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# ON THE HOMOGENIZATION OF SECOND ORDER DIFFERENTIAL EQUATIONS

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**Abstract.** We discuss the homogenization process of second order differential equations involving highly oscillating coefficients in the time and space variables. It generate memory or nonlocal effect. For initial value problems, the memory kernels are described by Volterra integral equations; and for boundary value problems, they are characterized by Fredholm integral equations. When the equation is translation (in time or in space) invariant, the memory or nonlocal kernel can be represented explicitly in terms of the Young's measure.

## 1. Introduction

Homogenization is concerned with the understanding of oscillations in differential equations. In more intuitive terms, it seeks to understand the equations governing macroscopic quantities in presence of microscopic variations of physical quantities. In order to understand this kind of problems, Luc Tartar started in the 1980's with a simplified model where such memory or nonlocal effect appears [22-24]. The basic fact is that if the microscopic constitutive law has highly oscillating coefficients, the macroscopic constitutive law will present an integral or memory term, with a kernel depending on the way those oscillations are produced. This explains the mathematical meaning for the absorption and spontaneous emission rules in quantum mechanics, that effective equations often have extra nonlocal terms in space and time.

In linear homogenization theory it is possible for a differential equation with an integral term (a memory or nonlocal effect) to arise from an equation with pure differential structure: viscoelastic behavior of composite material is an example

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Dedicated to Professor Hwai-Chiuan Wang with our best wishes in his retirement.

(Enrique Sanchez-Palencia [20]). The transport and dispersion processes in heterogeneous media and the miscible displacement of fluids in a porous medium are interesting examples (Amirat-Hamdache-Ziani [4], Koch et al. [13, 14]). The memory or nonlocal effect can be induced by oscillations in time and/or spatial properties. Indeed, the nonlocal theory provides a convenient setting in which to study transport in macroscopically heterogeneous system [13, 14].

In order to understand the main reasons behind the strange rules of absorption and emission used by physicists, Tartar looked at the following differential equation

$$\partial_t u^\epsilon + a^\epsilon(x) u^\epsilon = f(x,t) \quad \text{in} \quad (0,T)\times\Omega, \quad u^\epsilon|_{t=0} = u_0 \quad \text{in} \quad \Omega\,. \tag{1.1}$$

Here $\Omega$ is an open set of $\mathbf{R}^n$, $\{a^\epsilon\}$ is a sequence of measurable functions bounded in $L^\infty(\Omega)$. The sequence $\{a^\epsilon\}$ takes values between $\alpha$ and $\beta$ and converges to $a^0$ in $L^\infty(\Omega)$ weak $*$. The Young's measure of $a^\epsilon$ contains all the information need. He showed that the limiting equation would have a convolution term:

$$\partial_t u^0 + a^0(x)u^0 - \int_0^t K(x,t-s)u^0(x,s)ds = f(x,t)\,, \quad u^0(x,0) = u_0(x)\,. \tag{1.2}$$

The reason that Eq. (1.2) is natural is that the operator $\partial_t + a^\epsilon$ is linear and commutes with translation in $t$, and a theorem of Laurent Schwartz says that every operator which commutes with translation is a convolution operator. Following Tartar's approach [22, 23], the boundary value problem of the general second order differential equation with time-independent coefficients is discussed thoroughly by N. Antonic [7]. The memory kernel is described by using the eigenfunction expansion and a representation theorem for Nevanlinna functions. Using a factorization of the second-order operator and the Dunford–Taylor integral representation theorem, Y. Amirat, K. Hamdache and A. Ziani [3, 4] derived a nonlocal limiting equation with source terms. For the $\Gamma$-convergence approach to the memory effect, we shall refer to M. Mascarenhas [17] (see also De Giorgi [8] for the original motivation). The $2 \times 2$ Dirac system was studied in [10]. The authors also investigated the $3 \times 3$ systems in [12]. For an introduction to the progress of some current research and applications of homogenization, we refer to the survey paper by Y. Amirat, K. Hamdache and A. Ziani [5].

Homogenization problems which introduce the memory or nonlocal effect are difficult and, despite three decades research, the available results are still restricted to particular types of equations. The purpose of this paper is to present the memory or nonlocal effect induced by homogenization of second order differential equations including initial and boundary value problems. We shall give a survey and some short proofs in this paper. For some of the detail proofs, the reader is referred to [9-11].

The organization of the paper is as follows: In section 2, we study the homogenization of initial value problems of second order differential equations with time dependent coefficients. The second order differential equation with the damping term is also investigated via Liouville transforms. The memory or nonlocal kernel is described by Volterra equations. In second 3, we investigate the case when the coefficient is time-independent. Due to the time translation invariant of the equation, the Volterra equation obtained in section 2 is an integral equation of the convolution type. We can solve the Volterra integral equation by Laplace transform and reduce it to the moment relation. The homogenized equation of the initial value problem is an integro-differential equation with the nonlocal or memory kernel given explicitly as the product of the fundamental solution (in fact, the Green function) of the associated second order differential equation with the Young's measure. The explicit upper and lower bounds of the effective coefficients are also obtained and proved to be optimal. This reflects the fact that the homogenized equation also obeys the finiteness of the speed of propagation. We introduce the kinetic variable to obtain the kinetic formulation of the homogenized equation. Using the kinetic formulation we show that the homogenization procedure does not obey the principle of equipartition of energy.

In the final section, We investigate the homogenization of the boundary value problem of a second order differential equation. The nonlocal term can be described by the eigenfunction expansion. The Fredholm alternative is also discussed. In particular, when the coefficients are independent of $x$, the nonlocal kernel can be characterized explicitly in terms of the Young's measure.

## 2. Second Order Time-dependent Equation

We start from the second order time-dependent equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u^\epsilon(x,t) \; -a^\epsilon(x,t)u^\epsilon(x,t) = f(x,t)\,, & (x,t)\in\Omega\times(0,T)\,, \\ u^\epsilon(x,0) \quad = 0, \quad \partial_t u^\epsilon(x,0)=0, & x\in\Omega\,. \end{cases} \tag{2.1}$$

where the sequence of measurable functions defined by $a^\epsilon(x,t) = a(\frac{x}{\epsilon},t)$ satisfies the bounds

$$\alpha \le a^\epsilon(x,t) \le \beta\,, \qquad \text{a.e. in } [0,\pi]\times[0,T]\,, \tag{2.2}$$

and is equicontinuous in $t$, i.e., there is a function $\varphi$ such that $\varphi(\sigma)\to 0$ as $\sigma\to 0$ and

$$|a^\epsilon(x,t) - a^\epsilon(x,s)| \le \varphi(|t-s|)\,. \tag{2.3}$$

For any continuous function $f(x,t)$ the solution sequences of Eq. (2.1) can be represented as

$$u^\epsilon(x,t) = \int_0^t G^\epsilon(x,s,t) f(x,s)\, ds\,, \tag{2.4}$$

where $G^\epsilon(x,s,t)$ is the Green's function for the initial value problem of the differential operator

$$\mathcal{L} \equiv \frac{\partial^2}{\partial t^2} - a^\epsilon(x,t).$$

It is zero if $t < s$. For $t \geq s$, it is the solution of the differential equation (for fixed $s$ and variable $t$) $\mathcal{L}[G^\epsilon] = 0$ with the initial conditions

$$G^\epsilon(x,s,s) = 0\,, \quad \frac{\partial}{\partial t} G^\epsilon(x,s,s) = 1\,.$$

We define the functions $H^\epsilon$ by

$$H^\epsilon(x,s,t) = a^\epsilon(x,t) G^\epsilon(x,s,t) \quad \text{in} \quad \Omega \times (0,T) \times (0,T).$$

As in Tartar [23], the sequence of measurable functions $\{a^\epsilon\}_\epsilon$ is uniformly bounded in $L^\infty(\Omega \times (0,T))$. According to the Banach-Alaoglu-Bourbaki theorem, a norm bounded set is relatively compact in weak-$*$ topology. Consequently, we may extract a subsequence still denoted by $\{a^\epsilon\}_\epsilon$ with

$$a^\epsilon \overset{w}{\rightharpoonup} a^0 \quad \text{weak} * \quad \text{in} \quad L^\infty\big(\Omega \times (0,T)\big)\,.$$

Compactness requires more than just boundedness here because of the strong topology in $t$. For this reason we appeal to the Arzela-Ascloi theorem which asserts that $\{f_n\}$ is a relatively compact set in $C([0,\infty); w^*\text{-}L^\infty(\Omega))$ if and only if

(i) $\{f_n(t)\}$ is a relatively compact set in $w^*\text{-}L^\infty(\Omega)$ for all $t \geq 0$;

(ii) $\{f_n\}$ is a equicontinuous in $C([0,\infty); w^*\text{-}L^\infty(\Omega))$, i.e. for every $g \in L^1(\Omega)$ the sequence $\{\langle g f_n \rangle\}$ is equicontinuous in $C([0,\infty))$.

We know from (2.3) that $a^\epsilon$ is equicontinuous in $t$, which implies that the functions $G^\epsilon(x,s,t)$ are bounded with bounded derivatives in $s$ and $t$, and that $B^\epsilon$ are bounded with bounded derivatives in $s$ and are equicontinuous in $t$. Since $a^\epsilon$ is uniformly bounded and equicontinuous in $t$, this implies that the functions $G^\epsilon(x,s,t)$ are bounded with bounded derivatives in $s$ and $t$ and that $H^\epsilon$ are bounded with bounded derivatives in $s$ and are equicontinuous in $t$. Using equicontinuity properties in $t$ and then applying the Arzela-Ascoli theorem, we may extract a subsequence such that

$$(2.5)\qquad \begin{cases} a^\epsilon(\cdot,t) \overset{w}{\rightharpoonup} a^0(\cdot,t) & \text{in} L^\infty(\Omega) \ \text{weak}* \ \ \forall t \in (0,T) \\ G^\epsilon(\cdot,s,t) \overset{w}{\rightharpoonup} G^0(\cdot,s,t) & \text{in} \\ H^\epsilon(\cdot,s,t) \overset{w}{\rightharpoonup} H^0(\cdot,s,t) & \text{in } L^\infty(\Omega) \ \text{weak}* \ \ \forall t \in (0,T)\,. \end{cases}$$

Because of the quadratic nature of the second term of the right hand side of (2.1) the weak convergence in (2.5) does not imply

$$a^\epsilon(\cdot,t)G^\epsilon(\cdot,s,t) \overset{w}{\rightharpoonup} a^0(\cdot,t)G^0(\cdot,s,t) \qquad \text{in} \quad L^\infty(\Omega) \ \text{weak}* \ \ \forall t \in (0,T)\,.$$

Consequently, the problem of passage to the limit involves further investigation. Indeed, there is persistence of oscillations of the solutions due to the lack of compactness. For $(x,s,t) \in \Omega \times (0,T) \times (0,T)$ we define the function $C$ by

$$(2.6)\qquad C(x,s,t) \equiv H^0(x,s,t) - a^0(x,t)G^0(x,s,t)\,.$$

Substituting (2.4), the explicit expression of $u^\epsilon$, into (2.1) we find that $u^\epsilon$ satisfying the integro-differential equation

$$(2.7)\qquad \frac{\partial^2}{\partial t^2}u^\epsilon(x,t) - \int_0^t a^\epsilon(x,t)G^\epsilon(x,s,t)f(x,s)\,ds = f(x,t)\,.$$

On the other hand, it follows from (2.2) and (2.4) that the weak limit of $u^\epsilon$ and $a^\epsilon u^\epsilon$ are given respectively by

$$(2.8)\qquad u^0(x,t) = \int_0^t G^0(x,s,t)f(x,s)\,ds\,,$$

$$(2.9)\qquad w^0(x,t) = \int_0^t H^0(x,s,t)f(x,s)ds\,.$$

Taking the limit in (2.7) and using (2.6), we see that $u^0$ satisfies

$$(2.10)\qquad \frac{\partial^2}{\partial t^2}u^0(x,t) - \int_0^t [C(x,s,t) + a^0(x,t)G^0(x,s,t)]f(x,s)\,ds = f(x,t),$$

hence we obtain the following relation

$$(2.11)\qquad \frac{\partial^2}{\partial t^2}u^0(x,t) - a^0(x,t)u^0(x,t) - \int_0^t C(x,s,t)f(x,s)\,ds = f(x,t)\,.$$

To describe the memory or nonlocal kernel we let

$$(2.12)\qquad g(x,t) \equiv f(x,t) + \int_0^t C(x,s,t)f(x,s)\,ds\,.$$

Solving this Volterra integral equation of $f(x,t)$, we obtain

$$f(x,t) = g(x,t) - \int_0^t D(x,s,t) g(x,s)\, ds \tag{2.13}$$

where the kernel $D$ is a solution of the following integral equation

$$D(x,s,t) = C(x,s,t) - \int_s^t C(x,s,\sigma) D(x,\sigma,t)\, d\sigma\,. \tag{2.14}$$

Let

$$g(x,t) = \frac{\partial^2}{\partial t^2} u^0(x,t) - a^0(x,t) u^0(x,t)\,, \tag{2.15}$$

and using (2.11) and (2.13) we find that the weak limit $u^0$ satisfies the integro-differential equation

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\ u^0(x,t) - a^0(x,t)u^0(x,t) \\ &\quad - \int_0^t D(x,s,t) \left[\frac{\partial^2}{\partial s^2} u^0(x,s) - a^0(x,s) u^0(x,s)\right] ds = f(x,t)\,, \end{aligned} \tag{2.16}$$

which, after integrating by part, becomes

$$\frac{\partial^2}{\partial t^2} u^0(x,t) - a^0(x,t)u^0(x,t) - \int_0^t K(x,s,t) u^0(x,s)\, ds = f(x,t)\,, \tag{2.17}$$

where the kernel $K$ is given by

$$K(x,s,t) = \frac{\partial^2}{\partial s^2} D(x,s,t) - a^0(x,s) D(x,s,t)\,, \tag{2.18}$$

with $(x,s,t) \in \Omega \times (0,T) \times (0,T)$. We thus have proved:

**Theorem 2.1.** *Under hypothesis* (2.2)-(2.3) *the homogenized equation of the equation* (2.1) *is*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u^0(x,t) & -a^0(x,t)u^0(x,t) - \displaystyle\int_0^t K(x,s,t) u^0(x,s)\, ds = f(x,t)\,, \\ u^0(x,0) & = 0, \quad \partial_t u^0(x,0) = 0, \quad x \in \Omega\,. \end{cases} \tag{2.19}$$

*The memory kernel $K$ is given by* (2.18) *and the kernel $D$ defined on $\Omega \times (0,T) \times (0,T)$ is the solution of the Volterra equation*

$$\begin{cases} C(x,s,t) & = D(x,s,t) + \displaystyle\int_s^t C(x,s,\sigma) D(x,\sigma,t)\, d\sigma\,, \\ D(x,t,t) & = 0\,, \quad \partial_t D(x,t,t) = 0\,, \end{cases} \tag{2.20}$$

*with $C$ defined by* (2.5) – (2.6).

Using the Liouville transformation we have the following

**Corollary 2.2.** *Under the same hypothesis as Theorem 2.1 the homogenized equation of*

$$\text{(2.21)} \qquad \begin{cases} \dfrac{\partial^2}{\partial t^2}u^\epsilon(x,t) + b(x)\dfrac{\partial}{\partial t}u^\epsilon(x,t) - a^\epsilon(x,t)u^\epsilon(x,t) = f(x,t)\,, \\ u^\epsilon(x,0) = 0, \quad \partial_t u^\epsilon(x,0) = 0, \quad x \in \Omega\,. \end{cases}$$

*is given by*

$$\text{(2.22)} \qquad \begin{cases} \dfrac{\partial^2}{\partial t^2}u^0(x,t) + b(x)\dfrac{\partial}{\partial t}u^0(x,t) - a^0(x,t)u^0(x,t) \\ \qquad\qquad + \displaystyle\int_0^t K(x,s,t)u^0(x,s)ds = f(x,t)\,, \\ u^0(x,0) = 0, \quad \partial_t u^0(x,0) = 0, \quad x \in \Omega \end{cases}$$

*The memory kernel $K$ is given by*

$$\text{(2.23)} \qquad K(x,s,t) = \frac{\partial^2}{\partial s^2}D(x,s,t) - b(x)\frac{\partial}{\partial s}D(x,s,t) - a^0(x,s)D(x,s,t)$$

*where the kernel $D$ defined on $\Omega \times (0,T) \times (0,T)$ is the solution of the Volterra equation* (2.20).

## 3. Second Order Time-independent Equation

In this section we consider the time-independent case;

$$\text{(3.1)} \qquad \begin{cases} \dfrac{\partial^2}{\partial t^2}u^\epsilon(x,t) - a^\epsilon(x)u^\epsilon(x,t) = f(x,t)\,, \quad (x,t) \in \Omega \times (0,T)\,, \\ u^\epsilon(x,0) = 0, \quad \partial_t u^\epsilon(x,0) = 0, \qquad x \in \Omega\,, \end{cases}$$

where $\{a^\epsilon(x)\}$ is a sequence of measurable functions that satisfies

$$\text{(3.2)} \qquad 0 < \alpha \le a^\epsilon(x) \le \beta, \quad a^\epsilon \overset{w}{\rightharpoonup} a^0 \quad \text{in} \quad L^\infty(\Omega) \quad \text{weak}*$$

we then have the following theorem.

**Theorem 3.1.** *There is a subsequence of $\{a^\epsilon\}_\epsilon$ and a kernel $K$ defined on $\Omega \times [0,\infty)$ such that for every bounded and measurable function $f$, the sequence*

$\{u^\epsilon\}_\epsilon$ *of the solutions of problem* (3.1) *converges weak* $*$ *to the unique solution* $u^0$ *of*

$$\text{(3.3)} \qquad \begin{cases} \dfrac{\partial^2}{\partial t^2} u^0(x,t) & -a^0(x)u^0(x,t) + K * u^0 = f(x,t)\,, \\ u^\epsilon(x,0) & = 0, \quad \partial_t u^\epsilon(x,0) = 0\,, \end{cases}$$

*where* $a^0$ *is the weak* $*$ *limit of* $a^\epsilon$ *and the memory kernel* $K$ *admits the integral representation*

$$\text{(3.4)} \qquad K(x,t) = \int_I \frac{-1}{\sqrt{\lambda}} \sinh t\sqrt{\lambda}\, d\mu_x(\lambda)\,.$$

*Here* $\{\mu_x\}$ *is a family of parametrized nonnegative measure associated with* $\{a^\epsilon\}$ *and supported in* $I = [\alpha, \beta]$*. Moreover,* $a^0$ *satisfies the optimal estimates*

$$\text{(3.5)} \qquad 0 < \alpha + \int_I \frac{1}{\lambda - \alpha}\, d\mu_x(\lambda) \le a^0(x) \le \beta - \int_I \frac{1}{\beta - \lambda}\, d\mu_x(\lambda)\,.$$

*Proof.* The display of the kernel $K$ relies upon the use of the Laplace transform in $t$ of Eq. (3.1). The solution of (3.1) can be represented as

$$\text{(3.6)} \qquad u^\epsilon(x,t) = \int_0^t G^\epsilon(x, t-s) f(x,s) ds\,.$$

where the Green's function $G^\epsilon$ is given by

$$\text{(3.7)} \qquad G^\epsilon(x,t) = \begin{cases} \dfrac{1}{\sqrt{a^\epsilon(x)}} \sinh t\sqrt{a^\epsilon(x)}\,, \quad t > 0\,, \quad 0\,, \qquad t < 0\,. \end{cases}$$

In order to study the limit we shall use the Young's measure associated with the sequence $\{a^\epsilon\}$: there exists a probability measure $\nu_x$ such that the weak limit of $a^\epsilon$ is given explicitly by

$$\text{(3.8)} \qquad a^0 = w^*\text{-}\lim a^\epsilon = \langle \lambda, \nu_x \rangle = \int_I \lambda d\nu_x(\lambda)\,.$$

Hence by Young's fundamental theorem, we obtain

$$\text{(3.9)} \qquad \begin{aligned} G^0(x,t-s) &= w^*\text{-}\lim A^\epsilon(x,t-s) = \int_I \frac{1}{\sqrt{\lambda}} \sinh(t-s)\sqrt{\lambda} d\nu_x(\lambda)\,, \\ H^0(x,t-s) &= w^*\text{-}\lim B^\epsilon(x,t-s) = \int_I \frac{\lambda}{\sqrt{\lambda}} \sinh(t-s)\sqrt{\lambda} d\nu_x(\lambda)\,. \end{aligned}$$

Therefore the fluctuation part is

$$C(x,t-s) = H^0 - a^0 G^0 = \int_I (\lambda - a^0)\frac{1}{\sqrt{\lambda}}\sinh(t-s)\sqrt{\lambda}d\nu_x(\lambda)\,. \tag{3.10}$$

Taking the Laplace transform of both sides of (2.14) and (3.10) respectively we have

$$LD(x,p) = \frac{LC(x,p)}{1+LC(x,p)}\,, \tag{3.11}$$

$$LC(x,p) = -1 + \int_I \frac{p^2 - a^0}{p^2-\lambda}d\nu_x(\lambda)\,. \tag{3.12}$$

Hence

$$LD(x,p) = \frac{1}{p^2-a^0}\cdot\left[\int_I (p^2-\lambda)d\nu_x(\lambda) - \left(\int_I \frac{1}{p^2-\lambda}d\nu_x(\lambda)\right)^{-1}\right]. \tag{3.13}$$

By the representation theorem, there exists a parameterized measure $d\mu_x$, defined on $I = [\alpha,\beta]$ associated with the Young's measure $d\nu_x$, satisfying the *moments relation* :

$$\int_I \frac{1}{p^2-\lambda}d\mu_x(\lambda) = \left[\int_I (p^2-\lambda)d\nu_x(\lambda) - \left(\int_I \frac{1}{p^2-\lambda}d\nu_x(\lambda)\right)^{-1}\right] \tag{3.14}$$

Combining (3.13) and (3.14) yields

$$LD(x,p) = \frac{1}{p^2-a^0}\cdot\int_I \frac{1}{p^2-\lambda}d\mu_x(\lambda) \tag{3.15}$$

which can be inverted immediately, giving

$$\begin{aligned} D(x,t-s) &= \left(\frac{1}{\sqrt{a^0}}\sinh(t-s)\sqrt{a^0}\right) * \left(\int_I \frac{1}{2\sqrt{\lambda}}\sinh(t-s)\sqrt{\lambda}d\mu_x(\lambda)\right) \\ &= \int_0^{t-s} \frac{1}{\sqrt{a^0}}\sinh(t-s-\sigma)\sqrt{a^0}\left[\int_I \frac{1}{2\sqrt{\lambda}}\sinh\sigma\sqrt{\lambda}d\mu_x(\lambda)\right]d\sigma\,. \end{aligned} \tag{3.16}$$

Note that the convolution is taken in the time variable $t$. Thus, the memory kernel $K$ is given by

$$\begin{aligned} K(x,t) &= K(x,t-s)\Big|_{s=0} = \frac{\partial^2}{\partial s^2}D(x,t-s) - a^0(x)D(x,t-s)\Big|_{s=0} \\ &= \int_I \frac{-1}{\sqrt{\lambda}}\sinh t\sqrt{\lambda}\,d\mu_x(\lambda) \end{aligned} \tag{3.17}$$

Performing the Laplace transform of $K$ with the aid of the moment relations (3.14) yields

$$LK(x,p) = \left\langle \frac{1}{p^2-\lambda}, \mu_x \right\rangle = \langle p^2-\lambda, \nu_x\rangle - \left\langle \frac{1}{p^2-\lambda}, \nu_x \right\rangle^{-1} \tag{3.18}$$

where $p$ is the Laplace variable corresponding to $t$. Now we are in a position to estimate the upper and lower bounds for the weak limit $a^0(x)$ of $a^\epsilon(x)$. From the moment relation (3.14) and the explicit form of $a^0$ in (3.8), we can define the complex-valued function $F(z)$ by

$$F(z) = z - a^0(x) - \int_I \frac{d\mu_x(\lambda)}{z-\lambda}\,. \tag{3.19}$$

Setting $z = \xi + i\eta$ and taking the real part after differentiation, we obtain

$$F'(\xi) = 1 + \int_I \frac{d\mu_x(\lambda)}{(\xi-\lambda)^2}$$

i.e., $F(\xi)$ is an increasing function for $\xi < \alpha$ and $\beta < \xi$. This leads to

$$\alpha - a^0(x) - \int_I \frac{d\mu_x(\lambda)}{\alpha-\lambda} \le 0 \le \beta - a^0(x) - \int_I \frac{d\mu_x(\lambda)}{\beta-\lambda}\,,$$

or

$$\alpha + \int_I \frac{d\mu_x(\lambda)}{\lambda-\alpha} \le a^0(x) \le \beta - \int_I \frac{d\mu_x(\lambda)}{\beta-\lambda}\,.$$

This completes the proof.

As an immediate consequence of Theorem 3.1, we have the following result on the effect of the damping term.

**Corollary 3.2.** *Under the same hypothesis as Theorem 3.1, the homogenized equation of*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u^\epsilon(x,t) + b(x)\dfrac{\partial}{\partial t}u^\epsilon(x,t) - a^\epsilon(x)u^\epsilon(x,t) = f(x,t)\,, \\ u^\epsilon(x,0) = 0, \quad \partial_t u^\epsilon(x,0) = 0\,, \end{cases} \tag{3.20}$$

*is given by*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u^0(x,t) + b(x)\dfrac{\partial}{\partial t}u^0(x,t) - a^0(x)u^0(x,t) \\ \qquad + \displaystyle\int_0^t K(x,t-s)u^0(x,s)ds = f(x,t)\,, \\ u^\epsilon(x,0) = 0, \quad \partial_t u^\epsilon(x,0) = 0, \end{cases} \tag{3.21}$$

*where the weak limit of* $\{a^\epsilon\}_\epsilon$ *is the same as* (3.8). *The memory kernel* $K$ *is given explicitly by*

$$K(x,t) = \left\langle \exp\Big(-t\frac{b(x)}{2}\Big)\frac{-1}{\sqrt{\frac{b^2(x)}{4}+\lambda}}\sinh t\sqrt{\lambda+\frac{b^2(x)}{4}}, \mu_x \right\rangle \tag{3.22}$$

*and* $a^0$ *satisfies the optimal estimate*

$$\alpha + \int_I \frac{1}{\lambda-\alpha}\, d\mu_x(\lambda) - \frac{1}{4}b^2(x) \le a^0(x) \le \beta - \int_I \frac{1}{\beta-\lambda}\, d\mu_x(\lambda) - \frac{1}{4}b^2(x). \tag{3.23}$$

*The additional correction term* $-\frac{1}{4}b^2(x)$ *is due to the effect of damping.*

Let $S^\epsilon(x,t)$ be the semi-group associated with Eq. (3.1). The weak limit $S(x,t,\lambda)$ of $\{S^\epsilon(x,t)\}$ is not a semi-group in the same class. To see what is the correct class we go to the kinetic formulation. More precisely, we prove the following statement.

**Theorem 3.3.** *The effective equation* (3.3) *admits a kinetic formulation*

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\omega(x,t,\lambda) - \lambda\omega(x,t,\lambda) + u^0(x,t) = 0, \\ &\omega(x,0,\lambda) = 0, \quad \partial_t \omega(x,0,\lambda) = 0\,, \end{aligned} \tag{3.24}$$

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}u^0(x,t) - a^0(x)u^0(x,t) + \int_I \omega(x,t,\lambda)\, d\mu_x(\lambda) = f, \\ &u^0(x,0) = 0\,, \quad \partial_t u^0(x,0) = 0\,, \end{aligned} \tag{3.25}$$

*for* $x \in \Omega$, $t \in (0,T)$ *and* $\lambda \in I$. *The effective equation does not possess the principle of equipartition of energy. Indeed, the total energy of the homogenized equation is decomposed into two parts* $E^0 = E^0_{ma} + E^0_{mi}$ *which are given by*

$$E^0_{ma} = \frac{1}{2}|\partial_t u^0|^2 + \Big(-\frac{1}{2}a^0(x)|u^0|^2 - fu^0\Big) \tag{3.26}$$

$$E^0_{mi} = \int_I \Big(\frac{1}{2}|\partial_t\omega|^2 - \frac{1}{2}\lambda|\omega|^2 + \omega u^0\Big)\, d\mu_x(\lambda)\,. \tag{3.27}$$

*Proof.* We introduce a new variable $\lambda$, describing the oscillations of the sequence $\{a^\epsilon\}_\epsilon$, which lies in the interval $I$ and the auxiliary function $\omega$ defined by

$$\omega(x,t,\lambda) \equiv \int_0^t u^0(x,s)\frac{-1}{\sqrt{\lambda}}\sinh(t-s)\sqrt{\lambda}\, ds\,. \tag{3.28}$$

Here $u^0$ is the weak limit of a subsequence of $\{u^\epsilon\}$ satisfying (3.11). Direct computation shows that $\omega$ satisfies (3.24). If $u^\epsilon$ stands for a position, then Eq. (3.1) is Newton's equation. For convenience we assume that the force field does not depend on the time variable $t$ explicitly. Then the kinetic and potential energies are given by

$$T^\epsilon = \frac{1}{2}|\partial_t u^\epsilon|^2, \quad V^\epsilon = -\frac{1}{2}a^\epsilon(x)|u^\epsilon|^2 - fu^\epsilon \tag{3.29}$$

respectively, and the total energy is

$$E^\epsilon \equiv T^\epsilon + V^\epsilon = \frac{1}{2}|\partial_t u^\epsilon|^2 + \Big( -\frac{1}{2}a^\epsilon(x)|u^\epsilon|^2 - fu^\epsilon \Big) \tag{3.30}$$

It is difficult to characterize the limiting behavior of the total energy from this expression. However, if we multiply (3.25) by $\partial_t u^0$ and then apply (3.24), we find that the total energy $E^0 = E^0_{ma} + E^0_{mi}$ of the homogenized equation is constituted of two parts; the macroscopic energy $E^0_{ma}$ and the microscopic energy $E^0_{mi}$ defined by (3.26) and (3.27). It is obvious that $E^\epsilon \not\rightharpoonup E^0_{ma}$. Thus the principle of equipartition of energy does not hold. These calculations establish the theorem.

**Remark.** The semi-group $S(x,t,\lambda)$ is in a larger class which includes a variable $\lambda$, which plays the role of the kinetic variable, and the auxiliary function $\omega$. When the microscopic velocity is not equal to the macroscopic velocity, the *averaged kinetic energy* $E^0$ is more than the *kinetic energy* $E^0_{ma}$, computed from the microscopic velocity. The difference $E^0_{mi}$ is then called the *internal energy* (see Tartar [23]).

With the same arguments, we prove the following.

**Theorem 3.4.** *The effective equation* (3.21) *admits a kinetic formulation*

$$\begin{cases} ]ds\partial_{tt}u^0(x,t) + b(x)\partial_t u^0(x,t) - a^0(x)u^0(x,t) + \displaystyle\int_I \omega(x,t,\lambda)\, d\mu_x(\lambda) = f, \\ \qquad u^0(x,0) = 0\,, \quad \partial_t u^0(x,0) = 0 \\ \partial_{tt}\omega(x,t,\lambda) + b(x)\partial_t\omega(x,t,\lambda) - \lambda\omega(x,t,\lambda) + u^0(x,t) = 0, \\ \qquad \omega(x,0,\lambda) = 0, \quad \partial_t\omega(x,0,\lambda) = 0 \end{cases} \tag{3.31}$$

*for $x \in \Omega$, $t \in (0,T)$ and $\lambda \in I$. The total energy $E^0$ of the homogenized equation is decomposed into two parts $E^0 = E^0_{ma} + E^0_{mi}$ given respectively by*

$$E^0_{ma} = \frac{1}{2}|\partial_t u^0|^2 + \Big( -\frac{1}{2}a^0(x)|u^0|^2 - fu^0 \Big) + b(x)\int_0^t |\partial_\tau u^0(x,\tau)|^2 d\tau\,, \tag{3.32}$$

$$E^0_{mi} = \int_I \left(\frac{1}{2}|\partial_t \omega|^2 - \frac{1}{2}\lambda|\omega|^2 + \omega u \right. \\ \left. + b(x) \int_I \int_0^t |\partial_\tau \omega(x,\tau,\lambda)|^2 d\tau d\mu_x(\lambda)\,. \right. \tag{3.33}$$

The principle of equipartition of energy does not hold in this case neither.

## 4. Boundary Value Problem

In this section we shall study the behavior of solutions $u^\epsilon$, as $\epsilon$ goes to zero, of the second order differential equation

$$\begin{aligned} \mathcal{L}^\epsilon u^\epsilon(x,t) \equiv\ & -\partial_x^2 u^\epsilon(x,t) + a^\epsilon(x,t)u^\epsilon(x,t) = \lambda u^\epsilon(x,t) + f(x,t), \\ & u^\epsilon(0,\cdot) = u^\epsilon(\pi,\cdot) = 0\,, \end{aligned} \tag{4.1}$$

where $f(x,t), a(x,t) \in C([0,\pi]\times[0,T])$, $(x,t) \in [0,\pi]\times[0,T]$. Here $\lambda$ is a parameter, while the sequence of measurable functions defined by $a^\epsilon(x,t) = a(x,\frac{t}{\epsilon})$ satisfy the bounds

$$\alpha \le a^\epsilon(x,t) \le \beta\,, \qquad \text{a.e. in } [0,\pi]\times[0,T] \tag{4.2}$$

and is equicontinuous in $x$, i.e., there is a function $\varphi$ such that $\varphi(\sigma) \to 0$ as $\sigma \to 0$ and

$$|a^\epsilon(x,t) - a^\epsilon(z,t)| \le \varphi(|x-z|) \tag{4.3}$$

Let us now state the three main theorems.

**Theorem 4.1.** *Under hypotheses* $(4.2)-(4.3)$, *there exists a compact operator* $\mathbf{C}$ *on* $L^2([0,\pi])$ *with the eigenfunctions* $\{\phi_n(x)\}_n$ *and the eigenvalues* $\{\mu_n(\lambda,t)\}_n$ *satisfying*

$$\phi_n(x) - \frac{1}{\mu_n(\lambda,t)}\mathbf{C}\phi_n(x) = 0, \quad n = 1,2,3,\cdots \tag{4.4}$$

*and* $a^0(x,t)$ *is the weak star limit of* $\{a^\epsilon(x,t)\}$ *in* $L^\infty[0,\pi]$, $t\in[0,T]$. *If* $\mu_n(\lambda,t) \neq 1, n = 1,2,\cdots$, *(i.e., 1 is not the eigenvalue of* $\mathbf{C}$*) then there exists a kernel* $K(\lambda,x,y,t)$ *defined on* $[0,\pi]\times[0,\pi]\times[0,T]$ *such that for every bounded and measurable function* $f$, *the sequence* $u^\epsilon$ *of solutions of (4.1) converges weakly to the solution* $u^0(x,t)$ *of*

$$\begin{aligned} \mathcal{L}^0 u^0(x,t) &= \lambda u^0(x,t) + f(x,t) \\ u^0(0,t) &= u^0(\pi,t) = 0, \quad t\in[0,T]\,. \end{aligned} \tag{4.5}$$

*Here $\mathcal{L}^0$ is the integro-differential operator defined by*

$$\begin{aligned} \mathcal{L}^0 u^0(x,t): &= -\frac{\partial^2}{\partial x^2} u^0(x,t) + a^0(x,t)u^0(x,t) \\ &\quad + \int_0^\pi K(\lambda, x, y, t) u^0(y,t)\, dy\,, \end{aligned} \tag{4.6}$$

*and the kernel $K(\lambda, x, y, t)$ is given by*

$$K(\lambda, x, y, t) := \sum_{n=1}^{\infty} \frac{\mu_n(\lambda,t)}{1-\mu_n(\lambda,t)} K_n(\lambda, y, t)\phi_n(x) \tag{4.7}$$

*with*

$$K_n(\lambda, y, t) = -\partial_y^2 \phi_n(y) + (a^0(y,t) - \lambda)\phi_n(y)\,. \tag{4.8}$$

We also have an explicit illustration of the Fredholm Alternative.

**Theorem 4.2.** *Under the same hypothesis as Theorem 4.1, let $\Lambda \equiv \{n : 1-\mu_n(\lambda,t) \neq 0\}$, $\Lambda^c \equiv \{n : 1-\mu_n(\lambda,t) = 0\}$ and if moreover, $f$ is orthogonal to the eigenspace (i.e., $\langle f, \phi_n\rangle = 0, n = 1, 2, 3, \cdots$), then the limit $u^0$ of the sequence of solutions $\{u^\epsilon\}$ of the problem* (4.1) *satisfies the equation*

$$\begin{aligned} \mathcal{L}^0 u^0(x,t) &= \lambda u^0(x,t) + f(x,t) - g(x,t) \\ u^0(0,t) &= u^0(\pi,t) = 0, \quad t \in [0,T]\,. \end{aligned} \tag{4.9}$$

*Here $\mathcal{L}^0$ is the integro-differential operator defined the same as* (4.6)*, and the kernel $K(\lambda, x, y, t)$ is given explicitly in terms of the eigenfunctions*

$$K(\lambda, x, y, t) = \sum_{n\in\Lambda} \frac{\mu_n(\lambda,t)}{1-\mu_n(\lambda,t)} K_n(\lambda, y, t)\phi_n(x) \tag{4.10}$$

*where $K_n$ is defined in* (4.8) *and*

$$g(x,t) = \sum_{n\in\Lambda^c} c_n(t)\phi_n(x) \tag{4.11}$$

*with $c_n$ being arbitrary constants.*

When the oscillating coefficients $\{a^\epsilon\}_\epsilon$ are independent of $x$, we can apply the Young's measure theory developed by Tartar [20, 21] to characterize the kernel $K$.

**Theorem 4.3.** *Under the assumptions of Theorem 4.1, and if* $a^\epsilon(x,t)$ *is independent of* $x$, $a^\epsilon(x,t) = a^\epsilon(t)$, *there exists a compact operator* $\mathbf{C}$ *with eigenfunctions and eigenvalues given by* $\left\{\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx\right\}$ *and* $\left\{\int_I \frac{\omega - a^0(t)}{n^2-(\lambda-\omega)} d\nu_t(\omega)\right\}$ *where* $\nu_t$ *denotes the Young's measure corresponding to a subsequence of* $\{a^\epsilon\}$. *There is a subsequence of* $\{a^\epsilon\}$ *and a kernel* $K$ *associated with* $\{a^\epsilon\}$ *such that the solution sequence* $\{u^\epsilon\}$ *of* (4.1) *converges weakly to the solution* $u^0(x,t)$ *of*

$$
\begin{aligned}
-\frac{\partial^2}{\partial x^2}u^0(x,t) & + a^0(t)u^0(x,t) + \int_0^\pi K(\lambda,x,y,t)u^0(y,t)\,dy \\
& = \lambda u^0(x,t) + f(x,t)
\end{aligned}
\tag{4.12}
$$

$$
u^0(0,t) = u^0(\pi,t) = 0, \quad x \in [0,T]\,. \tag{4.13}
$$

*The kernel* $K(\lambda,x,y,t)$ *is given by*

$$
K(\lambda,x,y,t) = \int_I \left(\sum_{n=1}^{\infty} \frac{-\phi_n(x)\phi_n(y)}{n^2-(\lambda-\omega)}\right) d\mu_t(\omega). \tag{4.14}
$$

*Here the nonnegative parameterized measure* $\mu_t$ *is associated with the Young's measure* $\nu_t$ *through the moment relation* (4.49) *below.*

*Proof. of Theorem 4.1* The solution sequence $\{u^\epsilon(x,t)\}$ of (4.1) is represented by

$$
u^\epsilon(x,t) = \int_0^\pi G^\epsilon(\lambda,x,y,t)f(y,t)dy, \tag{4.15}
$$

where $G^\epsilon(\lambda,x,y,t)$ is the Green's function of (4.1). We define the functions $H^\epsilon$ by

$$
H^\epsilon(\lambda,x,y,t) = (a^\epsilon(x,t)-\lambda)G^\epsilon(\lambda,x,y,t) \tag{4.16}
$$

$(x,y,t) \in [0,\pi]\times[0,\pi]\times[0,T]$. Since $a^\epsilon$ is uniformly bounded and equicontinuous in $x$, this implies that the functions $G^\epsilon(\lambda,x,y,t)$ are bounded and that $H^\epsilon$ are bounded equicontinuous in $x$. Using equicontinuity properties in $x$ and then applying the Arzela-Ascoli theorem, we may extract a subsequence such that

$$
\begin{aligned}
a^\epsilon(x,\cdot) & \overset{w}{\rightharpoonup} a^0(x,\cdot) && \text{in } L^\infty[0,T] \text{ weak}* \ \forall x \in [0,\pi] \\
G^\epsilon(\lambda,x,y,\cdot) & \overset{w}{\rightharpoonup} G^0(\lambda,x,y,\cdot) && \text{in } L^\infty[0,T] \text{ weak}* \ \forall x,y \in [0,\pi] \\
H^\epsilon(\lambda,x,y,\cdot) & \overset{w}{\rightharpoonup} H^0(\lambda,x,y,\cdot) && \text{in } L^\infty[0,T] \text{ weak}* \ \forall x,y \in [0,\pi].
\end{aligned}
\tag{4.17}
$$

For $(x, y, t) \in [0, \pi] \times [0, \pi] \times [0, T]$, we define the function $C$ by

$$C(\lambda, x, y, t) \equiv H^0(\lambda, x, y, t) - (a^0(x, t) - \lambda)G^0(\lambda, x, y, t) \,. \tag{4.18}$$

which describes the corrector of the weak limit. It is clear from (4.15) that $u^\epsilon$ satisfies the following integro-differential equation

$$-\frac{\partial^2}{\partial x^2} u^\epsilon(x, t) + \int_0^\pi (a^\epsilon(x, t) - \lambda)G^\epsilon(\lambda, x, y, t) f(y, t) \, dy = f(x, t) \,. \tag{4.19}$$

Again from (4.15) we see that $u^\epsilon$ and $(a^\epsilon - \lambda)u^\epsilon$ converge weakly respectively to

$$u^0(x, t) = \int_0^\pi G^0(\lambda, x, y, t) f(y, t) \, dy \,, \tag{4.20}$$

$$w^0(\lambda, x, t) \equiv \int_0^\pi H^0(\lambda, x, y, t) f(y, t) dy \,. \tag{4.21}$$

Taking the limit in (4.19), we see that $u^0$ satisfies

$$-\frac{\partial^2}{\partial x^2} u^0(x, t) + (a^0(x, t) - \lambda)u^0(x, t) + \int_0^\pi C(\lambda, x, y, t) f(y, t) \, dy = f(x, t) \,. \tag{4.22}$$

To describe the memory or nonlocal kernel, let

$$h(\lambda, x, t) \equiv f(x, t) - \int_0^\pi C(\lambda, x, y, t) f(y, t) \, dy \,, \quad \text{in } [0, \pi] \times [0, T] \tag{4.23}$$

and let $\mathbf{C}$ be the integral operator generated by $C$:

$$\mathbf{C}(f) \equiv \int_0^\pi C(\lambda, x, y, t) f(y) \, dy, \quad \forall f \in L^2([0, \pi]), \tag{4.24}$$

we see that the integral operator $\mathbf{C}$ is compact and self-adjoint. The Spectral Theorem can be used to obtain the explicit formula for the solution of the Fredlhom integral equation (4.23). Indeed, we have the eigenfunction expansion

$$\mathbf{C}(f) = \sum_{n=1}^{\infty} \mu_n(\lambda, t) \langle f, \phi_n \rangle \phi_n \tag{4.25}$$

where $\{\mu_n(\lambda, t)\}_n$ is a sequence of non-zero real number and $\{\phi_n\}_n$ is orthonormal and $\langle f, \phi_n \rangle = \int_0^\pi f(y)\phi_n(y) dy$ is the inner product. Solving the integral equation (4.23) by eigenfunction expansion, we obtain

$$f(x, t) = h(\lambda, x, t) + \int_0^\pi D(\lambda, x, y, t) h(\lambda, y, t) dy \tag{4.26}$$

where

$$D(\lambda, x, y, t) = \sum_{n=1}^{\infty} \frac{\mu_n(\lambda, t)}{1 - \mu_n(\lambda, t)} \phi_n(y)\phi_n(x) \tag{4.27}$$

is the resolvent kernel corresponding to the kernel $C(\lambda, x, y, t)$. If we let

$$h(\lambda, x, t) = -\partial_x^2 u^0(x, t) + (a^0(x, t) - \lambda)u^0(x, t) \tag{4.28}$$

so that from (4.26) and integration by parts, we derive

$$\begin{aligned} -\partial_x^2 u^0(x, t) \ &+ a^0(x, t)u^0(x, t) + \int_0^{\pi} K(\lambda, x, y, t)u^0(y, t)\, dy \\ &= \lambda u^0(x, t) + f(x, t) \end{aligned} \tag{4.29}$$

where the kernel $K$ is defined by the equation

$$\begin{aligned} K(\lambda, x, y, t) \ &= -\partial_y^2 D(\lambda, x, y, t) + (a^0(y, t) - \lambda)D(\lambda, x, y, t) \\ &= \sum_{n=1}^{\infty} \frac{\mu_n(\lambda, t)}{1 - \mu_n(\lambda, t)} K_n(y, t)\phi_n(x) \end{aligned} \tag{4.30}$$

with $K_n(\lambda, y, t) = -\partial_y^2 \phi_n(y) + (a^0(y, t) - \lambda)\phi_n(y)$. This completes the proof of Theorem 4.1.

*Proof. of Theorem 4.2.* If, in the summation of (4.27), there are values of $m$ for which $1 - \mu_m(\lambda, t) = 0$, let $\Lambda \equiv \{m : 1 - \mu_m(\lambda, t) \neq 0\}$ and $\Lambda^c \equiv \{m : 1 - \mu_m(\lambda, t) = 0\}$. Then $\Lambda^c$ is a finite set and if $m \in \Lambda^c$ and (4.23) has a solution, then $\langle h, \phi_m \rangle = 0$. Since $\{\phi_m : m \in \Lambda^c\}$ is a basis for the set of $\psi$ satisfying

$$\psi - \mathbf{C}(\psi) = \psi - \mathbf{C}^*(\psi) = 0$$

we have the usual condition for the existence of a solution. Indeed, by the Hilbert-Schmidt formula, if $h$ satisfies this condition, the solution of the Fredholm integral equation (4.23) is

$$f(x, t) = h(\lambda, x, t) + \sum_{n \in \Lambda} \frac{\mu_n(\lambda, t)}{1 - \mu_n(\lambda, t)} \langle h, \phi_n \rangle \phi_n(x) + \sum_{n \in \Lambda^c} c_n(t)\phi_n(x) \tag{4.31}$$

where $\{c_n : n \in \Lambda^c\}$ are arbitrarily chosen constants. We may interchange the order of integration and summation to obtain

$$f(x, t) = h(\lambda, x, t) + \int_0^{\pi} D_1(\lambda, x, y, t)h(\lambda, y, t)dy + g(x, t) \tag{4.32}$$

where

$$
\begin{aligned}
D_1(\lambda,x,y,t) &= \sum_{n\in\Lambda} \frac{\mu_n(\lambda,t)}{1-\mu_n(\lambda,t)}\phi_n(y)\phi_n(x) \\
g(x,t) &= \sum_{n\in\Lambda^c} c_n(t)\phi_n(x)
\end{aligned} \tag{4.33}
$$

Similarly, consider the function $h$

$$
h(\lambda,x,t) = -\partial_x^2 u^0(x,t) + (a^0(x,t)-\lambda)u^0(x,t), \tag{4.34}
$$

integration by part produces

$$
\begin{aligned}
-\partial_x^2 u^0(x,t) &+ a^0(x,t)u^0(x,t) + \int_0^\pi K(\lambda,x,y,t)u^0(y,t)]dy \\
&= \lambda u^0(x,t) + f(x,t) - g(x,t).
\end{aligned} \tag{4.35}
$$

The kernel $K$ is given by

$$
\begin{aligned}
K(\lambda,x,y,t) &= \partial_y^2 D_1(x,y,t) - (a^0(y,t)-\lambda)D_1(\lambda,x,y,t) \\
&= \sum_{n\in\Lambda^c} \frac{\mu_n(\lambda,t)}{1-\mu_n(\lambda,t)}K_n(\lambda,y,t)\phi_n(x)\,.
\end{aligned} \tag{4.36}
$$

This proves Theorem 4.2.

*Proof of Theorem 4.3.* When the coefficient $a^\epsilon$ is independent of $x$, $a^\epsilon(x,t) = a^\epsilon(t)$, then for $(x,t) \in [0,\pi]\times[0,T]$ Eq. (4.1) – (4.2) become

$$
\begin{cases}
-\dfrac{\partial^2}{\partial x^2}u^\epsilon(x,t) + a^\epsilon(t)u^\epsilon(x,t) = \lambda u^\epsilon(x,t) + f(x,t)\,, \\
u^\epsilon(0,t) = 0, \quad u^\epsilon(\pi,t) = 0.
\end{cases} \tag{4.37}
$$

Let $\delta$ be the Dirac delta function. Then for

$$
\left[-\frac{\partial^2}{\partial x^2} + (a^\epsilon(t)-\lambda)\right]G^\epsilon(\lambda,x,y,t) = \delta(x-y) \tag{4.38}
$$

subject to the boundary conditions, the Green's function is given by

$$
\begin{aligned}
G^\epsilon(\lambda,x,y,t) = & \frac{-1}{\sqrt{\lambda-a^\epsilon(t)}\sin\sqrt{\lambda-a^\epsilon(t)}\pi} \\
& \begin{cases}
\sin\sqrt{\lambda-a^\epsilon(t)}(\pi-y)\sin\sqrt{\lambda-a^\epsilon(t)}x, \\
\qquad 0\le x\le y\,, \\
\sin\sqrt{\lambda-a^\epsilon(t)}(\pi-x)\sin\sqrt{\lambda-a^\epsilon(t)}y, \\
\qquad y\le x\le \pi\,,
\end{cases}
\end{aligned} \tag{4.39}
$$

Thus, the solution of (4.37) can be represented as

$$u^\epsilon(x,t) = \int_0^\pi G^\epsilon(\lambda, x, y, t) f(y,t) dy. \tag{4.40}$$

It follows from the Young's fundamental theorem that

$$\begin{aligned} a^0 &= w^*\text{-}\lim a^\epsilon = \langle \omega, \nu_t \rangle = \int_I \omega d\nu_t(\omega)\,, \\ G^0(\lambda, x, y, t) &= w^*\text{-}\lim G^\epsilon(\lambda, x, y, t) = \int_I G(\lambda, x, y, \omega) d\nu_t(\omega)\,, \\ B^0(\lambda, x, y, t) &= w^*\text{-}\lim B^\epsilon(\lambda, x, y, t) = \int_I (\omega - \lambda) G(\lambda, x, y, \omega) d\nu_t(\omega)\,, \end{aligned} \tag{4.41}$$

where $I = [\alpha, \beta]$ , $\nu_t$ is the associated Young's measure and

$$\begin{aligned} G(\lambda, x, y, \omega) = \; & \frac{-1}{\sqrt{\lambda - \omega} \sin \sqrt{\lambda - \omega} \pi} \\ & \begin{cases} \sin \sqrt{\lambda - \omega}(\pi - y) \sin \sqrt{\lambda - \omega} x, & 0 \le x \le y\,, \\ \sin \sqrt{\lambda - \omega}(\pi - x) \sin \sqrt{\lambda - \omega} y, & y \le x \le \pi\,, \end{cases} \end{aligned} \tag{4.42}$$

is the Green's function for

$$\begin{aligned} -\partial_x^2 v(x,t) + (\omega - \lambda) v(x,t) &= f(x,t), \\ v(0,t) = v(\pi, t) &= 0. \end{aligned} \tag{4.43}$$

Now, the fluctuation part is

$$C(\lambda, x, y, t) = B^0 - (a^0 - \lambda) G^0 = \int_I (\omega - a^0(t)) G(\lambda, x, y, \omega) d\nu_t(\omega)\,. \tag{4.44}$$

To characterize the kernel, we express the Green's function $G(\lambda, x, y, \omega)$ as the sum of a series

$$G(\lambda, x, y, \omega) = \sum_{n=1}^{\infty} \frac{1}{s_n} \phi_n(x) \phi_n(y) \tag{4.45}$$

which involves all the eigenvalues $s_n = n^2 - (\lambda - \omega)$ and eigenvectors $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$. Define the operator $\mathbf{C}$ by

$$\mathbf{C}\phi_n(x) := \int_0^\pi C(\lambda, x, y, t) \phi_n(y) dy = \mu_n(t) \phi_n(x)\,, \tag{4.46}$$

with $\mu_n(t)$ given by

$$\mu_n(t) = \int_I \frac{\omega - a^0(t)}{n^2 - (\lambda - \omega)} d\nu_t(\omega)\,. \tag{4.47}$$

After simple calculations, we have

$$\begin{aligned} &\left[\int_I \frac{\omega - a^0(t)}{n^2 - (\lambda - \omega)} d\nu_t(\omega) \Big/ \int_I \frac{1}{n^2 - (\lambda - \omega)} d\nu_t(\omega)\right] \\ &= \left[-(n^2 - (\lambda - a^0(t))) + \left(\int_I \frac{1}{n^2 - (\lambda - \omega)} d\nu_t(\omega)\right)^{-1}\right]. \end{aligned} \tag{4.48}$$

Recall the *moments relation* :

$$\int_I \frac{1}{p + \omega} d\mu_t(\omega) = \left[\int_I (p + \omega) d\nu_t(\omega) - \left(\int_I \frac{1}{p + \omega} d\nu_t(\omega)\right)^{-1}\right] \tag{4.49}$$

where $d\mu_t$ is the parameterized measure defined in $I = [\alpha, \beta]$ associated with the Young's measure $d\nu_t$. Combining (4.48) and (4.49) yields

$$\frac{\mu_n(t)}{1 - \mu_n(t)} = \frac{-1}{n^2 - (\lambda - a^0(t))} \int_I \frac{1}{n^2 - (\lambda - \omega)} d\mu_t(\omega). \tag{4.50}$$

From (4.27) the resolvent kernel can be represented as

$$\begin{aligned} D\quad &(\lambda, x, y, t) \\ &= \sum_{n=1}^{\infty} \left(\frac{-1}{n^2 - (\lambda - a^0(t))} \int_I \frac{1}{n^2 - (\lambda - \omega)} d\mu_t(\omega)\right) \phi_n(x)\phi_n(y) \end{aligned} \tag{4.51}$$

Therefore the nonlocal kernel is

$$\begin{aligned} K(\lambda, x, y, t) &= -\partial_y^2 D(\lambda, x, y, t) + (a^0(t) - \lambda) D(\lambda, x, y, t) \\ &= \sum_{n=1}^{\infty} \left(\int_I \frac{-1}{n^2 - (\lambda - \omega)} d\mu_t(\omega)\right) \phi_n(x)\phi_n(y). \end{aligned} \tag{4.52}$$

This completes the proof of Theorem 4.3.

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