $u = -1$ to $u = +1$, we further introduce a stretched variable s in the r-coordinate by $r = \varepsilon s$.

Lemma 3.2. *In terms of the coordinate system* (s, y), *the differential equation in* (3.2) *is expressed as*

$$(3.4)\qquad \begin{aligned} 0 = {} & \frac{\partial^2 u}{\partial s^2} + f(u) + \varepsilon\kappa(y)\frac{\partial u}{\partial s} \\ & + \varepsilon^2\left[\Delta^\Gamma - \{\kappa_1(y)^2 + \kappa_2(y)^2\}s\frac{\partial u}{\partial s} - 2\nabla_p^\Gamma(s\frac{\partial u}{\partial s}) - \nabla_p^\Gamma u\right] \\ & + \sum_{j\ge 3}\varepsilon^j P_j(s, y)u, \end{aligned}$$

where

$\kappa_k(y)$ $(k = 1, 2)$ *are principal curvatures of* Γ *at* $\gamma_0(y)$;

$\kappa(y)$ *is the sum of principal curvatures of* Γ *at* $\gamma_0(y)$ (*mean curvature*);

Δ^Γ *is the Laplace-Beltrami operator on* Γ;

∇_p^Γ *is the gradient operator in* p*-direction*;

$P_j(s, y)$ *are differential operators in* (s, y).

This will be proved in §4.

Note that $O(\varepsilon)$-term in (3.4) is actually absent because Γ is minimal (cf. **(A1)**).

Let us now substitute a formal expression

$$(3.5)\qquad u_{\text{in}}^\varepsilon(s, y) = \sum_{j\ge 0}\varepsilon^j u^j(s, y)$$

into (3.4). Equating like powers of ε in the resulting equation, we obtain a series of equations for u^j ($j \ge 0$).

$$(3.6)\qquad u_{ss}^0 + f(u^0) = 0,$$

$$(3.7)\qquad u_{ss}^j + f'(u^0(s))u^j = h_j(s, y; u^0, \ldots, u^{j-1}) \qquad (j \ge 1).$$

We consider these equations on $(s,y) \in \mathbb{R} \times \overline{\mathbb{D}}$ with boundary conditions

$$|u^0(s,y) - (\pm 1)| = O(e^{-d_0|s|}) \text{ as } s \to \pm\infty, \tag{3.8}$$

$$|u^j(s,y)| = O(e^{-d_0|s|}) \text{ as } s \to \pm\infty \text{ for } j \geq 1, \tag{3.9}$$

where $d_0 > 0$ is some constant satisfying $d_0 < \sqrt{F''(\pm 1)}$. These conditions, called *inner-outer matching conditions*, are imposed in order to join inner solutions to the outer solution u_{out} in a compatible way.

Lemma 3.3.

(i) *The equation* (3.6) *with the boundary condition* (3.8) *has a unique solution* $u^0(s,y) = Q(s+\alpha(y))$, *where* $Q(z)$ *is the standing-wave appeared in* §*1 and* $\alpha(y)$ *is an arbitrary function of* $y \in \overline{\mathbb{D}}$.

(ii) *The equation* (3.7) *with the boundary condition* (3.9) *for* $j=1$ *has a unique solution* $u^1(s,y) = c_1(y)u^0_s(s,y)$, *where* $c_1(y)$ *is an arbitrary function.*

(iii) *The equation* (3.7) *with the boundary condition* (3.9) *for* $j=2$ *has a unique family of solutions*

$$u^2(s,y) = c_2(y)u^0_s(s,y) + \overline{u}^2(s,y),$$

if and only if $\alpha(y) \equiv \alpha_*$ *with* α_* *being the unique value for which*

$$\int_{-\infty}^{\infty} s\,(Q_s(s+\alpha_*))^2\,ds = 0$$

is realized, where $c_2(y)$ *is an arbitrary function and* $\overline{u}^2$ *is a function which depends only on* c_1 *and* $u^0(s,y)$. *Therefore,* $u^0(s,y) \equiv u^0(s) = Q(s+\alpha_*)$ *and* $u^1(s,y) = c_1(y)u^0_s(s)$.

(iv) *For* $j \geq 3$, *the equation* (3.7) *with the boundary condition* (3.9) *has a unique family of solutions*

$$u^j(s,y) = c_j(y)u^0_s(s) + \overline{u}^j(s,y;c_1,\ldots,c_{j-1})$$

if and only if c_{j-2} *satisfies*

$$\Delta^\Gamma c_1 + (\kappa_1(y)^2 + \kappa_2(y)^2)c_1 = \overline{h}_1(y) \quad in\ \Gamma, \tag{3.11}$$

for $j=3$ *and*

$$\Delta^\Gamma c_{j-2} + (\kappa_1(y)^2 + \kappa_2(y)^2)c_{j-2} = \overline{h}_{j-2}(y;c_1,\ldots,c_{j-3}) \quad in\ \Gamma, \tag{3.12}$$

for $j \geq 4$.

By using (3.4), the proof of Lemma 3.3 is almost identical to that in §2.1 of [10], and hence omitted.

We thus obtain the inner expansion $u_{\rm in}^{\epsilon}(s,y) = \sum \varepsilon^j u^j(s,y)$, as soon as the functions $c_j(y)$ ($j \geq 1$) satisfying (3.11) and (3.12) are found. In order to determine c_j uniquely, we need to supply boundary conditions to (3.11) and (3.12). These boundary conditions will naturally emerge in the boundary correction.

3.2.3. Boundary correction

If we arbitrarily choose c_j satisfying (3.11) and (3.12), then we obtain an inner expansion $u_{\rm in}^{\varepsilon}$. This approximation in general does not satisfy the boundary condition in (3.2). In order to remedy the defect, we add boundary corrections to the inner expansion:

$$u^{\varepsilon}(s,y) = u_{\rm in}^{\varepsilon}(s,\theta,\varepsilon\eta) + \sum_{j\geq 1} \varepsilon^j b^j(s,\theta,\eta) \tag{3.13}$$

where a stretched coordinate in ρ-direction η is introduced by $\rho = \varepsilon\eta$. As we will see in the following, in order for b^j to be determined so that the expression in (3.13) satisfies the homogenous Neumann boundary conditions, c_j have to satisfy certain boundary conditions which are the desired conditions supplementing (3.11) and (3.12).

Lemma 3.4. *In terms of the coordinate system* (s,θ,η), *the equation in* (3.2) *is expressed as*

$$\begin{aligned}
0 &= \frac{\partial^2 u}{\partial s^2} + \frac{1}{l_2(\theta)^2}\frac{\partial^2 u}{\partial \eta^2} + f(u) \\
&\quad + \varepsilon\Big\{-2\eta\frac{A(\theta)}{l_2(\theta)^4}\frac{\partial^2 u}{\partial \eta^2} - 2s\frac{C(\theta)}{l_1(\theta)^2 l_2(\theta)^2}\frac{\partial^2 u}{\partial s \partial \eta} \\
&\quad + \left(\frac{B(\theta)}{l_1(\theta)^2 l_2(\theta)^2} - \frac{A(\theta)}{l_2(\theta)^4} - \frac{C(\theta)}{l_1(\theta)^2 l_2(\theta)^2}\right)\frac{\partial}{\partial \eta}\Big\} \\
&\quad + \sum_{j\geq 2}\varepsilon^j \tilde{P}_j(s,\theta,\eta)u,
\end{aligned} \tag{3.14}$$

where

$$l_1(\theta) = \left|\frac{\partial \gamma_0(\theta,0)}{\partial \theta}\right|, \quad l_2(\theta) = \left|\frac{\partial \gamma_0(\theta,0)}{\partial \rho}\right|,$$

$$A(\theta) = \left\langle \frac{\partial^2 \gamma_0(\theta,0)}{\partial \rho^2}, \frac{\partial \gamma_0(\theta,0)}{\partial \rho}\right\rangle,$$

$$B(\theta)=\left\langle\frac{\partial^2\gamma_0(\theta,0)}{\partial\theta\partial\rho},\ \frac{\partial\gamma_0(\theta,0)}{\partial\theta}\right\rangle,\quad C(\theta)=\left\langle p(\theta,0),\ \frac{\partial\gamma_0(\theta,0)}{\partial\rho}\right\rangle$$

$\tilde{P}_j$ *are second order differential operators in* (s,θ,η).

The boundary condition in (3.2) *is recast as*

(3.15)
$$0=\frac{\partial u}{\partial\eta}+\varepsilon s\left\{-\left\langle p,\frac{\partial\gamma_0}{\partial\rho}\right\rangle\frac{\partial u}{\partial s}+2\frac{1}{l_2(\theta)^2}\left\langle\frac{\partial\gamma_0}{\partial\rho},\frac{\partial\nu}{\partial\rho}\right\rangle\frac{\partial u}{\partial\eta}\right\}+\sum_{j\geq 2}\varepsilon^j\hat{P}_j(s,\theta)u,$$

where $\hat{P}_j(s,\theta)$ *are first order differential operators in* (s,θ,η) *with coefficients depending only on* (s,θ).

The proof will be given in §4.

Substituting (3.13) into (3.14) and (3.15), equating like powers of ε in the resulting equation and taking into account the equations satisfied by the inner expansion, we obtain the following boudary value problems for $j=1,2,\ldots$.

(3.16)
$$\frac{\partial^2 b^j}{\partial s^2}+\frac{1}{l_2(\theta)^2}\frac{\partial^2 b^j}{\partial\eta^2}=\ell_j(s,\theta,\eta),\quad (s,\theta,\eta)\in\mathbb{R}\times\partial\mathbb{D}\times[0,\infty),$$

(3.17)
$$\frac{\partial b^j}{\partial\eta}(s,\theta,0)=\hat{\ell}_j(s,\theta),\quad (s,\theta)\in\mathbb{R}\times\partial\mathbb{D},$$

(3.18)
$$b^j(s,\theta,\eta)=O(e^{-d_0|s|}e^{-d_0\eta})\ \text{ as }\ s\to\pm\infty\ \text{ and }\ \eta\to\infty,$$

where $d_0>0$ is the same constant as appeared in (3.8) and (3.9). In the last three equations, θ is considered as a parameter. The conditions in (3.18), called *boundary-inner matching* conditions, are imposed so that the boundary correction terms are joined smoothly to the inner expansion.

Lemma 3.5.

(i) *For* $j=1$, *the problem* (3.16)-(3.17)-(3.18) *has a unique solution* $b^1(s,\theta,\eta)$ *because of our choice of* $u^0(s)$ *as in* Lemma 3.3 (iii)

$$\int_{-\infty}^{\infty}s(u_s^0(s))^2ds=0.$$

(ii) *For* $j\geq 2$, *the problem* (3.16)-(3.17)-(3.18) *has a unique solution* $b^j(s,\theta,\eta)$ *if and only if* $c_j(y)$ *appeared in Lemma 3.3 satisfies*

(3.19) $$\frac{\partial c_1}{\partial \mathbf{n}} - \left\langle \frac{\partial \mathbf{n}}{\partial \nu},\ \nu \right\rangle c_1 = k_1(\theta) \ \ on\ \ \partial\mathbb{D} \qquad (j=1),$$

(3.20) $$\frac{\partial c_j}{\partial \mathbf{n}} - \left\langle \frac{\partial \mathbf{n}}{\partial \nu},\ \nu \right\rangle c_j = k_j(\theta; c_1, \ldots, c_{j-1}) \ \ on\ \ \partial\mathbb{D} \qquad (j \geq 2),$$

where k_j ($j \geq 1$) are some functions of variables indicated.

By using (3.14) and (3.15), the proof of this lemma is almost identical to that in §2.2 of [10]. Therefore, we omit the proof.

Since (2.1) has no 0-eigenvalue thanks to the condition **(A2)**, we find that the boundary value problems (3.11)-(3.19) and (3.12)-(3.20) have unique solutions $c_j(y)$ for $j = 1, 2, \ldots$.

3.2.4. Completion of approximate solutions

Let $\delta_0 := \frac{1}{2}\min\{\delta, r_0\} > 0$. We choose a smooth cut-off function $\Theta(\tau)$ which verifies the following conditions;

$$\Theta(\tau) = 1, \ \text{ if } \ |\tau| \leq \frac{\delta_0}{2}, \quad \Theta(\tau) = 0, \ \text{ if } \ |\tau| \geq \delta_0, \quad \text{and } \ 0 \leq \Theta(\tau) \leq 1.$$

We denote the inverse of $\gamma(r, y)$ by

$$\Omega_\delta^{r_0} \ni x \longmapsto (\hat{r}(x), \hat{y}(x)) \in (-r_0, r_0) \times \mathbb{D}.$$

When we need to express this inverse map in the coordinates (r, θ, ρ), we write it as

$$\Omega_\delta^{r_0} \ni x \longmapsto (\hat{r}(x), \hat{\theta}(x), \hat{\rho}(x)) \in (-r_0, r_0) \times \partial\mathbb{D} \times (-\delta, \delta).$$

Now, choose $k \geq 4$ and define a k-th order inner expansion by

$$u_{\text{in}}^{\varepsilon,k}(s, y) = \sum_{j=0}^{k} \varepsilon^j u^j(s, y).$$

Our k-th order approximate solution $u_{\text{app}}^{\varepsilon,k}(x)$ to a solution of (3.2) is defined by

(3.21) $$\begin{aligned} u_{\text{app}}^{\varepsilon,k}(x) &= u_{\text{out}}(x) + \Theta(\hat{r}(x))\big[u_{\text{in}}^{\varepsilon,k}(\hat{r}(x)/\varepsilon, \hat{y}(x)) - u_{\text{out}}(x)\big] \\ &\quad + \Theta(\hat{r}(x))\Theta(\hat{\rho}(x)) \sum_{j=1}^{k+1} b^j(\hat{r}(x)/\varepsilon, \hat{\theta}(x), \hat{\rho}(x)/\varepsilon) \end{aligned}$$

In the sequel, we use weighted Hölder-norms defined by

$$\|u\|_0 = \sup_{x\in\overline{\Omega}} |u(x)|, \quad \|u\|_{C^\alpha_\varepsilon(\overline{\Omega})} = \|u(x)\|_0 + \varepsilon^\alpha \sup_{x,x'\in\overline{\Omega}} \frac{|u(x)-u(x')|}{|x-x'|^\alpha},$$

for $< \alpha < 1$. We also use higher order weighted Hölder-norms

$$\|u\|_{C^{2,\alpha}_\varepsilon(\overline{\Omega})} = \|u\|_0 + \varepsilon\|\partial_x u\|_0 + \varepsilon^2\|\partial_x^2 u\|_0 + \varepsilon^{2+\alpha} \sup_{x,x'\in\overline{\Omega}} \frac{|\partial_x^2 u(x) - \partial_x^2 u(x')|}{|x-x'|^\alpha}.$$

We have the following.

Lemma 3.6. *The function $u^{\varepsilon,k}_{\mathrm{app}}(x)$ defined above satisfies*

$$\|\varepsilon^2\Delta u^{\varepsilon,k}_{\mathrm{app}} + f(u^{\varepsilon,k}_{\mathrm{app}})\|_{C^\alpha_\varepsilon(\overline{\Omega})} = O(\varepsilon^{k+1-\alpha}) \ \ as \ \ \varepsilon \to 0, \tag{3.22}$$

$$\left\|\frac{\partial u^{\varepsilon,k}_{\mathrm{app}}}{\partial \mathbf{n}}\right\|_{C^{2,\alpha}_\varepsilon(\partial\Omega)} = O(\varepsilon^{k+1-\alpha}) \ \ as \ \ \varepsilon \to 0. \tag{3.23}$$

The proof of this lemma is almost trivial from our construction of the approximate solutions. This can be made rigorous by following the method of [11], and hence the proof is omitted.

3.3. Existence of Boundary-interior Layers

Let us modify the approximation (3.21) so that the boundary conditions are satisfied exactly.

For $x \in \Omega$ with $\mathrm{dist}(x, \partial\Omega) < \delta_0$, we define $x' \in \partial\Omega$ as the unique point x' so that $\mathrm{dist}(x, \partial\Omega) = \mathrm{dist}(x, x')$. Then, our correction-function $u^{\varepsilon,k}_{\mathrm{cor}}(x)$ is defined by
(3.24)

$$u^{\varepsilon,k}_{\mathrm{cor}}(x) := \begin{cases} \Theta(\mathrm{dist}(x,x'))\mathrm{dist}(x,x')\dfrac{\partial u^{\varepsilon,k}_{\mathrm{app}}}{\partial \mathbf{n}}(x'), & \text{if } \ \mathrm{dist}(x,\partial\Omega) < \delta, \\ 0, & \text{if } \ \mathrm{dist}(x,\partial\Omega) \ge \delta. \end{cases}$$

Let $\overline{u}^{\varepsilon,k}_{\mathrm{app}}(x)$ be defined by

$$\overline{u}^{\varepsilon,k}_{\mathrm{app}}(x) = u^{\varepsilon,k}_{\mathrm{app}}(x) - u^{\varepsilon,k}_{\mathrm{cor}}(x) \tag{3.25}$$

Then Lemma 3.6 is improved to the following.

Lemma 3.7. *The function $\overline{u}^{\varepsilon,k}_{\mathrm{app}}(x)$ defined above satisfies*

$$\|\varepsilon^2\Delta\overline{u}^{\varepsilon,k}_{\rm app}+f(\overline{u}^{\varepsilon,k}_{\rm app})\|_{C^\alpha_\varepsilon(\overline{\Omega})}=O(\varepsilon^{k+1-\alpha})\ \ as\ \ \varepsilon\to 0, \tag{3.26}$$

$$\left\|\frac{\partial\overline{u}^{\varepsilon,k}_{\rm app}}{\partial\mathbf{n}}\right\|_{C^{2,\alpha}_\varepsilon(\partial\Omega)}=0. \tag{3.27}$$

The proof is easy because the correction defined in (3.24) make the boundary condition satisfied exactly and its $C^{2,\alpha}_\varepsilon(\overline{\Omega})$-norm is $O(\epsilon^{k+1-\alpha})$ as $\varepsilon\to 0$.

We now show that (3.2) has a solution near the approximation $\overline{u}^{\varepsilon,k}_{\rm app}$ for sufficiently large $k\geq 4$. Let us look for a solution of (3.2) as a perturbation of the approximation;

$$u^\varepsilon(x)=\overline{u}^{\varepsilon,k}_{\rm app}(x)+\phi(x). \tag{3.28}$$

Then (3.2) is rewritten as

$$\begin{cases} L^\varepsilon\phi+N^\varepsilon(\phi)+R^\varepsilon=0\ \text{ in }\ \Omega,\\ \dfrac{\partial\phi}{\partial\mathbf{n}}=0\ \text{ on }\ \partial\Omega,\end{cases} \tag{3.29}$$

where

$$\begin{cases} \text{(a)}\quad L^\varepsilon\phi=\varepsilon^2\Delta\phi+f(\phi),\\ \text{(b)}\quad N^\varepsilon(\phi)=f(\overline{u}^{\varepsilon,k}_{\rm app}+\phi)-f(\overline{u}^{\varepsilon,k}_{\rm app})-f'(\overline{u}^{\varepsilon,k}_{\rm app})\phi,\\ \text{(c)}\quad R^\varepsilon=\varepsilon^2\Delta\overline{u}^{\varepsilon,k}_{\rm app}+f(\overline{u}^{\varepsilon,k}_{\rm app}).\end{cases} \tag{3.30}$$

In order to show the solvability of (3.29), the following eigenvalue problem plays a decisive role.

$$\begin{cases} L^\varepsilon\varphi^\varepsilon=\lambda^\varepsilon\varphi^\varepsilon\quad\text{in }\ \Omega,\\ \dfrac{\partial\varphi^\varepsilon}{\partial\mathbf{n}}=0\quad\text{on }\ \partial\Omega.\end{cases} \tag{3.31}$$

We call an eigenvalue λ^ε of (3.31) *non-critical*, if

$$\limsup_{\varepsilon\to 0}\frac{\lambda^\varepsilon}{\varepsilon^2}=-\infty.$$

Otherwise, an eigenvalue λ^ε is called *critical*.

Theorem 3.1. *The critical eigenvalues of* (3.31) *has the following behavior*

$$\lambda^\varepsilon=\varepsilon^2\overline{\lambda}+o(\varepsilon^2)\quad as\ \ \varepsilon\to 0,$$

and $\overline{\lambda}$ *is an eigenvalue of* (2.1).

The proof of this theorem is carried out by the method developed in [1] (cf. §3 of [10]). Based upon this theorem and elliptic estimates in [7], we obtain the following result (cf. §3 of [10]).

Corollary 3.1. *The operator* L^ε *is invertible as a map*

$$L^\varepsilon : C^{2,\alpha}_\varepsilon(\overline{\Omega}) \longrightarrow C^\alpha_\varepsilon(\overline{\Omega}),$$

and there exists a constant $C > 0$, *independent of* ε, *such that*

$$\|(L^\varepsilon)^{-1}\|_{C^\alpha_\varepsilon(\overline{\Omega})\to C^{2,\alpha}_\varepsilon(\overline{\Omega})} \leq \frac{C}{\varepsilon^4}.$$

Let us now show the solvability of (3.29). We choose $k = 8$ and set $\phi = \varepsilon^4\tilde{\phi}$ in (3.29). Thanks to Corollary 3.1, we can rewrite it as

$$\tilde{\phi} = -\varepsilon^{-4}(L^\varepsilon)^{-1}\Big[N^\varepsilon(\varepsilon^4\tilde{\phi}) + R^\varepsilon\Big] := \mathcal{F}^\epsilon(\tilde{\phi}). \tag{3.32}$$

One can then show, as in [10], that $\mathcal{F}^\varepsilon$ is a contraction mapping in an $O(\varepsilon^{1-\alpha})$-neighborhood of the origin in $C^\alpha_\varepsilon(\overline{\Omega})$. This completes the existence part of proof for Theorem 2.1.

Since the difference of the true solution and its approximation is $O(\varepsilon^4)$ measured in $L^\infty(\overline{\Omega})$-norm, Theorem 3.1 applies to an eigenvalue problem associated with the linearization of (1.1) around the true solution. Therefore the stability property of U^ε is determined by the spectrum of L^ε. This completes the proof of the stability properties in Theorem 2.1

4. Proof of Technical Results

In this section, we prove technical results used in §3.

4.1. Proof of Lemma 3.1.

Let $\gamma_0, \mathbb{D}, \mathbb{D}_\delta$ and Γ_δ be as in §3.1. We choose $r_0 > 0$ so that

$$\big(\Omega^{r_0}_\delta \cap \Omega\big) \cap \{\gamma_0(y) + r\nu(y) \mid |r| < r_0, |y| = 1+\delta\} = \varnothing.$$

We define $\overline{\gamma} : (-r_0, r_0) \times \mathbb{D}_\delta \to \mathbb{R}^3$ by

$$\overline{\gamma}(r, y) = \gamma_0(y) + r\nu(y),$$

and denote by S the preimage of $\Omega^{r_0}_\delta \cap \partial\Omega$;

$$\mathrm{S} = \overline{\gamma}^{-1}(\Omega^{r_0}_\delta \cap \partial\Omega). \tag{4.1}$$

Since $\partial\Omega \perp \Gamma$ by **(A1)**, we have

$$\mathrm{S} \perp_\partial \ (\{0\} \times \mathbb{D}) \,. \tag{4.2}$$

We also denote by C the preimage of $\Omega_\delta^{r_0} \cap \Omega$;

$$\mathrm{C} = \overline{\gamma}^{-1}(\Omega_\delta^{r_0} \cap \overline{\Omega}), \tag{4.3}$$

and by $\mathrm{C}(r)$ the r-slice of C;

$$\mathrm{C}(r) = \{y \in \mathbb{D}_\delta \mid (r, y) \in \mathrm{C}\} \quad (|r| < r_0). \tag{4.4}$$

Since $\partial\Omega$ and γ_0 are smooth, $\mathrm{C}(r)$ is a smooth domain, diffeomorphic to $\mathrm{C}(0) = \overline{\mathbb{D}}$. Therefore, there exists a smooth family of diffeomorphisms

$$Y(r, \cdot) : \overline{\mathbb{D}} \to \mathrm{C}(r) \tag{4.5}$$

parametrized by $r \in (-r_0, r_0)$. Moreover, thanks to (4.2), we can choose Y so that

$$Y(0, y) = y, \quad \frac{\partial Y}{\partial r}(0, y) = 0 \quad (y \in \overline{\mathbb{D}}). \tag{4.6}$$

Let us now define the desired γ by

$$\gamma(r, y) := \overline{\gamma}(r, Y(r, y)) = \gamma_0(Y(r, y)) + r\nu(Y(r, y)) \tag{4.7}$$

for $(r, y) \in (-r_0, r_0) \times \overline{\mathbb{D}}$. It is now straightforward to verify that γ in (4.7) satisfies Lemma 3.1 (i). By elementary computations and (4.6), we find that

$$\begin{aligned}
&\frac{\partial \gamma}{\partial r}(0, y) = \nu(y), \\
p(y) := &\frac{\partial^2 \gamma}{\partial r^2}(0, y) = \sum_{j=1}^{2} \frac{\partial \gamma_0}{\partial Y^j} \frac{\partial^2 Y^j}{\partial r^2}(0, y) \perp \nu(y), \\
q(y) := &\frac{\partial^3 \gamma}{\partial r^3}(0, y) = \sum_{j=1}^{2} \left(\frac{\partial \gamma_0}{\partial Y^j} \frac{\partial^3 Y^j}{\partial r^3}(0, y) + 2 \frac{\partial \nu}{\partial Y^j} \frac{\partial^2 Y^j}{\partial r^2}(0, y) \right) \perp \nu(y),
\end{aligned}$$

proving the statements (ii) and (iii).

To prove Lemma 3.1 (iv), we use the coordinates (θ, ρ) introduced in §3.1. Recall that $(\theta, \rho = 0)$ parametrizes $\partial\mathbb{D}$ and ρ is chosen so that

$$0 = \left\langle \frac{\partial \gamma_0}{\partial \theta}, \frac{\partial \gamma_0}{\partial \rho} \right\rangle \quad \text{at } \rho = 0.$$

For $y \in \overline{\mathbb{D}}$ near $\partial\mathbb{D}$, we express $\gamma(r, y)$ by

$$\gamma(r, y) = \gamma(r, \theta, \rho).$$

We also denote by $\mathbf{n}(r,\theta)$ the unit inward normal vector of $\partial\Omega$ at $\gamma(r,\theta,0)$. Note that at $\rho=0$ (i.e. on $\partial\Omega\cap\Omega_\delta^{r_0}$), vectors

$$\frac{\partial\gamma}{\partial r},\quad \frac{\partial\gamma}{\partial\theta},\quad \frac{\partial\gamma}{\partial\rho}\in\mathbb{R}^3$$

constitute a basis for $\mathbb{R}^3$. Hence $\mathbf{n}(r,\theta)$ is expressed as

$$\mathbf{n}=a\frac{\partial\gamma}{\partial r}+b\frac{\partial\gamma}{\partial\theta}+c\frac{\partial\gamma}{\partial\rho}\quad \text{at } \rho=0, \tag{4.8}$$

where $c>0$. Since $\left\{\frac{\partial\gamma}{\partial r},\frac{\partial\gamma}{\partial\theta}\right\}$ spans the tangent space of $\partial\Omega$ at $x=\gamma(r,\theta,0)$, we have

$$\left\langle\frac{\partial\gamma}{\partial r},\mathbf{n}\right\rangle=0,\quad \left\langle\frac{\partial\gamma}{\partial\theta},\mathbf{n}\right\rangle=0,\quad \langle\mathbf{n},\mathbf{n}\rangle=1\quad \text{at } \rho=0. \tag{4.9}$$

From (4.8) and (4.9), we easily obtain

$$a=\frac{\tilde g^{13}}{\sqrt{\tilde g^{33}}},\quad b=\frac{\tilde g^{23}}{\sqrt{\tilde g^{33}}},\quad c=\frac{\tilde g^{33}}{\sqrt{\tilde g^{33}}}=\sqrt{\tilde g^{33}}, \tag{4.10}$$

where $(\tilde g^{ij})=(\tilde g_{ij})^{-1}$ with

$$\begin{cases}\tilde g_{11}=\langle\frac{\partial\gamma}{\partial r},\frac{\partial\gamma}{\partial r}\rangle, \tilde g_{12}=\tilde g_{21}=\langle\frac{\partial\gamma}{\partial r},\frac{\partial\gamma}{\partial\theta}\rangle,\\ \tilde g_{22}=\langle\frac{\partial\gamma}{\partial\theta},\frac{\partial\gamma}{\partial\theta}\rangle, \tilde g_{13}=\tilde g_{31}=\langle\frac{\partial\gamma}{\partial r},\frac{\partial\gamma}{\partial\rho}\rangle, \qquad \text{at } \rho=0.\\ \tilde g_{33}=\langle\frac{\partial\gamma}{\partial\rho},\frac{\partial\gamma}{\partial\rho}\rangle, \tilde g_{23}=\tilde g_{32}=\langle\frac{\partial\gamma}{\partial\theta},\frac{\partial\gamma}{\partial\theta}\rangle,\end{cases} \tag{4.11}$$

Therefore, $\partial/\partial\mathbf{n}$ is given by

$$\frac{\partial}{\partial\mathbf{n}}=\frac{1}{\sqrt{\tilde g^{33}}}\left(\tilde g^{13}\frac{\partial}{\partial r}+\tilde g^{23}\frac{\partial}{\partial\theta}+\tilde g^{33}\frac{\partial}{\partial\rho}\right). \tag{4.12}$$

Let us now expand $\tilde g^{jk}(r,\theta,0)$ in r at $r=0$. From the expansion of $\gamma(r,\theta,0)$ in Lemma 3.1 (iii), we have

$$\begin{cases}\dfrac{\partial\gamma}{\partial r}=\nu(\theta,0)+rp(\theta,0)+\dfrac{r^2}{2}q(\theta,0)+O(r^3),\\ \dfrac{\partial\gamma}{\partial\theta}=\dfrac{\gamma_0}{\partial\theta}(\theta,0)+r\dfrac{\partial\nu}{\partial\theta}(\theta,0)+\dfrac{r^2}{2}\dfrac{\partial p}{\partial\theta}(\theta,0)+O(r^3),\\ \dfrac{\partial\gamma}{\partial\rho}=\dfrac{\gamma_0}{\partial\rho}(\theta,0)+r\dfrac{\partial\nu}{\partial\rho}(\theta,0)+\dfrac{r^2}{2}\dfrac{\partial p}{\partial\rho}(\theta,0)+O(r^3).\end{cases} \tag{4.13}$$

By using the orthogonalities $\frac{\partial\gamma_0}{\partial y} \perp \nu$, $\frac{\partial\nu}{\partial y} \perp \nu, p \perp \nu$ and $\frac{\partial\gamma_0}{\partial\theta} \perp \frac{\partial\gamma_0}{\partial\rho}$, together with (4.13), we find

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & |\frac{\partial\gamma_0}{\partial\theta}|^2 & 0 \\ 0 & 0 & |\frac{\partial\gamma_0}{\partial\rho}|^2 \end{pmatrix} + r \begin{pmatrix} 0 & \langle p, \frac{\partial\gamma_0}{\partial\theta}\rangle & \langle p, \frac{\partial\gamma_0}{\partial\rho}\rangle \\ \langle p, \frac{\partial\gamma_0}{\partial\theta}\rangle & -2\tilde{L} & -2\tilde{M} \\ \langle p, \frac{\partial\gamma_0}{\partial\rho}\rangle & -2\tilde{M} & -2\tilde{N} \end{pmatrix} + O(r^2),$$

where

$$-\tilde{L} = \langle \frac{\partial\gamma_0}{\partial\theta}, \frac{\partial\nu}{\partial\theta}\rangle, \quad -\tilde{N} = \langle \frac{\partial\gamma_0}{\partial\rho}, \frac{\partial\nu}{\partial\rho}\rangle,$$
$$-2\tilde{M} = \langle \frac{\partial\gamma_0}{\partial\theta}, \frac{\partial\nu}{\partial\rho}\rangle + \langle \frac{\partial\gamma_0}{\partial\rho}, \frac{\partial\nu}{\partial\theta}\rangle.$$

Therefore, we obtain

$$\begin{pmatrix} \tilde{g}^{11} & \tilde{g}^{12} & \tilde{g}^{13} \\ \tilde{g}^{21} & \tilde{g}^{22} & \tilde{g}^{23} \\ \tilde{g}^{31} & \tilde{g}^{32} & \tilde{g}^{33} \end{pmatrix} = \begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & |\frac{\partial\gamma_0}{\partial\theta}|^{-2} & 0 \\ 0 & 0 & |\frac{\partial\gamma_0}{\partial\rho}|^{-2} \end{pmatrix}$$
$$+r \begin{pmatrix} 0 & -|\frac{\partial\gamma_0}{\partial\theta}|^{-2}\langle p, \frac{\partial\gamma_0}{\partial\theta}\rangle & -|\frac{\partial\gamma_0}{\partial\rho}|^{-2}\langle p, \frac{\partial\gamma_0}{\partial\rho}\rangle \\ -|\frac{\partial\gamma_0}{\partial\theta}|^{-2}\langle p, \frac{\partial\gamma_0}{\partial\theta}\rangle & 2|\frac{\partial\gamma_0}{\partial\theta}|^{-4}\tilde{L} & 2|\frac{\partial\gamma_0}{\partial\theta}|^{-2}|\frac{\partial\gamma_0}{\partial\rho}|^{-2}\tilde{M} \\ -|\frac{\partial\gamma_0}{\partial\rho}|^{-2}\langle p, \frac{\partial\gamma_0}{\partial\rho}\rangle & 2|\frac{\partial\gamma_0}{\partial\rho}|^{-2}|\frac{\partial\gamma_0}{\partial\theta}|^{-2}\tilde{M} & 2|\frac{\partial\gamma_0}{\partial\rho}|^{-4}\tilde{N} \end{pmatrix} + O(r^2).$$

This completes the proof of Lemma 3.1.

4.2. Proof of Lemma 3.2.

We first express the Laplacian $\Delta = \sum_{i=1}^{3}(\frac{\partial}{\partial x^i})^2$ in terms of $(r, y) \in (-r_0, r_0) \times \mathbb{D}$. Since $x = \gamma(r, y)$, the standard metric in $\Omega_\delta^{r_0} \subset \mathbb{R}^3$ is pulled back to $g_{jk}(r, y)$;

(4.14) $$g_{jk}(r, y) = \left\langle \frac{\partial\gamma}{\partial y^j}, \frac{\partial\gamma}{\partial y^k} \right\rangle, \quad j, k = 0, 1, 2,$$

where y^0 stands for r. Therefore, the Laplacian Δ is pulled back to the Laplace-Beltrami operator

(4.15) $$\Delta = \frac{1}{\sqrt{g}} \sum_{j,k=0}^{2} \frac{\partial}{\partial y^j} \left(\sqrt{g} g^{jk} \frac{\partial}{\partial y^k} \right)$$

on $(-r_0, r_0) \times \mathbb{D}$, where $g = g(r, y) = \det(g_{jk}(r, y))$ and $(g^{jk}) = (g_{jk})^{-1}$. Separating out the summation with respect to $j, k = 1, 2$ in (4.15), we have

$$\Delta = g^{00}\frac{\partial^2}{\partial r^2} + \Delta^{\Gamma}(r) + 2\sum_{j=1}^{2} g^{j0}\frac{\partial^2}{\partial y^j \partial r} + R\frac{\partial}{\partial r} + \sum_{k=1}^{2}\eta^k\frac{\partial}{\partial y^k} \tag{4.16}$$

where

$$\Delta^{\Gamma}(r) = \frac{1}{\sqrt{g}}\sum_{j,k=1}^{2}\frac{\partial}{\partial y^j}\left(\sqrt{g}g^{jk}\frac{\partial}{\partial y^k}\right), \tag{4.17}$$

$$R = \frac{1}{2g}\frac{\partial g}{\partial r}g^{00} + \frac{\partial g^{00}}{\partial r} + \frac{1}{2g}\sum_{j=1}^{2}\frac{\partial g}{\partial y^j}g^{j0} + \sum_{j=1}^{2}\frac{\partial g^{j0}}{\partial y^j}, \tag{4.17-R}$$

$$\eta^k = \frac{1}{2g}\frac{\partial g}{\partial r}g^{k0} + \frac{\partial g^{k0}}{\partial r} \quad (k = 1, 2). \tag{4.17-η}$$

Note that (4.17) is the pull-back of the Laplace-Beltrami operator on $\Gamma(r) = \{x = \gamma(r, y) \mid y \in \mathbb{D}\}$ to the r-slice $\mathrm{C}(r)$.

In order to prove Lemma 3.2, we will express $\varepsilon^2\Delta$ in terms of the variables (s, y) ($s = r/\varepsilon$), and give explicit forms to the coefficients of $\varepsilon^0, \varepsilon^1, \varepsilon^2$. For this purpose, we need to find coefficients of r^i ($i = 0, 1, 2$) in the following expansions.

$$\begin{cases} \text{(i)}g^{00}(r, y) = g^{00}_{(0)}(y) + rg^{00}_{(1)}(y) + \dfrac{r^2}{2}g^{00}_{(2)}(y) + O(r^3), \\ \text{(ii)}g^{j0}(r, y) = g^{j0}_{(0)}(y) + rg^{j0}_{(1)}(y) + O(r^2) \quad (j = 1, 2), \\ \text{(iii)}R(r, y) = R^{(0)}(y) + rR^{(1)}(y) + O(r^2), \\ \text{(iv)}\eta^k(r, y) = \eta^k_{(0)}(y) + O(r) \quad (k = 1, 2). \end{cases} \tag{4.18}$$

It is convenient to use an isothermal representation $\gamma_0 : \mathbb{D} \to \Gamma$. Namely, we use γ_0 that satisfies

$$\left|\frac{\partial\gamma_0}{\partial y^1}\right|^2 = \left|\frac{\partial\gamma_0}{\partial y^2}\right|^2 = \lambda^2(y), \quad \left\langle\frac{\partial\gamma_0}{\partial y^1}, \frac{\partial\gamma_0}{\partial y^2}\right\rangle = 0. \tag{4.19}$$

Therefore, the tangent vectors $\frac{\partial\gamma_0}{\partial y^i}$ $(i = 1, 2)$ have the same length $\lambda(y) > 0$ and are mutually orthogonal. In the isothermal representation, it is known [13] that the following identities hold true.

$$(4.20)\quad \begin{aligned} &\text{(i)} && \left(\frac{\partial}{\partial y^1}\right)^2 \gamma_0 = \frac{\partial \log \lambda}{\partial y^1}\frac{\partial \gamma_0}{\partial y^1} - \frac{\partial \log \lambda}{\partial y^2}\frac{\partial \gamma_0}{\partial y^2} + L\nu, \\ &\text{(ii)} && \frac{\partial^2}{\partial y^1 \partial y^2}\gamma_0 = \frac{\partial \log \lambda}{\partial y^2}\frac{\partial \gamma_0}{\partial y^1} + \frac{\partial \log \lambda}{\partial y^1}\frac{\partial \gamma_0}{\partial y^2} + M\nu, \\ &\text{(iii)} && \left(\frac{\partial}{\partial y^2}\right)^2 \gamma_0 = -\frac{\partial \log \lambda}{\partial y^1}\frac{\partial \gamma_0}{\partial y^1} + \frac{\partial \log \lambda}{\partial y^2}\frac{\partial \gamma_0}{\partial y^2} + N\nu, \end{aligned}$$

where

$$\begin{aligned} L &= \left\langle \left(\frac{\partial}{\partial y^1}\right)^2 \gamma_0, \nu \right\rangle = -\left\langle \frac{\partial \gamma_0}{\partial y^1}, \frac{\partial \nu}{\partial y^1} \right\rangle, \\ M &= \left\langle \frac{\partial^2 \gamma_0}{\partial y^1 \partial y^2}, \nu \right\rangle = -\left\langle \frac{\partial \gamma_0}{\partial y^1}, \frac{\partial \nu}{\partial y^2} \right\rangle = -\left\langle \frac{\partial \gamma_0}{\partial y^2}, \frac{\partial \nu}{\partial y^1} \right\rangle, \\ N &= \left\langle \left(\frac{\partial}{\partial y^2}\right)^2 \gamma_0, \nu \right\rangle = -\left\langle \frac{\partial \gamma_0}{\partial y^2}, \frac{\partial \nu}{\partial y^2} \right\rangle. \end{aligned}$$

It is also known [13] that derivatives of the normal vector ν are given by

$$(4.21)\qquad \frac{\partial \nu}{\partial y^1} = -\frac{L}{\lambda^2}\frac{\partial \gamma_0}{\partial y^1} - \frac{M}{\lambda^2}\frac{\partial \gamma_0}{\partial y^2},$$

$$(4.22)\qquad \frac{\partial \nu}{\partial y^2} = -\frac{M}{\lambda^2}\frac{\partial \gamma_0}{\partial y^1} - \frac{N}{\lambda^2}\frac{\partial \gamma_0}{\partial y^2}.$$

Therefore, with our sign convention for curvatures, the sum and product of principal curvatures κ_i ($i = 1, 2$) are given by

$$\kappa = \kappa_1 + \kappa_2 = -\frac{L+N}{\lambda^2}, \quad \kappa_1 \kappa_2 = \frac{LN - M^2}{\lambda^4}.$$

Proposition 4.1. *The coefficients in* (4.18) *are as follows.*

$$g^{00}_{(0)}(y) = 1, \quad g^{00}_{(1)}(y) = g^{00}_{(2)}(y) = 0,$$

$$g^{j0}_{(0)}(y) = 0,$$

$$g^{j0}_{(1)}(y) = -\lambda^2 \left\langle p, \frac{\partial \gamma_0}{\partial y^j} \right\rangle = -\frac{1}{\sqrt{g(0,y)}} \left\langle p, \frac{\partial \gamma_0}{\partial y^j} \right\rangle \qquad (j = 1, 2),$$

$$R^{(0)}(y) = \kappa, \quad R^{(1)}(y) = -(\kappa_1^2 + \kappa_2^2),$$

$$\eta^k_{(0)}(y) = g^{k0}_{(1)}(y) \qquad (k = 1, 2).$$

Proof. The proof consists of simply computing relvant quantities by using (4.19), (4.20) and (4.21)-(4.22). We omit the computational details, since they are long but elementary. ∎

Since the gradient operator on Γ is pulled back to

$$\nabla^\Gamma = \sum_{j=1}^{2} \frac{1}{\sqrt{g(0,y)}} \frac{\partial \gamma_0}{\partial y^j} \frac{\partial}{\partial y^j},$$

the directional derivative in $p(y)$-direction is expressed as

$$\nabla_p^\Gamma = \frac{1}{\sqrt{g(0,y)}} \sum_{j=1}^{2} \left\langle p, \frac{\partial \gamma_0}{\partial y^j} \right\rangle \frac{\partial}{\partial y^j}. \tag{4.23}$$

We are now ready to complete the proof of Lemma 3.2. By uisng (4.18), we have

$$\begin{aligned} \varepsilon^2 \Delta = g^{00} \frac{\partial^2}{\partial s^2} &+ \varepsilon \left\{ 2 \sum_{j=1}^{2} g^{j0} \frac{\partial}{\partial y^j} \frac{\partial}{\partial s} + R \frac{\partial}{\partial s} \right\} \\ &+ \varepsilon^2 \left\{ \Delta^\Gamma + \sum_{j=1}^{2} \eta^j \frac{\partial}{\partial y^j} \right\} + O(\varepsilon^3), \end{aligned} \tag{4.24}$$

where coefficients are evaluated at $(r, y) = (\varepsilon s, y)$. Therefore, by using (4.18), (4.23) and Proposition 4.1, we easily see that (4.24) is written as in (3.4). This completes the proof of Lemma 3.2.

4.3. Proof of Lemma 3.4.

We will prove Lemma 3.4. For this purpose, we use the coordinate system (θ, ρ) in place of y, introduced in §3.1. We then introduce stretched variables $s = r/\varepsilon, \eta = \rho/\varepsilon$.

4.3.1. Proof of (3.15).

From Lemma 3.1 (iv), we have at $(r, \theta) = (\varepsilon s, \theta)$

$$\begin{aligned} \varepsilon \sqrt{\tilde{g}^{33}} \frac{\partial}{\partial \mathbf{n}} &= \left(\tilde{g}^{13} \frac{\partial}{\partial s} + \tilde{g}^{33} \frac{\partial}{\partial \eta} \right) + \varepsilon \tilde{g}^{23} \frac{\partial}{\partial \theta} \\ &= -\varepsilon s \left| \frac{\partial \gamma_0}{\partial \rho} \right|^{-2} \left\langle p, \frac{\partial \gamma_0}{\partial \rho} \right\rangle \frac{\partial}{\partial s} + \left| \frac{\partial \gamma_0}{\partial \rho} \right|^{-2} \frac{\partial}{\partial \eta} \\ &\quad + 2\varepsilon s \left| \frac{\partial \gamma_0}{\partial \rho} \right|^{-4} \left\langle \frac{\partial \gamma_0}{\partial \rho}, \frac{\partial \nu}{\partial \rho} \right\rangle \frac{\partial}{\partial \eta} + O(\varepsilon^2) \end{aligned}$$

$$= \left|\frac{\partial\gamma_0}{\partial\rho}\right|^{-2} \left\{ \frac{\partial}{\partial\eta} + \varepsilon s \left[-\left\langle p, \frac{\partial\gamma_0}{\partial\rho}\right\rangle \frac{\partial}{\partial s} + \frac{2}{l_2(\theta)^2}\left\langle \frac{\partial\gamma_0}{\partial\rho}, \frac{\partial\nu}{\partial\rho}\right\rangle \frac{\partial}{\partial\eta}\right]\right\}$$
$$+O(\varepsilon^2),$$

which establishes (3.15).

4.3.2. Proof of (3.14).

We now establish (3.14). In §4.2, we have obtained

$$\begin{aligned}\varepsilon^2\Delta = {}& \frac{\partial^2}{\partial s^2} + \varepsilon\kappa(\theta,\rho)\frac{\partial}{\partial s} \qquad \text{(note } \kappa(\theta,\rho)\equiv 0\text{)}\\ &+\varepsilon^2\left\{\Delta^\Gamma - \left[\kappa_1^2+\kappa_2^2\right]s\frac{\partial}{\partial s} - 2s\nabla_p^\Gamma\frac{\partial}{\partial s} - \nabla_p^\Gamma\right\}\\ &+\sum_{j\ge 3}\varepsilon^j P_j(s,\theta,\rho).\end{aligned} \tag{4.25}$$

We will now express Δ^Γ and ∇_p^Γ in terms of (θ,ρ), and then in terms of (θ,η), where $\eta = \rho/\varepsilon$ is the stretched variable.

Proposition 4.2. *In terms of the coordinate system (θ,ρ) introduced in §3.1, the Laplace-Beltrami Δ^Γ and the directional derivative ∇_p^Γ are expressed as follows.*

$$\begin{aligned}\Delta^\Gamma = {}& \left|\frac{\partial\gamma_0}{\partial\theta}(\theta,\rho)\right|^{-2}\frac{\partial^2}{\partial\theta^2} + \left|\frac{\partial\gamma_0}{\partial\rho}(\theta,\rho)\right|^{-2}\frac{\partial^2}{\partial\rho^2}\\ &+\left\{\left|\frac{\partial\gamma_0}{\partial\theta}(\theta,\rho)\right|^{-2}\left|\frac{\partial\gamma_0}{\partial\rho}(\theta,\rho)\right|^{-2}\left\langle\frac{\partial^2\gamma_0}{\partial\theta\partial\rho}(\theta,\rho),\frac{\partial\gamma_0}{\partial\theta}(\theta,\rho)\right\rangle\right.\\ &\left.-\left|\frac{\partial\gamma_0}{\partial\rho}(\theta,\rho)\right|^{-4}\left\langle\frac{\partial^2\gamma_0}{\partial\rho^2}(\theta,\rho),\frac{\partial\gamma_0}{\partial\rho}(\theta,\rho)\right\rangle\right\}\frac{\partial}{\partial\rho}\\ &+\left\{\left|\frac{\partial\gamma_0}{\partial\theta}(\theta,\rho)\right|^{-2}\left|\frac{\partial\gamma_0}{\partial\rho}(\theta,\rho)\right|^{-2}\left\langle\frac{\partial^2\gamma_0}{\partial\theta\partial\rho}(\theta,\rho),\frac{\partial\gamma_0}{\partial\rho}(\theta,\rho)\right\rangle\right.\\ &\left.-\left|\frac{\partial\gamma_0}{\partial\theta}(\theta,\rho)\right|^{-4}\left\langle\frac{\partial^2\gamma_0}{\partial\theta^2}(\theta,\rho),\frac{\partial\gamma_0}{\partial\theta}(\theta,\rho)\right\rangle\right\}\frac{\partial}{\partial\theta}.\end{aligned} \tag{4.26}$$

$$\begin{aligned}\nabla_p^\Gamma = {}& \left|\frac{\partial\gamma_0}{\partial\theta}(\theta,\rho)\right|^{-2}\left|\frac{\partial\gamma_0}{\partial\rho}(\theta,\rho)\right|^{-2}\left\langle p(\theta,\rho),\frac{\partial\gamma_0}{\partial\rho}(\theta,\rho)\right\rangle\frac{\partial}{\partial\rho}\\ &+\left|\frac{\partial\gamma_0}{\partial\theta}(\theta,\rho)\right|^{-2}\left|\frac{\partial\gamma_0}{\partial\rho}(\theta,\rho)\right|^{-2}\left\langle p(\theta,\rho),\frac{\partial\gamma_0}{\partial\theta}(\theta,\rho)\right\rangle\frac{\partial}{\partial\theta}.\end{aligned} \tag{4.27}$$

To obatin (4.26) and (4.27), we simply compute relvant quantities according to the definitions of Δ^Γ and ∇_p^Γ. We omit the detail.

We substitute $\rho = \varepsilon\eta$ in (4.26) and (4.27), and expand them in the powers of ε. This gives rise to

$$\begin{aligned}\varepsilon^2\Delta^\Gamma &= \frac{1}{l_2(\theta)^2}\frac{\partial^2}{\partial\eta^2}\\ &\quad +\varepsilon\left\{-2\eta\frac{A(\theta)}{l_2(\theta)^4}\frac{\partial^2}{\partial\eta^2} + \left(\frac{B(\theta)}{l_1(\theta)^2 l_2(\theta)^2} - \frac{A(\theta)}{l_2(\theta)^4}\right)\frac{\partial}{\partial\eta}\right\} + O(\varepsilon^2),\\ \varepsilon^2\nabla_p^\Gamma &= \varepsilon\frac{C(\theta)}{l_1(\theta)^2 l_2(\theta)^2}\frac{\partial}{\partial\eta} + O(\varepsilon^2).\end{aligned}$$

Substituting these into (4.25), we immediately establish (3.14).

References

1. N. D. Alikakos, G. Fusco and V. Stefanopoulos, Critical spectrum and stability of interfaces for a class of reaction-diffusion equations. *J. Differential Equations* **126(1)** (1996), 106-167.
2. N. D. Alikakos, G. Fusco and M. Kowalczyk, Finite dimensional dynamics and interfaces intersecting the boundary: Equilibria and quasi-invariant manifold. *Indiana Univ. Math. J.* **45(4)** (1996), 1119-1155.
3. X-F. Chen, Generation and propagation of interfaces for reaction-diffusion equations. *J. Differential Equations* **96** (1992), 116-141.
4. S.-I. Ei, M.-H. Sato and E. Yanagida, Stability of stationary interfaces with contact angle in a generalized mean curvature flow. *Amer. J. Math.* **118(3)** (1996), 653-687.
5. S.-I. Ei and E. Yanagida, Slow dynamics of interfaces in the Allen-Cahn equation on a strip-like domain. *SIAM J. Math. Anal.* **29(3)** (1998), 555-595.
6. M. Gage and R. Hamilton, The heat equation shrinking convex plane curves. *J. Differential Geom.* **23** (1986), 69-96.
7. D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order. *Springer-Verlag* (1983), Berlin-Heidelberg-New York-Tokyo.
8. M. Grayson, The heat equation shrinks embedded plane curves to round points. *J. Differential Geom.* **26** (1987), no.2, 285-314.
9. J. K. Hale and K. Sakamoto, Existence and Stability of transition layers, *J. Appl. Math.* **5(3)** (1988), 367-405.
10. T. Iibun and K. Sakamoto, Internal Layers Intersectin the Boundary of Domain in the Allen-Cahn Equation, *J. J. Ind. Appl. Math.*, **18** (2001), 697-738.
11. H. Ikeda, On the asymptotic solutions for a weakly coupled elliptic boundary value problem with a small parameter, *Hiroshima Math. J.* **16(2)** (1986), 227-250.

12. Y. Kan-on and E.Yanagida, Existence of non-constant stable equilibia in competition-diffusion equations, *Hiroshima Math. J.* **23** (1993), 193-221.

13. K. Kenmotsu, *Surfaces with constant mean curvature*, Math. Monograph Series Vol. 221, American Mathematical Society (2003).

14. M. Kowalczyk, Exponentially Slow Dynamics and Interfaces Intersecting the Boundary, *J. Diff. Equations* **138** (1997), 55-85.

15. R. V. Kohn and P. Sternberg, Local minimisers and singular perturbaions. *Proc. Roy. Soc. Edinburgh Sect. A* **111(1,2)** (1989), 69-84.

16. M. Kowalczyk, On the existence and Morse index of solutions to the Allen-Cahn equations in two dimensions, (2002), Preprint .

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