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PARSEVAL'S FORMULA FOR DOUBLE LAGUERRE SERIES

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Abstract. Let $a \geq 0$. We give sufficient conditions on $\{c_{jk}\}$ to obtain Parseval's formula for the corresponding double Laguerre series $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\mathcal{L}_j^a(x)\mathcal{L}_k^a(y)$.

1. Introduction

For $a \geq 0$, define the Laguerre function $\mathcal{L}_n^a(t)$ by

$$\mathcal{L}_n^a(t) = \sqrt{\frac{n!}{\Gamma(n+a+1)}}e^{-t/2}t^{a/2}L_n^a(t) \qquad n = 0, 1, 2, \cdots,$$

where

$$L_n^a(t) = \frac{1}{n!}t^{-a}e^t\frac{d^n}{dt^n}(t^{n+a}e^{-t}).$$

It is known that $\{\mathcal{L}_n^a(t)\}_{n=0}^{\infty}$ forms an orthonormal basis in $L^2(\mathbb{R}^+, dt)$ with the inner product $\langle f, g\rangle = \int_0^{\infty} f(t)g(t)\,dt$ (cf. [6, 7]).

In this paper, we consider the following double Laguerre series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\mathcal{L}_j^a(x)\mathcal{L}_k^a(y), \qquad x, y \in \mathbb{R}^+. \tag{1.1}$$

Let $s_{mn}(x, y)$ be the rectangular partial sums of series (1.1) given by

$$s_{mn}(x, y) = \sum_{j=0}^{m}\sum_{k=0}^{n} c_{jk}\mathcal{L}_j^a(x)\mathcal{L}_k^a(y).$$

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We say that series (1.1) *converges regularly* to $f(x,y)$ if $s_{mn}(x,y) \to f(x,y)$ as $\min\{m,n\} \to \infty$, the row series $\sum_{j=0}^{\infty} c_{jk}\mathcal{L}_j^a(x)\mathcal{L}_k^a(y)$ converges for each fixed value of k, and the column series $\sum_{k=0}^{\infty} c_{jk}\mathcal{L}_j^a(x)\mathcal{L}_k^a(y)$ converges for each fixed value of j (cf. [4]). For $E \subseteq \mathbb{R}^+ \times \mathbb{R}^+$, the series (1.1) is said to *converge uniformly* on E to $f(x,y)$ if $s_{mn}(x,y) \to f(x,y)$ uniformly on E as $\min\{m,n\} \to \infty$.

We are interested in finding conditions on $\{c_{jk}\}$ and ϕ such that the following *Parseval's formula* holds:

$$\lim_{\substack{\epsilon,\delta\to 0+\\ \alpha,\beta\to\infty}} \int_\delta^\beta \int_\epsilon^\alpha f(x,y)\phi(x,y)\,dxdy = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\,\hat{\phi}^*(j,k), \tag{1.2}$$

where $f(x,y)$ is the limit function of $s_{mn}(x,y)$ and

$$\hat{\phi}^*(j,k) = \lim_{\substack{\epsilon,\delta\to 0+\\ \alpha,\beta\to\infty}} \int_\delta^\beta \int_\epsilon^\alpha \phi(x,y)\mathcal{L}_j^a(x)\mathcal{L}_k^a(y)\,dxdy.$$

For suitable ϕ, we shall prove that the following conditions are sufficient for the validity of (1.2):

$$|c_{jk}|\,(\bar{j}\bar{k})^{p/2-1/4}(\overline{\log j}\,\overline{\log k})^{1+\eta} = O(1) \qquad \text{as} \quad \max\{j,k\} \to \infty, \tag{1.3}$$

$$\sum_{j=0}^{\infty} |\Delta_{p0}c_{jn}|\,(\bar{j}\bar{n})^{p/2-1/4}(\overline{\log n})^{1+\eta} = O(1) \qquad \text{as} \quad n \to \infty, \tag{1.4}$$

$$\sum_{k=0}^{\infty} |\Delta_{0p}c_{mk}|\,(\bar{m}\bar{k})^{p/2-1/4}(\overline{\log m})^{1+\eta} = O(1) \qquad \text{as} \quad m \to \infty, \tag{1.5}$$

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\Delta_{pp}c_{jk}|\,(\bar{j}\bar{k})^{p/2-1/4} < \infty, \tag{1.6}$$

where $p \geq 1, \eta > 0, \bar{\xi} \equiv \max\{\xi, 1\}$, and

$$\Delta_{\alpha\beta}c_{jk} = \sum_{s=0}^{\alpha}\sum_{t=0}^{\beta}(-1)^{s+t}\binom{\alpha}{s}\binom{\beta}{t}c_{j+s,k+t}.$$

Conditions (1.4) – (1.6) describe certain concept of bounded variation, which are closely related to those in [2, Theorem 2]. Our main result is

Theorem 1.1. *Let $a \geq 0, p \geq 1$, and $\eta > 0$. Assume that $\{c_{jk}\}$ satisfies conditions (1.3) – (1.6). Then series (1.1) converges regularly to some function*

$f(x,y)$ for all $x,y>0$, and the convergence is uniform on any rectangle $\{\epsilon \le x \le \alpha, \delta \le y \le \beta\}$, where $0<\epsilon<\alpha<\infty$ and $0<\delta<\beta<\infty$. Moreover, if ϕ is measurable and locally bounded in $(0,\infty)\times(0,\infty)$, $\hat{\phi}^*(j,k)$ exists for all $j,k\ge 0$, and

(1.7)
$$\sup_{\substack{j,k\ge 0\\ 0<\epsilon<\alpha<\infty\\ 0<\delta<\beta<\infty}} (\bar{j}\bar{k})^{(a+p)/2}\left|\int_\delta^\beta\int_\epsilon^\alpha \phi(x,y)(xy)^{-p/2}\mathcal{L}_j^{a+p}(x)\mathcal{L}_k^{a+p}(y)\,dxdy\right| < \infty,$$

then (1.2) *holds.*

The proof of Theorem 1.1 will be given in the next section. In §3, we investigate the validity of (1.7) for functions of the type $\phi(x,y)=\phi_1(x)\phi_2(y)$, where ϕ_1 and ϕ_2 are of the form $\big(t/(1+t)\big)^{\kappa+1}(1+t)^\mu$.

Throughout this paper C, C_p, and C_{ap} denote constants, which are not necessarily the same at each occurrence.

2. Proof of Theorem 1.1

Set
$$d_{jk}=c_{jk}\sqrt{\frac{j!}{\Gamma(j+a+1)}}\sqrt{\frac{k!}{\Gamma(k+a+1)}}.$$
Then $|d_{jk}|\le C(\bar{j}\bar{k})^{-a/2}|c_{jk}|$. Therefore, (1.3) implies

$$(2.1)\qquad |d_{jk}|\,(\bar{j}\bar{k})^{(a+p)/2-1/4}\longrightarrow 0 \quad \text{as} \quad \max\{j,k\}\to\infty.$$

Using the inequality $1-\sqrt{1-y}\le y$ for $y\in[0,1]$, we get

$$\begin{aligned}\left|\Delta^\mu\left(\sqrt{\frac{j!}{\Gamma(j+a+1)}}\right)\right| &\le C_p\sqrt{\frac{j!}{\Gamma(j+a+1)}}\frac{a}{j+a+1}\\ &\le C_{ap}\,\bar{j}^{-a/2-1} \qquad \text{for} \quad 1\le\mu\le p,\end{aligned}$$

which implies

$$\begin{aligned}|\Delta_{p0}d_{jn}| &\le \Bigg\{\sqrt{\frac{(j+p)!}{\Gamma(j+p+a+1)}}|\Delta_{p0}c_{jn}|\\ &\qquad + C_{ap}\,\bar{j}^{-a/2-1}\sum_{s=0}^{p-1}\binom{p}{s}|\Delta_{s0}c_{jn}|\Bigg\}\sqrt{\frac{n!}{\Gamma(n+a+1)}}\\ &\le C_{ap}(\bar{j}\bar{n})^{-a/2}\left\{|\Delta_{p0}c_{jn}|+\bar{j}^{-1}\Big(\max_{j\le\mu\le j+p-1}|c_{\mu n}|\Big)\right\}.\end{aligned}$$

Putting this with (1.3), (1.4) together yields

$$\lim_{n\to\infty}\sum_{j=0}^{\infty}|\Delta_{p0}d_{jn}|(\bar{j}\bar{n})^{(a+p)/2-1/4}=0. \tag{2.2}$$

Similarly, conditions (1.3) and (1.5) imply

$$\lim_{m\to\infty}\sum_{k=0}^{\infty}|\Delta_{0p}d_{mk}|\,(\bar{m}\bar{k})^{(a+p)/2-1/4}=0. \tag{2.3}$$

We have

$$\begin{aligned}|\Delta_{pp}d_{jk}| \;&\le \sqrt{\frac{(j+p)!}{\Gamma(j+p+a+1)}}\sqrt{\frac{(k+p)!}{\Gamma(k+p+a+1)}}|\Delta_{pp}c_{jk}|\\ &\quad +C_{ap}\,\bar{j}^{-a/2-1}\sum_{s=0}^{p-1}\binom{p}{s}|\Delta_{sp}c_{jk}|\sqrt{\frac{(k+p)!}{\Gamma(k+p+a+1)}}\\ &\quad +C_{ap}\,\bar{k}^{-a/2-1}\sum_{t=0}^{p-1}\binom{p}{t}|\Delta_{pt}c_{jk}|\sqrt{\frac{(j+p)!}{\Gamma(j+p+a+1)}}\\ &\quad +C_{ap}(\bar{j}\bar{k})^{-a/2-1}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\binom{p}{s}\binom{p}{t}|\Delta_{st}c_{jk}|\\ &\le C_{ap}(\bar{j}\bar{k})^{-a/2}\Big\{|\Delta_{pp}c_{jk}|+\bar{j}^{-1}\Big(\max_{j\le\mu\le j+p-1}|\Delta_{0p}c_{\mu k}|\Big)\\ &\quad +\bar{k}^{-1}\Big(\max_{k\le\nu\le k+p-1}|\Delta_{p0}c_{j\nu}|\Big)+(\bar{j}\bar{k})^{-1}\Big(\max_{\substack{j\le\mu\le j+p-1\\ k\le\nu\le k+p-1}}|c_{\mu\nu}|\Big)\Big\}.\end{aligned}$$

Putting this with (1.3) – (1.6) together yields

$$\begin{aligned}&\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{pp}d_{jk}|(\bar{j}\bar{k})^{(a+p)/2-1/4}\\ &\quad\le C_{ap}\Big\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{pp}c_{jk}|(\bar{j}\bar{k})^{p/2-1/4}\\ &\qquad+\sum_{j=0}^{\infty}\bar{j}^{-1}\sum_{k=0}^{\infty}\Big(\max_{j\le\mu\le j+p-1}|\Delta_{0p}c_{\mu k}|\Big)(\bar{j}\bar{k})^{p/2-1/4}\\ &\qquad+\sum_{k=0}^{\infty}\bar{k}^{-1}\sum_{j=0}^{\infty}\Big(\max_{k\le\nu\le k+p-1}|\Delta_{p0}c_{j\nu}|\Big)(\bar{j}\bar{k})^{p/2-1/4}\\ &\qquad+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\bar{j}\bar{k})^{-1}\Big(\max_{\substack{j\le\mu\le j+p-1\\ k\le\nu\le k+p-1}}|c_{\mu\nu}|\Big)(\bar{j}\bar{k})^{p/2-1/4}\Big\}\\ &\quad<\infty.\end{aligned} \tag{2.4}$$

Applying [3, Theorem 2.1] to the series $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_{jk}L_j^a(x)L_k^a(y)$, we see that series (1.1) converges regularly to some function $f(x,y)$ for $x,y>0$, and the convergence is uniform on any rectangle $\{\epsilon \le x \le \alpha, \delta \le y \le \beta\}$, where $0<\epsilon<\alpha<\infty$ and $0<\delta<\beta<\infty$. As proved in [3, Theorem 2.1],

$$\sum_{j=0}^{m}\sum_{k=0}^{n}(\Delta_{pp}d_{jk})\, L_j^{a+p}(x)L_k^{a+p}(y) \longrightarrow \quad e^{(x+y)/2}(xy)^{-a/2}f(x,y) \qquad \text{as} \quad \min\{m,n\}\to\infty,$$

and the convergence is uniform on $\{\epsilon \le x \le \alpha, \delta \le y \le \beta\}$. Set

$$\Phi_{jk}^{st}(\epsilon,\delta,\alpha,\beta) = \int_\delta^\beta\int_\epsilon^\alpha \phi(x,y)L_j^{a+s}(x)L_k^{a+t}(y)e^{-(x+y)/2}(xy)^{a/2}\,dxdy.$$

Then as $\min\{m,n\}\to\infty$,

$$\sum_{j=0}^{m}\sum_{k=0}^{n}(\Delta_{pp}d_{jk})\Phi_{jk}^{pp}(\epsilon,\delta,\alpha,\beta) \longrightarrow \int_\delta^\beta\int_\epsilon^\alpha f(x,y)\phi(x,y)\,dxdy. \tag{2.5}$$

Employing the equation $\sum_{k=0}^{n} L_k^a(t) = L_n^{a+1}(t)$ (cf. [7, Eq. 5.1.13]), we get

$$\begin{aligned}\Phi_{jk}^{st}(\epsilon,\delta,\alpha,\beta) &= \sum_{\mu=0}^{j}\Phi_{\mu k}^{s-1,t}(\epsilon,\delta,\alpha,\beta) = \sum_{\nu=0}^{k}\Phi_{j\nu}^{s,t-1}(\epsilon,\delta,\alpha,\beta)\\ &= \sum_{\mu=0}^{j}\sum_{\nu=0}^{k}\Phi_{\mu\nu}^{s-1,t-1}(\epsilon,\delta,\alpha,\beta).\end{aligned} \tag{2.6}$$

On the other hand,

$$\hat{\phi}^*(j,k) = \Bigg(\lim_{\substack{\epsilon,\delta\to0+\\ \alpha,\beta\to\infty}} \Phi_{jk}^{00}(\epsilon,\delta,\alpha,\beta)\Bigg)\sqrt{\frac{j!}{\Gamma(j+a+1)}}\sqrt{\frac{k!}{\Gamma(k+a+1)}}. \tag{2.7}$$

Since $\hat{\phi}^*(j,k)$ exists for all $j,k\ge0$, it follows from (2.6) and (2.7) that the limit $\zeta_{jk}^{st} \equiv \lim_{\substack{\epsilon,\delta\to0+\\ \alpha,\beta\to\infty}} \Phi_{jk}^{st}(\epsilon,\delta,\alpha,\beta)$ exists for all $s,t,j,k\ge0$. The condition (1.7) on ϕ is equivalent to the existence of the constant C with

$$\sup_{\substack{j,k\ge0\\ 0<\epsilon<\alpha<\infty\\ 0<\delta<\beta<\infty}} \left|\Phi_{jk}^{pp}(\epsilon,\delta,\alpha,\beta)\right| \le C < \infty. \tag{2.8}$$

By (2.6) and (2.8), we can assume

$$|\zeta_{jk}^{st}| \le C \qquad \text{for all } j,k\ge0 \text{ and } 0\le s,t\le p. \tag{2.9}$$

We have $(a+p)/2 - 1/4 \geq 0$. It follows from (2.4), (2.5), and (2.8) that the limit

$$\lim_{\substack{\epsilon,\delta\to 0+\\ \alpha,\beta\to\infty}} \int_\delta^\beta \int_\epsilon^\alpha f(x,y)\phi(x,y)\,dxdy$$

exists and equals

$$\zeta \equiv \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (\Delta_{pp} d_{jk}) \zeta_{jk}^{pp}.$$

The double series defining ζ converges absolutely. For $m,n \geq 0$, we have

$$\begin{aligned}\lambda_{mn} &\equiv \sum_{j=0}^{m}\sum_{k=0}^{n} c_{jk}\,\hat{\phi}^*(j,k) = \lim_{\substack{\epsilon,\delta\to 0+\\ \alpha,\beta\to\infty}} \int_\delta^\beta \int_\epsilon^\alpha \phi(x,y)\, s_{mn}(x,y)\,dxdy \\ &= \lim_{\substack{\epsilon,\delta\to 0+\\ \alpha,\beta\to\infty}} \int_\delta^\beta \int_\epsilon^\alpha \phi(x,y)\, t_{mn}(x,y) e^{-(x+y)/2}(xy)^{a/2}\,dxdy,\end{aligned} \tag{2.10}$$

where $t_{mn}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} d_{jk} L_j^a(x) L_k^a(y)$. Summation by parts gives

$$\begin{aligned} t_{mn}(x,y) = &\sum_{j=0}^{m}\sum_{k=0}^{n} (\Delta_{pp} d_{jk}) L_j^{a+p}(x) L_k^{a+p}(y) \\ &+ \sum_{t=0}^{p-1}\sum_{j=0}^{m} (\Delta_{pt} d_{j,n+1}) L_j^{a+p}(x) L_n^{a+t+1}(y) \\ &+ \sum_{s=0}^{p-1}\sum_{k=0}^{n} (\Delta_{sp} d_{m+1,k}) L_m^{a+s+1}(x) L_k^{a+p}(y) \\ &+ \sum_{s=0}^{p-1}\sum_{t=0}^{p-1} (\Delta_{st} d_{m+1,n+1}) L_m^{a+s+1}(x) L_n^{a+t+1}(y). \end{aligned}$$

Plug this into (2.10) to obtain

$$\begin{aligned} \lambda_{mn} = &\sum_{j=0}^{m}\sum_{k=0}^{n} (\Delta_{pp} d_{jk})\, \zeta_{jk}^{pp} + \sum_{t=0}^{p-1}\sum_{j=0}^{m} (\Delta_{pt} d_{j,n+1})\, \zeta_{jn}^{p,t+1} \\ &+ \sum_{s=0}^{p-1}\sum_{k=0}^{n} (\Delta_{sp} d_{m+1,k})\, \zeta_{mk}^{s+1,p} + \sum_{s=0}^{p-1}\sum_{t=0}^{p-1} (\Delta_{st} d_{m+1,n+1})\, \zeta_{mn}^{s+1,t+1}. \end{aligned}$$

It follows from (2.1) – (2.3) and (2.9) that

$$\begin{aligned}\sum_{t=0}^{p-1}\sum_{j=0}^{m}|\Delta_{pt}d_{j,n+1}||\zeta_{jn}^{p,t+1}| &\le C\sum_{t=0}^{p-1}\sum_{\nu=0}^{t}\binom{t}{\nu}\sum_{j=0}^{m}|\Delta_{p0}d_{j,n+1+\nu}| \\ &\le 2^pC\Big(\sup_{k>n}\sum_{j=0}^{m}|\Delta_{p0}d_{jk}|\Big) \\ &\longrightarrow 0 \quad \text{as} \quad \min\{m,n\}\to\infty,\end{aligned}$$

$$\begin{aligned}\sum_{s=0}^{p-1}\sum_{k=0}^{n}|\Delta_{sp}d_{m+1,k}||\zeta_{mk}^{s+1,p}| &\le C\sum_{s=0}^{p-1}\sum_{\mu=0}^{s}\binom{s}{\mu}\sum_{k=0}^{n}|\Delta_{0p}d_{m+1+\mu,k}| \\ &\le 2^pC\Big(\sup_{j>m}\sum_{k=0}^{n}|\Delta_{0p}d_{jk}|\Big) \\ &\longrightarrow 0 \quad \text{as} \quad \min\{m,n\}\to\infty,\end{aligned}$$

and

$$\begin{aligned}\sum_{s,t=0}^{p-1}|\Delta_{st}d_{m+1,n+1}||\zeta_{mn}^{s+1,t+1}| &\le C\sum_{s,t=0}^{p-1}\sum_{\mu=0}^{s}\sum_{\nu=0}^{t}\binom{s}{\mu}\binom{t}{\nu}|\Delta_{00}d_{m+1+\mu,n+1+\nu}| \\ &\le 2^{2p}C\Big(\sup_{j>m,k>n}|d_{jk}|\Big) \\ &\longrightarrow 0 \quad \text{as} \quad \min\{m,n\}\to\infty.\end{aligned}$$

Hence $\lambda_{mn}\longrightarrow\zeta$ as $\min\{m,n\}\to\infty$. This completes the proof.

3. Investigation of Condition (1.7)

The condition (1.7) with $p=1$ is equivalent to

$$(1.7')\qquad \sup_{\substack{j,k\ge 0\\ 0<\epsilon<\alpha<\infty\\ 0<\delta<\beta<\infty}}\left|\int_\delta^\beta\int_\epsilon^\alpha \phi(x,y)L_j^{a+1}(x)L_k^{a+1}(y)e^{-(x+y)/2}(xy)^{a/2}\,dxdy\right|<\infty.$$

For one-variable case, it reduces to

$$(1.7'')\qquad \sup_{\substack{j\ge 0\\ 0<\epsilon<\alpha<\infty}}\left|\int_\epsilon^\alpha \phi(t)L_j^{a+1}(t)e^{-t/2}t^{a/2}\,dt\right|<\infty.$$

Theorem 3.1. *Let* $0\le a\le 1/2$. *Assume that*

$$(3.1)\qquad |\phi(t)|\le C\Big(\big(\frac{t}{1+t}\big)^{\kappa+1}(1+t)^{\mu}\Big)\qquad (t>0),$$

$$(3.2)\qquad |\phi'(t)| \le C\Big(\big(\frac{t}{1+t}\big)^{\kappa}(1+t)^{\mu}\Big) \qquad (t>0)$$

for some κ,μ *satisfying* $\kappa > -3/4$ *and* $\mu \le -a/2-1$. *Then condition* $(1.7'')$ *holds.*

Proof. Set $\psi(t)=\phi(t)e^{-t/2}t^{a/2}$. From [1, 5] we can find an absolute constant C such that

$$(3.3)\qquad \big|L_j^a(t)\big| \le Ce^{t/2}t^{-a/2-1/4}(\bar{j})^{a/2-1/4}(\bar{t})^{1/2} \qquad (j\ge 0, t>0).$$

Since $a/2-1/4\le 0, \kappa>-3/4$, and $\mu+1/4\le 0$, it follows from (3.1) and (3.3) that

$$(3.4)\qquad \sup_{t>0,j\ge 0}|\psi(t)L_{j+1}^a(t)| \le C\Big(\sup_{t>0}\big|\phi(t)t^{-1/4}(1+t)^{1/2}\big|\Big)<\infty.$$

We have

$$|\psi'(t)| \le e^{-t/2}t^{a/2}\Big\{|\phi'(t)|+\Big(\frac{1}{2}+\frac{a}{2t}\Big)|\phi(t)|\Big\}.$$

For $0<\epsilon<\alpha<\infty$ and $j\ge 0$, as proved in [3, Lemma 3.3], (3.1) and (3.2) imply

$$\begin{aligned}&\Big|\int_\epsilon^\alpha \psi'(t)L_{j+1}^a(t)\,dt\Big|\\ &\quad\le\Big(\int_0^{1/(j+1)}+\int_{1/(j+1)}^1+\int_1^{\nu/2}+\int_{\nu/2}^{3\nu/2}+\int_{3\nu/2}^\infty\Big)|\psi'(t)||L_{j+1}^a(t)|\,dt\\ &\quad\le C,\end{aligned}$$

where $\nu=4j+2a+6$. Hence

$$(3.5)\qquad \sup_{\substack{j\ge 0\\ 0<\epsilon<\alpha<\infty}}\Big|\int_\epsilon^\alpha \psi'(t)L_{j+1}^a(t)\,dt\Big|\le C.$$

Using $\frac{d}{dt}L_{j+1}^a(t)=-L_j^{a+1}(t)$ and integration by parts, we obtain

$$(3.6)\qquad \begin{aligned}\int_\epsilon^\alpha \phi(t)L_j^{a+1}(t)e^{-t/2}t^{a/2}\,dt &= -\int_\epsilon^\alpha \psi(t)\Big(\frac{d}{dt}L_{j+1}^a(t)\Big)\,dt\\ &= -\psi(t)L_{j+1}^a(t)\Big|_\epsilon^\alpha+\int_\epsilon^\alpha \psi'(t)L_{j+1}^a(t)\,dt.\end{aligned}$$

Putting $(3.4)-(3.6)$ together yields $(1.7'')$. This finishes the proof.

Consider the case $\phi(x,y)=\phi_1(x)\phi_2(y)$, where ϕ_1 and ϕ_2 are of the form given in Theorem 3.1. Then $(1.7')$ is satisfied and Theorem 1.1 can apply to such a case.

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