SHARP ERROR BOUNDS OF A QUADRATURE RULE WITH ONE MULTIPLE NODE FOR THE FINITE HILBERT TRANSFORM IN SOME CLASSES OF CONTINUOUS DIFFERENTIABLE FUNCTIONS

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Abstract. Some inequalities and approximations for the finite Hilbert transform by the use of Taylor's formula with the integral remainder are given.

1. INTRODUCTION

Cauchy principal value integrals of the form

$$(1.1) \quad (Tf)(a,b;t) = PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau := \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)}{\tau - t} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau - t} d\tau \right]$$

play an important role in fields like aerodynamics, the theory of elasticity and other areas of the engineering sciences. They are also helpful tools in some methods for the solution of differential equations (cf., e.g. [22]).

For different approaches in approximating the finite Hilbert transform (1.1) including: interpolatory, noninterpolatory, Gaussian, Chebychevian and spline methods, see for example the papers [1-12, 13-21, 23-32] and the references therein.

In contrast with all these methods, we point out here a new method in approximating the finite Hilbert transform by the use of the Taylor formula with integral remainder for functions whose n-th derivatives are absolutely continuous.

For a comprehensive list of papers on generalized Taylor formulae and Ostrowski's inequality, visit the site http://rgmia.vu.edu.au

Estimates for the error bounds and some numerical examples for the obtained approximation are also presented.

2. Inequalities on the Whole Interval $[a, b]$

The following result holds.

Theorem 1. *Let* $f : [a, b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ $(n \geq 2)$ *is absolutely continuous on* $[a, b]$. *Then we have the bounds*:

$$\left| (Tf)(a, b; t) - f(t) \ln\left(\frac{b-t}{t-a}\right) - \sum_{k=1}^{n-1} \frac{f^{(k)}(t)}{k!} \cdot \left[\frac{(b-t)^k + (-1)^{k+1}(t-a)^k}{k} \right] \right|$$

$$(2.1) \leq \begin{cases} \dfrac{\|f^{(n)}\|_{[a,b],\infty}}{n \cdot n!}[(b-t)^n + (t-a)^n], & if\ f^{(n)} \in L_\infty[a, b]; \\[3mm] \dfrac{q\|f^{(n)}\|_{[a,b],p}\left[(b-t)^{n-1+\frac{1}{q}} + (t-a)^{n-1+\frac{1}{q}}\right]}{(n-1)![(n-1)q+1]^{1+\frac{1}{q}}}, & if\ f^{(n)} \in L_p[a, b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[3mm] \dfrac{\|f^{(n)}\|_{[a,b],1}}{(n-1) \cdot (n-1)!}\left[(b-t)^{n-1} + (t-a)^{n-1}\right], \end{cases}$$

$$\leq \begin{cases} \dfrac{(b-a)^n}{n \cdot n!}\|f^{(n)}\|_{[a,b],\infty}, & if\ f^{(n)} \in L_\infty[a, b]; \\[3mm] \dfrac{q(b-a)^{n-1+\frac{1}{q}}}{(n-1)![(n-1)q+1]^{1+\frac{1}{q}}}\|f^{(n)}\|_{[a,b],p}, & if\ f^{(n)} \in L_p[a, b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[3mm] \dfrac{1}{(n-1) \cdot (n-1)!}(b-a)^{n-1}\|f^{(n)}\|_{[a,b],1}, \end{cases}$$

for any $t \in (a, b)$.

Proof. Start with Taylor's formula for a function $g : I \to \mathbb{R}$ (I is a compact interval) with the property that $g^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on I, then we have

$$g(x) = \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!}g^{(k)}(a) + \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}g^{(n)}(t)dt,$$

where $a, x \in \mathring{I}$ ($\mathring{I}$ is the interior of I). This implies that

$$\left| g(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!}g^{(k)}(a) \right| \leq \frac{1}{(n-1)!}\left| \int_a^x |x-t|^{n-1}|g^{(n)}(t)|dt \right|$$

$$=: \frac{1}{(n-1)!} \cdot M(x)$$

for any $a, x \in \mathring{I}$.

Before we estimate $M(x)$, let us introduce the following notations

$$\|h\|_{[a,x],p} := \left| \int_a^x |h(t)|^p dt \right|^{\frac{1}{p}} \quad \text{if} \ p \geq 1$$

and

$$\|h\|_{[a,x],\infty} := ess \sup_{\substack{t \in [a,x] \\ t \in ([x,a])}} |h(t)|,$$

where $a, x \in \mathring{I}$.

It is obvious now that

$$M(x) \leq \sup_{\substack{t \in [a,x] \\ (t \in [x,a])}} \left| g^{(n)}(t) \right| \left| \int_a^x |x - t|^{n-1} dt \right| = \left\| g^{(n)} \right\|_{[a,x],\infty} \frac{|x - a|^n}{n!}$$

for any $a, x \in \mathring{I}$.

Using Hölder's integral inequality, we may state that

$$M(x) \leq \left| \int_a^x |g^{(n)}(t)|^p dt \right|^{\frac{1}{p}} \left| \int_a^x |x - t|^{(n-1)q} dt \right|^{\frac{1}{q}}$$

$$= \left\| g^{(n)} \right\|_{[a,x],p} \frac{|x - a|^{n-1+\frac{1}{q}}}{[(n-1)q + 1]^{\frac{1}{q}}}$$

for any $a, x \in \mathring{I}$.

Also, we observe that

$$M(x) \leq |x - a|^{n-1} \left| \int_a^x |g^{(n)}(t)| dt \right| = \left\| g^{(n)} \right\|_{[a,x],1} |x - a|^{n-1}$$

for all $a, x \in \mathring{I}$.

In conclusion, we may state the following inequality which will be used in the sequel

$$\left| g(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} g^{(k)}(a) \right|$$

$$(2.2) \quad \leq \begin{cases} \dfrac{|x-a|^n}{n!} \|g^{(n)}\|_{[a,x],\infty} & \text{if} \ g^{(n)} \in L_\infty(\mathring{I}); \\[2ex] \dfrac{|x-a|^{n-1+\frac{1}{q}}}{(n-1)![(n-1)q+1]^{\frac{1}{q}}} \|g^{(n)}\|_{[a,x],p} & \text{if} \ g^{(n)} \in L_p(\mathring{I}), \\ & \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \dfrac{|x-a|^{n-1}}{(n-1)!} \|g^{(n)}\|_{[a,x],1} \end{cases}$$

for any $a, x \in \mathring{I}$.

Now, let us note for the function $f_0 : [a, b] \to \mathbb{R}$, $f_0(t) = 1$, we have that

$$(Tf_0)(a, b; t) = \ln\left(\frac{b - t}{t - a}\right), \quad t \in (a, b),$$

and then

$$(Tf)(a, b; t) = PV \int_a^b \frac{f(\tau) - f(t) + f(t)}{\tau - t} d\tau$$

$$= PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + f(t) \ln\left(\frac{b - t}{t - a}\right),$$

giving the equality

$$(2.3) \qquad (Tf)(a, b; t) - f(t) \ln\left(\frac{b - t}{t - a}\right) = PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau.$$

Writing (2.2) for $g = f$, $x = \tau$, $a = t$, we get

$$(2.4) \qquad \left| \frac{f(\tau) - f(t)}{\tau - t} - \sum_{k=1}^{n-1} \frac{(\tau - t)^{k-1}}{k!} f^{(k)}(t) \right|$$

$$\leq \begin{cases} \dfrac{|\tau - t|^{n-1}}{n!} \|f^{(n)}\|_{[t,\tau],\infty} & \text{if} \quad f^{(n)} \in L_\infty[a, b]; \\[2ex] \dfrac{\|\tau - t\|^{n-2+\frac{1}{q}}}{(n-1)![(n-1)q + 1]^{\frac{1}{q}}} \|f^{(n)}\|_{[t,\tau],p} & \text{if} \quad f^{(n)} \in L_p[a, b], \\[1ex] & \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \dfrac{|\tau - t|^{n-2}}{(n-1)!} \|f^{(n)}\|_{[t,\tau],1} \end{cases}$$

for any $t, \tau \in (a, b)$, $t \neq \tau$.

If we take the PV in (2.4), then we may write

$$(2.5) \qquad \left| PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau - \sum_{k=1}^{n-1} \frac{f^{(k)}(t)}{k!} PV \int_a^b (\tau - t)^{k-1} d\tau \right|$$

$$\leq PV \int_a^b \left| \frac{f(\tau) - f(t)}{\tau - t} - \sum_{k=1}^{n-1} \frac{(\tau - t)^{k-1}}{k!} f^{(k)}(t) \right| d\tau$$

$$\leq \begin{cases} \dfrac{1}{n!}PV\displaystyle\int_a^b |\tau-t|^{n-1}\|f^{(n)}\|_{[t,\tau],\infty}d\tau \\[2mm] \dfrac{1}{(n-1)![(n-1)q+1]^{\frac{1}{q}}}PV\displaystyle\int_a^b |\tau-t|^{n-2+\frac{1}{q}}\|f^{(n)}\|_{[t,\tau],p}d\tau \\[2mm] \dfrac{1}{(n-1)!}PV\displaystyle\int_a^b |\tau-t|^{n-2}\|f^{(n)}\|_{[t,\tau],1}d\tau \end{cases}$$

$$\leq \begin{cases} \dfrac{1}{n!}\|f^{(n)}\|_{[a,b],\infty}PV\displaystyle\int_a^b |\tau-t|^{n-1}d\tau \\[2mm] \dfrac{1}{(n-1)![(n-1)q+1]^{\frac{1}{q}}}\|f^{(n)}\|_{[a,b],p}PV\displaystyle\int_a^b |\tau-t|^{n-2+\frac{1}{q}}d\tau \\[2mm] \dfrac{1}{(n-1)!}\|f^{(n)}\|_{[a,b],1}PV\displaystyle\int_a^b |\tau-t|^{n-2}d\tau. \end{cases}$$

However,

$$PV\int_a^b |\tau-t|^{n-1}d\tau = \frac{1}{n}[(b-t)^n+(t-a)^n],$$

$$PV\int_a^b |\tau-t|^{n-2+\frac{1}{q}}d\tau = \frac{q}{[(n-1)q+1]}\left[(b-t)^{n-1+\frac{1}{q}}+(t-a)^{n-1+\frac{1}{q}}\right],$$

$$PV\int_a^b |\tau-t|^{n-2}d\tau = \frac{1}{n-1}\left[(b-t)^{n-1}+(t-a)^{n-1}\right]$$

and

$$PV\int_a^b (\tau-t)^{k-1}d\tau = \frac{1}{k}\left[(b-t)^k+(-1)^{k+1}(t-a)^k\right]$$

and then by (2.5) we deduce the desired result (2.1). ∎

It is obvious that the best inequality one would deduce from (2.1) is the one for $t=\frac{a+b}{2}$, getting the following corollary.

Corollary 1. *With the assumptions of Theorem* 1, *we have*

$$\left|(Tf)\left(a,b;\frac{a+b}{2}\right)-\sum_{k=1}^{n-1}\frac{(b-a)^k}{2^k\cdot k\cdot k!}\left[1+(-1)^{k+1}\right]f^{(k)}\left(\frac{a+b}{2}\right)\right|$$

$$(2.6)\ \leq \begin{cases} \dfrac{(b-a)^n}{2^{n-1}\cdot n\cdot n!}\|f^{(n)}\|_{[a,b],\infty}, & if\quad f^{(n)}\in L_\infty[a,b]; \\[3mm] \dfrac{q(b-a)^{n-1+\frac{1}{q}}}{2^{n-2+\frac{1}{q}}(n-1)![(n-1)q+1]^{1+\frac{1}{q}}}\|f^{(n)}\|_{[a,b],p}, & if\quad f^{(n)}\in L_p[a,b], \\ & \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[3mm] \dfrac{(b-a)^{n-1}}{2^{n-2}\cdot(n-1)\cdot(n-1)!}\|f^{(n)}\|_{[a,b],1}. & \end{cases}$$

It is important to note that for small intervals, we basically have the following representation:

Corollary 2. *Assume that $f \in C^\infty[a,b]$ and $0 < b - a \leq 1$. Then*

$$(Tf)(a,b;t) = f(t)\ln\left(\frac{b-t}{t-a}\right) + \sum_{k=1}^{\infty}\frac{f^{(k)}(t)}{k!}\cdot\left[\frac{(b-t)^k + (-1)^{k+1}(t-a)^k}{k}\right]$$

and the convergence is uniform by rapport of $t \in [a,b]$.

3. THE COMPOSITE CASE

The following lemma holds.

Lemma 1. *Let $g : [a,b] \to \mathbb{R}$ be such that $g^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$. Then for any $m \in \mathbb{N}$, $m \geq 1$, we have the inequality:*

$$\left|\frac{1}{b-a}\int_a^b g(u)du - \sum_{i=0}^{m-1}\sum_{k=1}^{n}\frac{(b-a)^{k-1}}{m^k k!}\cdot g^{(k-1)}\left(a + i\cdot\frac{b-a}{m}\right)\right|$$

$$(3.1) \qquad \leq \begin{cases} \dfrac{(b-a)^n}{m^n(n+1)!}\|g^{(n)}\|_{[a,b],\infty}, & \text{if} \quad g^{(n)} \in L_\infty[a,b]; \\[2ex] \dfrac{(b-a)^{n-1+\frac{1}{q}}}{m^n n!(nq+1)^{\frac{1}{q}}}\|g^{(n)}\|_{[a,b],p}, & \text{if} \quad g^{(n)} \in L_p[a,b], \\[1ex] & \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \dfrac{(b-a)^{n-1}}{m^n n!}\|g^{(n)}\|_{[a,b],1}. \end{cases}$$

Proof. Write Taylor's formula with the integral remainder for $\varphi(x) = \int_\alpha^x g(u)du$ and then choose $x = \beta$, to get:

$$\left|\int_\alpha^\beta g(u)du - \sum_{k=1}^{n}\frac{(\beta-\alpha)^k}{k!}g^{(k-1)}(\alpha)\right|$$

$$(3.2) \qquad \leq \begin{cases} \dfrac{|\beta-\alpha|^{n+1}}{(n+1)!}\|g^{(n)}\|_{[\alpha,\beta],\infty}, & \text{if} \quad g^{(n)} \in L_\infty[a,b]; \\[2ex] \dfrac{|\beta-\alpha|^{n+\frac{1}{q}}}{n!(nq+1)^{\frac{1}{q}}}\|g^{(n)}\|_{[\alpha,\beta],p}, & \text{if} \quad g^{(n)} \in = L_p[a,b], \\[1ex] & \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \dfrac{|\beta-\alpha|^n}{n!}\|g^{(n)}\|_{[\alpha,\beta],1} \end{cases}$$

for any $\alpha, \beta \in [a, b]$.

Now, if we consider the division

$$I_n : x_i = a + i \cdot \frac{b-a}{m}, \quad i = \overline{0, m},$$

and apply (3.2) on the intervals $[x_i, x_{i+1}] = (i = \overline{0, m-1})$, we can write:

$$\left| \int_{x_i,}^{x_{i+1}} g(u)du - \sum_{k=1}^{n} \frac{(b-a)^k}{m^k k!} \cdot g^{(k-1)} \left(a + i \cdot \frac{b-a}{m} \right) \right|$$

$$\leq \begin{cases} \dfrac{(b-a)^{n+1}}{m^{n+1}(n+1)!} \|g^{(n)}\|_{[x_i, x_{i+1}], \infty}, \\[3mm] \dfrac{(b-a)^{n+\frac{1}{q}}}{m^{n+\frac{1}{q}} n! (nq+1)^{\frac{1}{q}}} \|g^{(n)}\|_{[x_i, x_{i+1}], p}, \\[3mm] \dfrac{(b-a)^{n}}{m^{n} n!} \|g^{(n)}\|_{[x_i, x_{i+1}], 1}. \end{cases}$$

Summing over i from 0 to $m-1$ and using the generalized triangle inequality, we deduce (3.2). $\blacksquare$

The following main result holds.

Theorem 2. *Let* $f : [a, b] \to \mathbb{R}$ *be such that* $f^{(n)}$ $(n \geq 1)$ *is absolutely continuous on* $[a, b]$. *Then for any* $m \in \mathbb{N}$, $m \geq 1$, *we have:*

$$(Tf)(a, b; t) = f(t) \ln \left(\frac{b-t}{t-a} \right) + A_{n,m}(f, t) + R_{n,m}(f, t),$$

where

$$A_{n,m}(f, t)$$

$$= \sum_{k=1}^{n} \frac{f^{(k)}(t)}{m^k k!} \cdot \left[\frac{(b-t)^k + (-1)^{k+1}(t-a)^k}{k} \right] + (b-a) \sum_{i=1}^{m-1} \sum_{k=1}^{n} \frac{1}{m^k k!}$$

(3.3)

$$\times \left\{ \sum_{\nu=1}^{k-1} (-1)^{\nu-1}(k-1)\cdots(k-\nu) \left(\frac{m}{i} \right)^{\nu-1} \right.$$

$$\times \left[f^{(k-\nu)}; t + \frac{i}{m}(b-t), t - \frac{i}{m}(t-a) \right]$$

$$\left. + (-1)^{k-1} \left(\frac{m}{i} \right)^{k-1} (k-1)! \left[f; t + \frac{i}{m}(b-t), t - \frac{i}{m}(t-a) \right] \right\}$$

and the remainder $R_{n,m}(f,t)$ satisfies the estimate

$$|R_{n,m}(f,t)|$$

(3.4)

$$\leq \begin{cases} \dfrac{\|f^{(n+1)}\|_{[a,b],\infty}}{m^n(n+1)!\cdot(n+1)}\left[(b-t)^{n+1}+(t-a)^{n+1}\right], & if \quad f^{(n+1)}\in L_\infty[a,b]; \\[2ex] \dfrac{q\left[(b-t)^{n+\frac{1}{q}}+(t-a)^{n+\frac{1}{q}}\right]}{m^n n!(nq+1)^{1+\frac{1}{q}}}\|f^{(n+1)}\|_{[a,b],p}, & if \quad f^{(n+1)}\in L_p[a,b], \\ & \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{\|f^{(n+1)}\|_{[a,b],1}}{m^n n!\cdot n}[(b-t)^n+(t-a)^n]; \end{cases}$$

$$\leq \begin{cases} \dfrac{(b-a)^{n+1}}{m^n(n+1)\cdot(n+1)!},\|f^{(n+1)}\|_{[a,b],\infty} & if \quad f^{(n+1)}\in L_\infty[a,b]; \\[2ex] \dfrac{q(b-a)^{n+\frac{1}{q}}}{m^n n!(nq+1)^{1+\frac{1}{q}}}\|f^{(n+1)}\|_{[a,b],p}, & if \quad f^{(n+1)}\in L_p[a,b], \\ & \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{(b-a)^n}{m^n n!\cdot n}\|f^{(n+1)}\|_{[a,b],1}; \end{cases}$$

Proof. We have (see (2.3)) that:

$$(Tf)(a,b;t)-f(t)\ln\left(\frac{b-t}{t-a}\right)=PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}\,d\tau.$$

If we write the inequality (3.1) for $g=f'$, we get

$$\left|\frac{f(t)-f(\tau)}{\tau-t}-\sum_{i=0}^{m-1}\sum_{k=1}^{n}\frac{(\tau-t)^{k-1}}{m^k k!}\cdot f^{(k)}\left(t+i\cdot\frac{\tau-t}{m}\right)\right|$$

(3.5)

$$\leq \begin{cases} \dfrac{|\tau-t|^n}{m^n(n+1)!}\|f^{(n+1)}\|_{[t,\tau],\infty}, & if \quad f^{(n+1)}\in L_\infty[a,b]; \\[2ex] \dfrac{|\tau-t|^{n-1+\frac{1}{q}}}{m^n n!(nq+1)^{\frac{1}{q}}}\|f^{(n+1)}\|_{[t,\tau],p}, & if \quad f^{(n+1)}\in L_p[a,b], \\ & \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{|\tau-t|^{n-1}}{m^n n!}\|f^{(n+1)}\|_{[t,\tau],1}. \end{cases}$$

If we apply PV to (3.5), we may write that:

$$\left| PV \int_a^b \frac{f(t) - f(\tau)}{\tau - t} d\tau \right.$$

$$\left. - \sum_{i=0}^{m-1} \sum_{k=1}^n PV \int_a^b \frac{(\tau - t)^{k-1}}{m^k k!} \cdot f^{(k)} \left(t + i \cdot \frac{\tau - t}{m} \right) d\tau \right|$$

$$\leq \begin{cases} \dfrac{1}{m^n (n+1)!} PV \displaystyle\int_a^b |\tau - t|^n \| f^{(n+1)} \|_{[t,\tau],\infty} d\tau, \\[2ex] \dfrac{1}{m^n n! (nq+1)^{\frac{1}{q}}} PV \displaystyle\int_a^b |\tau - t|^{n-1+\frac{1}{q}} \| f^{(n+1)} \|_{[t,\tau],p} d\tau, \\[2ex] \dfrac{1}{m^n n!} PV \displaystyle\int_a^b |\tau - t|^{n-1} \| f^{(n+1)} \|_{[t,\tau],1} d\tau, \end{cases}$$

(3.7)

$$\leq \begin{cases} \dfrac{1}{m^n (n+1)!} \| f^{(n+1)} \|_{[a,b],\infty} PV \displaystyle\int_a^b |\tau - t|^n d\tau, \\[2ex] \dfrac{1}{m^n n! (nq+1)^{\frac{1}{q}}} \| f^{(n+1)} \|_{[a,b],p} PV \displaystyle\int_a^b |\tau - t|^{n-1+\frac{1}{q}} d\tau, \\[2ex] \dfrac{1}{m^n n!} \| f^{(n+1)} \|_{[a,b],1} PV \displaystyle\int_a^b |\tau - t|^{n-1} d\tau, \end{cases}$$

$$\leq \begin{cases} \dfrac{1}{m^n (n+1)!} \| f^{(n+1)} \|_{[a,b],\infty} \left[\dfrac{(b-t)^{n+1} + (t-a)^{n+1}}{n+1} \right], \\[2ex] \dfrac{1}{m^n n! (nq+1)^{\frac{1}{q}}} \| f^{(n+1)} \|_{[a,b],p} \left[\dfrac{(b-t)^{n+\frac{1}{q}} + (t-a)^{n+\frac{1}{q}}}{n+\frac{1}{q}} \right], \\[2ex] \dfrac{1}{m^n n!} \| f^{(n+1)} \|_{[a,b],1} \left[\dfrac{(b-t)^n + (t-a)^n}{n} \right]. \end{cases}$$

Now, let us denote

$$I_{i,k} := PV \int_a^b (\tau - t)^{k-1} f^{(k)} \left(t + \frac{i}{m} (\tau - t) \right) d\tau,$$

where $i = 0, \ldots, m-1$, $k = 1, \ldots, n$.

For $i = 0$, we have

$$I_{0,k} := PV \int_a^b (\tau - t)^{k-1} f^{(k)}(t) d\tau = f^{(k)}(t) \cdot \frac{(b-t)^k + (-1)^{k+1}(t-a)^k}{k}$$

for any $k = 1, \ldots, n$.

For $k = 1, \ldots, n$ and $i = 1, \ldots, m-1$, we have

$$I_{i,k} = \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} (\tau - t)^{k-1} f^{(k)} \left[t + \frac{i}{m}(\tau - t) \right] d\tau \right.$$

$$+ \left. \int_{t+\varepsilon}^b (\tau - t)^{k-1} f^{(k)} \left[t + \frac{i}{m}(\tau - t) \right] d\tau \right]$$

$$= \lim_{\varepsilon \to 0+} \left[\frac{m}{i} f^{(k-1)} \left[t + \frac{i}{m}(\tau - t) \right] (\tau - t)^{k-1} \Big|_a^{t-\varepsilon} \right.$$

$$- \frac{m}{i} \int_a^{t-\varepsilon} (k-1)(\tau - t)^{k-2} f^{(k-1)} \left[t + \frac{i}{m}(\tau - t) \right] d\tau$$

(3.8)
$$+ \frac{m}{i} f^{(k-1)} \left[t + \frac{i}{m}(\tau - t) \right] (\tau - t)^{k-1} \Big|_{t+\varepsilon}^b$$

$$- \left. \frac{m}{i} \int_{t+\varepsilon}^b (k-1)(\tau - t)^{k-2} f^{(k-1)} \left[t + \frac{i}{m}(\tau - t) \right] d\tau \right]$$

$$= \frac{m}{i} \left[f^{(k-1)} \left[t + \frac{i}{m}(b - t) \right] - f^{(k-1)} \left[t - \frac{i}{m}(t - a) \right] \right]$$

$$- \frac{m}{i}(k-1) PV \int_a^b (\tau - t)^{k-2} f^{(k-1)} \left(t + \frac{i}{m}(\tau - t) \right) d\tau$$

$$= (b - a) \left[f^{(k-1)}; t + \frac{i}{m}(b - t), t - \frac{i}{m}(t - a) \right] - \frac{m}{i}(k-1) I_{i,k-1}$$

for any $k = 2, \ldots, n$.

For $k = 1$, we have

$$I_{i,1} = PV \int_a^b f^{(1)} \left(t + \frac{i}{m}(\tau - t) \right) d\tau$$

$$= \frac{m}{i} \left[f \left[t + \frac{i}{m}(b - t) \right] - f \left[t - \frac{i}{m}(t - a) \right] \right]$$

$$= (b - a) \left[f; t + \frac{i}{m}(b - t), t - \frac{i}{m}(t - a) \right].$$

Using the recursive relation (3.8), we may write

$$I_{i,k} = (b - a) \left[f^{(k-1)}; t + \frac{i}{m}(b - t), t - \frac{i}{m}(t - a) \right] - \left(\frac{m}{i} \right)(k-1) I_{i,k-1}$$

(3.9)
$$= (b - a) \left[f^{(k-1)}; t + \frac{i}{m}(b - t), t - \frac{i}{m}(t - a) \right] - \left(\frac{m}{i} \right)(k-1)$$

$$\times \left[(b - a) \left[f^{(k-2)}; t + \frac{i}{m}(b - t), t - \frac{i}{m}(t - a) \right] - \left(\frac{m}{i} \right)(k-2) I_{i,k-2} \right]$$

$$= (b-a) \left[f^{(k-1)}; t + \frac{i}{m}(b-t), t - \frac{i}{m}(t-a) \right]$$

$$-(b-a)\left(\frac{m}{i}\right)(k-1)\left[f^{(k-2)}; t + \frac{i}{m}(b-t), t - \frac{i}{m}(t-a) \right]$$

$$+\left(\frac{m}{i}\right)^2 (k-1)(k-2)I_{i,k-2}$$

$$\cdots\cdots\cdots\cdots$$

$$= (b-a) \sum_{\nu=1}^{k-1} (-1)^{\nu-1}(k-1)\cdots(k-\nu)\left(\frac{m}{i}\right)^{\nu-1}$$

$$\times \left[f^{(k-\nu)}; t + \frac{i}{m}(b-t), t - \frac{i}{m}(t-a) \right] + (-1)^{k-1}\left(\frac{m}{i}\right)^{k-1}(k-1)!I_{i,1}$$

$$= (b-a) \left[\sum_{\nu=1}^{k-1} (-1)^{\nu-1}(k-1)\cdots(k-\nu)\left(\frac{m}{i}\right)^{\nu-1} \right.$$

$$\times \left[f^{(k-\nu)}; t + \frac{i}{m}(b-t), t - \frac{i}{m}(t-a) \right]$$

$$\left. +(-1)^{k-1}\left(\frac{m}{i}\right)^{k-1}(k-1)!\left[f; t + \frac{i}{m}(b-t), t - \frac{i}{m}(t-a) \right] \right].$$

Replacing $I_{i,k}$ in (3.7), we deduce the estimate (3.4) with $A_{m,n}$ as defined by (3.3). ∎

4. A Numerical Example

Consider the function $f : [1,2] \to R, f(t) = \sin t$. The exact finite Hilbert transform of f provided by Maple is

(4.1)
$$(Tf)(1,2;t) = -Si(t-2)\cos t + Ci(2-t)\sin t$$
$$+Si(t-1)\cos t - Ci(t-1)$$

where

$$Si(x) := \int_0^x \frac{\sin u}{u} du, \, Ci(x) := \gamma + \ln x + \int_0^x \frac{\cos u - 1}{u} du.$$

The plot of the finite Hilbert transform is incorporated in Figure 1.

Define by $A_n(1,2;t)$ the approximation for the finite Hilbert transform provided by the formula (2.1), i.e.,

$$A_n(1,2;t) = \sin t \cdot \ln\left(\frac{2-t}{t-1}\right) + \sum_{k=1}^{n-1} \frac{[\sin t]^{(k)}}{k!} \cdot \left[\frac{(b-t)^k + (-1)^{k+1}(t-a)^k}{k} \right].$$
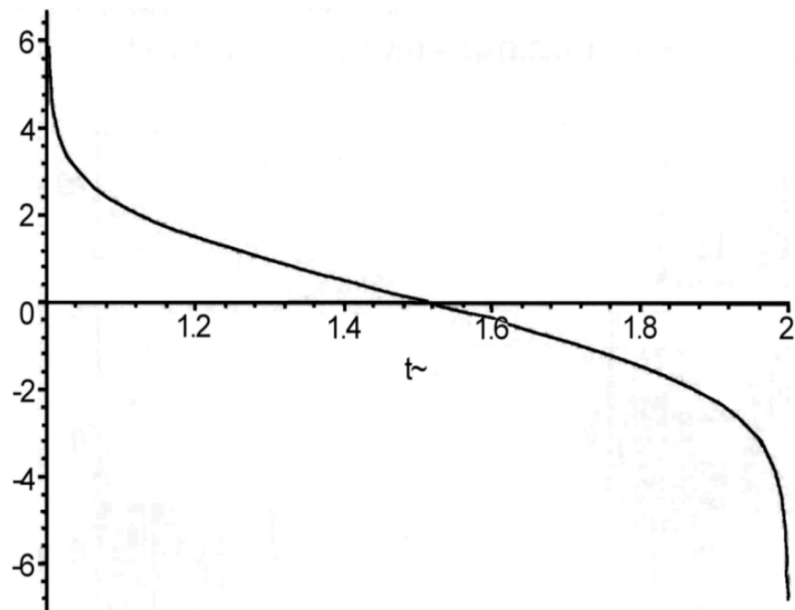
Fig. 1. The plot of the finite Hilbert transform.

Figure 2 contains the plot of the error.

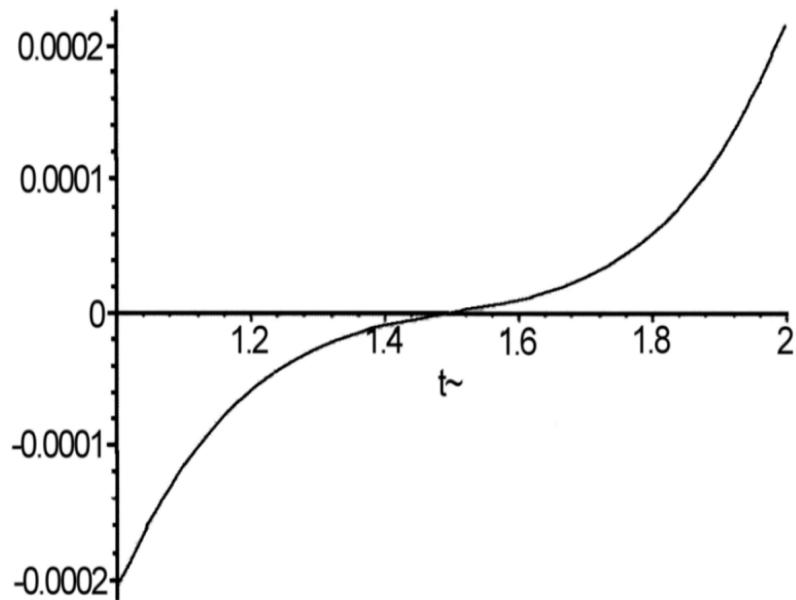$$Er_5(1,2;t) := (Tf)(1,2;t) - A_5(1,2;t), t \in [1,2].$$

Fig. 2. Contains the plot of the error.

Figure 3 Contains the plot of the error.

$$Er_{10}(1, 2; t) := (Tf)(1, 2; t) - A_{10}(1, 2; t), t \in [1, 2].$$
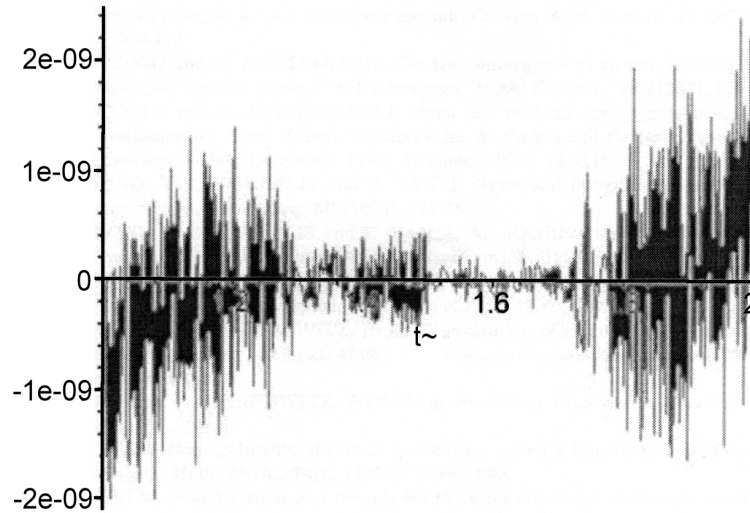

Fig. 3. Contains the plot of the error.

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