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A CLASS OF THIRD ORDER MULTI-POINT BOUNDARY VALUE PROBLEM

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Abstract. This paper deals with a class of third order multi-point boundary value problem at resonance case. Some existence theorems are obtained by using the coincidence degree theory of Mawhin.

1. Introduction

In this paper, we are concerned with the following third order ordinary differential equation:

$$x'''(t) = f(t, x(t), x'(t), x''(t)), \;\; t \in (0, 1), \tag{1.1}$$

with the following multi-point boundary conditions:

$$x(0) = \alpha x(\xi), \;\; x''(0) = 0, \;\; x'(1) = \sum_{j=1}^{m-2} \beta_j x'(\eta_j). \tag{1.2}$$

Where $f : [0, 1] \times R^3 \longrightarrow R$ is a continuous function, $\alpha \geq 0$, β_j ($j = 1, \cdots, m-2,$) $\in R$, $0 < \xi < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, all $\beta_i's$ have the same sign.

Similar to [1, 2], if the linear equation $x'''(t) = 0$, with boundary conditions (1.2) has only zero solution, and the differential operator defined in a suitable Banach space, with boundary conditions taken into account, is invertible, the so-called non-resonance case; otherwise, is non-invertible, then the so-called resonance case.

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For the resonance case, it is more delicate. Ma [3] studied existence and multiplity results for the following boundary value problem:

$$x''' + k^2x' + g(x, x') = p(t), \tag{1.3}$$

$$x'(0) = x'(\pi) = x(\eta) = 0, \tag{1.4}$$

by combining the well-known Lyapunov-Schmidt procedure with the continuum theory for O-epi maps. In the case $k = 1$, the solvability of (1.3), (1.4) has been considered by Nagle and Pothoven [4] under the condition that g is bounded on one side. Gupta [5] studied the existence of boundary value problem, similar to (1.3), (1.4) of the type

$$x''' + \pi^2x' + g(t, x, x', x'') = p(t), \tag{1.5}$$

$$x'(0) = x'(1) = x(\eta) = 0, 0 \leq \eta \leq 1, \tag{1.6}$$

under some appropriate conditions.

Feng [1], Liu [6] and Gupta [7] studied the existence results for some second order multi-point boundary value problems at resonance case.

Inspired by the work of the above papers, in the present article, we use the coincidence degree theory of Mawhin [8] to discuss the existence of solution for third order multi-point BVP (1.1), (1.2) at resonance case, and establish some existence theorems under sub-linear growth restriction of f. For some recent results on third order nonlinear boundary value problems and second order multi-point boundary value problems at resonance case we refer the reader to [9-12].

2. Main Results

We first recall some notation and an abstract existence result.

Let Y, Z be real Banach spaces and let $L : \text{dom}L \subset Y \longrightarrow Z$ be a linear operator which is Fredholm map of index zero (that is, ImL, the image of L, Ker$L$, the kernel of L is finite dimensional with the same dimension as the $Z/\text{Im}L$.) and $P : Y \longrightarrow Y$, $Q : Z \longrightarrow Z$ be continuous projectors such that ImP = Ker$L$, Ker$Q$ = ImL and $Y = \text{Ker}L \oplus \text{Ker}P$, $Z = \text{Im}L \oplus \text{Im}Q$. It follows that $L|_{\text{dom}L \cap \text{Ker}P} : \text{dom}L \cap \text{Ker}P \longrightarrow \text{Im}L$ is invertible, we denote the inverse of that map by K_P. Let Ω be an open bounded subset of Y such that $\text{dom}L \cap \Omega \neq \emptyset$, the map $N : Y \longrightarrow Z$ is said to be L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \longrightarrow Y$ is compact. Let $J : \text{Im}Q \longrightarrow \text{Ker}L$ be a linear isomorphism.

The theorem we use in the following is the Theorem IV.13 of [8].

Theorem 2.1. *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\mathrm{dom}L \backslash \mathrm{Ker}L) \cap \partial\Omega] \times (0, 1)$.*
(ii) *$Nx \notin \mathrm{Im}L$ for every $x \in \mathrm{Ker}L \cap \partial\Omega$.*
(iii) *$\deg(JQN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) \neq 0$, where $Q : Z \longrightarrow Z$ is a projection as above with $\mathrm{Im}L = \mathrm{Ker}Q$.*

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \overline{\Omega}$.

In the following, we shall use the classical spaces $C[0,1]$, $C^1[0,1]$, $C^2[0,1]$ and $L^1[0,1]$. For $x \in C^2[0,1]$, we use the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$ and $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\}$, and denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$. We will use the Sobolev space $W^{3,1}(0,1)$ which may be defined by

$$W^{3,1}(0,1) = \{x : [0,1] \longrightarrow R | x, x', x'' \text{ are absolutely continuous on } [0,\ 1] \text{ with } x''' \in L^1[0,1]\}.$$

Now we prove existence results for BVP (1.1), (1.2) in the following two cases:

(i) $\alpha = 0, \sum_{j=1}^{m-2} \beta_j = 1$;
(ii) $\alpha = 1, \sum_{j=1}^{m-2} \beta_j = 1$.

Let $Y = C^2[0,1]$, $Z = L^1[0,1]$, L is the linear operator from $\mathrm{dom}L \subset Y$ to Z with

$$\mathrm{dom}L = \{x \in W^{3,1}(0,1) : x(0) = \alpha x(\xi), x''(0) = 0, x'(1) = \sum_{j=1}^{m-2} \beta_j x'(\eta_j)\}$$

and $Lx = x'''$, $x \in \mathrm{dom}L$. We define $N : Y \longrightarrow Z$ by setting

$$Nx = f(t, x(t), x'(t), x''(t)), t \in (0,1).$$

Then BVP (1.1), (1.2) can be written as $Lx = Nx$.

Our first result is the following one dealing with BVP (1.1), (1.2) in case (i).

Theorem 2.2. *Let $f : [0,1] \times R^3 \longrightarrow R$ be a continuous function, assume that*

(1) *There exist functions a, b, c, $r \in L^1[0,1]$, such that for all $(x, y, z) \in R^3$, $t \in [0,1]$, satisfying*

$$|f(t, x, y, z)| \leq a(t)|x| + b(t)|y| + c(t)|z| + r(t). \tag{2.1}$$

(2) *There exists a constant $M > 0$, such that for $x \in domL$, if $|x'(t)| > M$, for all $t \in [0,1]$, then*

$$\sum_{j=1}^{m-2} \beta_j \left[\int_0^{\eta_j} (1-\eta_j) f(v, x(v), x'(v), x''(v)) dv + \int_{\eta_j}^1 (1-v) f(v, x(v), x'(v), x''(v)) dv\right] \neq 0. \tag{2.2}$$

(3) *There exists a constant $M^* > 0$, such that either*

$$c \cdot \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau} f(v, cv, c, 0) dv d\tau < 0, \quad \text{for all } |c| > M^*, \tag{2.3}$$

or else

$$c \cdot \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau} f(v, cv, c, 0) dv d\tau > 0, \quad \text{for all } |c| > M^*. \tag{2.4}$$

Then BVP (1.1), (1.2) with $\alpha = 0$, $\sum_{j=1}^{m-2} \beta_j = 1$, has at least one solution in $C^2[0, 1]$ if

$$\|a\|_1 + \|b\|_1 + \|c\|_1 < \frac{1}{2}.$$

We prove this result via the following lemmas.

In the following, we assume that the conditions in Theorem 2.2 are all satisfied.

Lemma 2.1. *If $\alpha = 0, \sum_{j=1}^{m-2} \beta_j = 1$, then $L : \text{dom}L \subset Y \longrightarrow Z$ is a Fredholm operator of index zero. Furthermore, the linear continuous projector operator $Q : Z \longrightarrow Z$ can be defined by*

$$Qy = \frac{2}{1 - \sum_{j=1}^{m-2} \beta_j \eta_j^2} \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau} y(v) dv d\tau,$$

and the linear operator $K_P : \text{Im}L \longrightarrow \text{dom}L \cap \text{Ker}P$ can be written by

$$K_P y = \int_0^t \int_0^s \int_0^{\tau} y(v) dv d\tau ds.$$

Furthermore

$$\|K_P\| \leq \|y\|_1, \text{ for every } y \in \text{Im}L.$$

Proof. It is clear that

$$\text{Ker}L = \{x \in \text{dom}L : x = ct, c \in R, t \in [0, 1]\}.$$

Now we show that

$$\text{Im}L = \{y \in Z : \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau} y(v) dv d\tau = 0\}. \tag{2.5}$$

Since the problem

$$x''' = y \tag{2.6}$$

has a solution $x(t)$ satisfied $x(0) = \alpha x(\xi), x''(0) = 0, x'(1) = \sum_{j=1}^{m-2} \beta_j x'(\eta_j)$, if and only if

$$\sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_{0}^{\tau} y(v) dv d\tau = 0. \tag{2.7}$$

In fact, if (2.6) has a solution $x(t)$ satisfied $x(0) = \alpha x(\xi), x''(0) = 0, x'(1) = \sum_{j=1}^{m-2} \beta_j x'(\eta_j)$, then from (2.6) we have

$$\begin{aligned} x(t) &= x(0) + x'(0)t + \frac{1}{2}x''(0)t^2 + \int_0^t \int_0^s \int_0^\tau y(v) dv d\tau ds \\ &= x'(0)t + \int_0^t \int_0^s \int_0^\tau y(v) dv d\tau ds. \end{aligned}$$

According to $\sum_{j=1}^{m-2} \beta_j = 1$, we obtain

$$\sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_{0}^{\tau} y(v) dv d\tau = 0.$$

On the other hand, if (2.7) holds, setting

$$x(t) = ct + \int_0^t \int_0^s \int_0^\tau y(v) dv d\tau ds,$$

where c is an arbitrary constant, then $x(t)$ is a solution of (2.6), and $x(0) = x''(0) = 0$, $x'(1) = \sum_{j=1}^{m-2} \beta_j x'(\eta_j)$. Hence (2.5) is valid.

For $y \in Z$, define

$$Qy(t) = \frac{2}{1 - \sum_{j=1}^{m-2} \beta_j \eta_j^2} \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_{0}^{\tau} y(v) dv d\tau, 0 \le t \le 1.$$

Let $y_1 = y - Qy$, it is easily shown that

$$\sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_{0}^{\tau} y_1(v) dv d\tau = 0,$$

then $y_1 \in \text{Im}L$. Hence $Z = \text{Im}L + Z_1$, where $Z_1 = \{x(t) \equiv c : t \in [0,1], c \in R\}$, also $\text{Im}L \cap Z_1 = \{0\}$. So we have $Z = \text{Im}L \oplus Z_1$, and

$$\dim \text{Ker}L = \dim Z_1 = \text{co}\dim \text{Im}L = 1.$$

Thus L is a Fredholm operator of index zero.

Now we define a projector P from Y to Y by setting

$$Px = x'(0)t.$$

Then the generalized inverse $K_P : \text{Im}L \longrightarrow \text{dom}L \cap \text{Ker}P$ of L can be written by

$$K_P y = \int_0^t \int_0^s \int_0^\tau y(v)dvd\tau ds.$$

In fact, for $y \in \text{Im}L$, we have

$$(LK_P)y(t) = [(K_P y)(t)]''' = y(t),$$

and for $x \in \text{dom}L\cap\text{Ker}P$, we know

$$(K_P L)x(t) = \int_0^t \int_0^s \int_0^\tau x'''(v)dvd\tau ds = x(t) - x(0) - x'(0)t - \frac{1}{2}x''(0)t^2,$$

in view of $x \in \text{dom}L\cap\text{Ker}P$, $x(0) = x''(0) = 0$ and $Px = 0$, thus

$$(K_P L)x(t) = x(t).$$

This shows that $K_P = (L|_{\text{dom}L\cap\text{Ker}P})^{-1}$. Also we have

$$\|K_P y\|_\infty \leq \int_0^1 \int_0^1 \int_0^1 |y(v)|dvd\tau ds = \|y\|_1,$$

and from $(K_P y)'(t) = \int_0^t \int_0^\tau y(v)dvd\tau$, $(K_P y)''(t) = \int_0^t y(v)dv$, we obtain

$$\|(K_P y)'\|_\infty \leq \int_0^1 \int_0^1 |y(v)|dvd\tau = \|y\|_1,$$

$$\|(K_P y)''\|_\infty \leq \int_0^1 |y(v)|dv = \|y\|_1,$$

then $\|K_P y\| \leq \|y\|_1$. This completes the proof of Lemma 2.1.

Lemma 2.2. *Let* $\Omega_1 = \{x \in domL\backslash KerL : Lx = \lambda Nx \text{ for some } \lambda \in [0,1]\}$. *Then* Ω_1 *is a bounded subset of* Y.

Proof. Suppose that $x \in \Omega_1$ and $Lx = \lambda Nx$. Thus $\lambda \neq 0$ and $QNx = 0$, so that

$$\sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^\tau f(v, x(v), x'(v), x''(v))dvd\tau = 0,$$

namely,

$$\sum_{j=1}^{m-2}\beta_j\left[\int_0^{\eta_j}(1-\eta_j)f(v,x(v),x'(v),x''(v))dv\right.$$
$$\left.+\int_{\eta_j}^{1}(1-v)f(v,x(v),x'(v),x''(v))dv\right]=0.$$

Thus, by condition (2), there exists $t_0\in[0,1]$, such that $|x'(t_0)|\le M$. In view of

$$x'(0)=x'(t_0)-\int_0^{t_0}x''(t)dt,\quad x''(t)=x''(0)-\int_0^{t}x'''(t)dt,$$

then, we have

$$|x'(0)|\le M+\int_0^1\int_0^1|x'''|dt=M+\|x'''\|_1=M+\|Lx\|_1\le M+\|Nx\|_1. \tag{2.8}$$

Again for $x\in\Omega_1, x\in \text{dom}L\backslash\text{Ker}L$, then $(I-P)x\in\text{dom}L\cap\text{Ker}P, LPx=0$, thus from Lemma 2.1, we know

$$\|(I-P)x\|=\|K_PL(I-P)x\|\le\|L(I-P)x\|_1=\|Lx\|_1\le\|Nx\|_1. \tag{2.9}$$

From (2.8) and (2.9), we have

$$\|x\|\le\|Px\|+\|(I-P)x\|=|x'(0)|+\|(I-P)x\|\le 2\|Nx\|_1+M. \tag{2.10}$$

From (2.1) and (2.10), we obtain

$$\|x\|\le 2[\|a\|_1\|x\|_\infty+\|b\|_1\|x'\|_\infty+\|c\|_1\|x''\|_\infty+\|r\|_1+\frac{M}{2}]. \tag{2.11}$$

Thus, from $\|x\|_\infty\le\|x\|$ and (2.11), we have

$$\|x\|_\infty\le\frac{2}{1-2\|a\|_1}\left[\|b\|_1\|x'\|_\infty+\|c\|_1\|x''\|_\infty+\|r\|_1+\frac{M}{2}\right]. \tag{2.12}$$

From $\|x'\|_\infty\le\|x\|$, (2.11) and (2.12), one has

$$\begin{aligned}\|x'\|_\infty&\le\|x\|\\&\le 2\left[1+\frac{2\|a\|_1}{1-2\|a\|_1}\right]\left[\|b\|_1\|x'\|_\infty+\|c\|_1\|x''\|_\infty+\|r\|_1+\frac{M}{2}\right]\\&=\frac{2}{1-2\|a\|_1}\left[\|b\|_1\|x'\|_\infty+\|c\|_1\|x''\|_\infty+\|r\|_1+\frac{M}{2}\right],\end{aligned}$$

i.e.,

$$\|x'\|_\infty \le \frac{2}{1-2\|a\|_1-2\|b\|_1}\left[\|c\|_1\|x''\|_\infty+\|r\|_1+\frac{M}{2}\right]. \tag{2.13}$$

Again from $\|x''\|_\infty \le \|x\|$, (2.11), (2.12) and (2.13), we get

$$\begin{aligned}\|x''\|_\infty \le \|x\| &\le \left[2\|b\|_1+\frac{4\|a\|_1\|b\|_1}{1-2\|a\|_1}\right]\|x'\|_\infty \\ &\quad+\left[\frac{4\|a\|_1}{1-2\|a\|_1}+2\right]\left[\|c\|_1\|x''\|_\infty+\|r\|_1+\frac{M}{2}\right] \\ &\le \left[\frac{4\|b\|_1}{(1-2\|a\|_1-2\|b\|_1)(1-2\|a\|_1)}+\frac{2}{1-2\|a\|_1}\right] \\ &\quad\cdot\left[\|c\|_1\|x''\|_\infty+\|r\|_1+\frac{M}{2}\right] \\ &= \frac{2}{1-2\|a\|_1-2\|b\|_1}\left[\|c\|_1\|x''\|_\infty+\|r\|_1+\frac{M}{2}\right],\end{aligned}$$

i.e.,

$$\|x''\|_\infty \le \frac{2C_1}{1-2\|a\|_1-2\|b\|_1-2\|c\|_1}, \tag{2.14}$$

where $C_1=\|r\|_1+\frac{M}{2}$. From (2.14), there exist $M_1>0$, such that

$$\|x''\|_\infty \le M_1, \tag{2.15}$$

thus from (2.15) and (2.13), there exist $M_2>0$, such that

$$\|x'\|_\infty \le M_2, \tag{2.16}$$

therefore from (2.16) and (2.12), there exist $M_3>0$, such that

$$\|x\|_\infty \le M_3. \tag{2.17}$$

Hence

$$\|x\|=\max\{\|x\|_\infty,\|x'\|_\infty,\|x''\|_\infty\}\le \max\{M_1,M_2,M_3\}.$$

Again from (2.1), (2.15), (2.16) and (2.17), we have

$$\|x'''\|_1=\|Lx\|_1\le\|Nx\|_1\le\|a\|_1M_3+\|b\|_1\|M_2+\|c\|_1M_1+\|r\|_1.$$

We show that Ω_1 is bounded.

Lemma 2.3. *The set* $\Omega_2=\{x\in KerL : Nx\in ImL\}$ *is bounded.*

Proof. Let $x \in \Omega_2$, then $x \in \text{Ker}L = \{x \in \text{dom}L : x = ct, c \in R, t \in [0,1]\}$, and $QNx = 0$, therefore

$$\sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_{0}^{\tau} f(v, cv, c, 0) dv d\tau = 0,$$

that is

$$\sum_{j=1}^{m-2} \beta_j \left[\int_{0}^{\eta_j} (1-\eta_j) f(v, cv, c, 0) dv + \int_{\eta_j}^{1} (1-v) f(v, cv, c, 0) dv \right] = 0.$$

From condition (2), $\|x\|_\infty = |c| \le M$, so $\|x\| = |c| \le M$, thus Ω_2 is bounded.

Lemma 2.3. *If the first part of Condition* (3) *of Theorem* 2.2 *holds, that is, there exists* $M^* > 0$*, such that*

$$c \cdot \frac{2}{1 - \sum_{j=1}^{m-2} \beta_j \eta_j^2} \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_{0}^{\tau} f(v, cv, c, 0) dv d\tau < 0, \tag{2.18}$$

for all $|c| > M^*$. *Let*

$$\Omega_3 = \{x \in \text{Ker}L : -\lambda x + (1-\lambda) JQNx = 0, \lambda \in [0,1]\},$$

where $J : \text{Im}Q \longrightarrow \text{Ker}L$ *is the linear isomorphism given by* $J(c) = ct$*,* $\forall c \in R$*,* $t \in [0,1]$*. Then* Ω_3 *is bounded.*

Proof. Suppose that $x = c_0 t \in \Omega_3$, then we obtain

$$\lambda c_0 t = (1-\lambda) \cdot \frac{2t}{1 - \sum_{j=1}^{m-2} \beta_j \eta_j^2} \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_{0}^{\tau} f(v, c_0 v, c_0, 0) dv d\tau, 0 \le t \le 1,$$

or equivalently

$$\lambda c_0 = (1-\lambda) \cdot \frac{2}{1 - \sum_{j=1}^{m-2} \beta_j \eta_j^2} \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_{0}^{\tau} f(v, c_0 v, c_0, 0) dv d\tau.$$

If $\lambda = 1$, then $c_0 = 0$. Otherwise, if $|c_0| > M^*$, in view of (2.18), one has

$$\lambda c_0^2 = c_0 \cdot (1-\lambda) \cdot \frac{2}{1 - \sum_{j=1}^{m-2} \beta_j \eta_j^2} \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau} f(v, c_0 v, c_0, 0) dv d\tau < 0,$$

which contradicts $\lambda c_0^2 \geq 0$. Then $|x| = |c_0 t| \leq |c_0| \leq M^*$, we obtain $\|x\| \leq M^*$, therefore $\Omega_3 \subset \{x \in \text{Ker}L : \|x\| \leq M^*\}$ is bounded.

The proof of Theorem 2.2 is now an easy consequence of the above lemmas and Theorem 2.1.

Proof of Theorem 2.2. Let $\Omega = \{x \in Y : \|x\| < d\}$ such that $\bigcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$. By the Ascoli-Arzela theorem, it can be shown that $K_P(I-Q)N : \overline{\Omega} \longrightarrow Y$ is compact, thus N is L-compact on $\overline{\Omega}$. Then by the above Lemmas, we have

(i) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\text{dom}L \backslash \text{Ker}L) \cap \partial\Omega] \times (0, 1)$.

(ii) $Nx \notin \text{Im}L$ for every $x \in \text{Ker}L \cap \partial\Omega$.

(iii) Let $H(x, \lambda) = -\lambda x + (1-\lambda) JQNx$, with J as in Lemma 2.4. We know $H(x, \lambda) \neq 0$, for $x \in \text{Ker}L \cap \partial\Omega$. Thus, by the homotopy property of degree, we get

$$\begin{aligned} \deg(JQN|_{\text{Ker}L}, \Omega \cap \text{Ker}L, 0) &= \deg(H(\cdot, 0), \Omega \cap \text{Ker}L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \text{Ker}L, 0) \\ &= \deg(-I, \Omega \cap \text{Ker}L, 0). \end{aligned}$$

According to definition of degree on a space which is isomorphic to $R^n, n < \infty$, and

$$\Omega \bigcap \text{Ker}L = \{ct : |c| < d\}.$$

We have

$$\begin{aligned} \deg(-I, \Omega \cap \text{Ker}L, 0) &= \deg(-J^{-1} IJ, J^{-1}(\Omega \cap \text{Ker}L), J^{-1}\{0\}) \\ &= \deg(-I, (-d, d), 0) = -1 \neq 0, \end{aligned}$$

and then

$$\deg(JQN|_{\text{Ker}L}, \Omega \cap \text{Ker}L, 0) \neq 0.$$

Then by Theorem 2.1, $Lx = Nx$ has at least one solution in $\text{dom}L \cap \overline{\Omega}$, so that the BVP (1.1), (1.2) has at least one solution in $C^2[0, 1]$. The proof is completed.

Remark 2.1. If the second part of Condition (3) of Theorem 2.2 holds, that is,

$$c\cdot\frac{2}{1-\sum\limits_{j=1}^{m-2}\beta_j\eta_j^2}\sum_{j=1}^{m-2}\beta_j\int_{\eta_j}^{1}\int_0^{\tau}f(v,cv,c,0)dvd\tau>0, \tag{2.19}$$

for all $|c|>M^*$, then in Lemma 2.4, we take

$$\Omega_3=\{x\in \mathrm{Ker}L:\lambda x+(1-\lambda)JQNx=0,\lambda\in[0,1]\},$$

and exactly as there, we can prove that Ω_3 is bounded. Then in the proof of Theorem 2.2, we have

$$\deg(JQN|_{\mathrm{Ker}L},\Omega\cap\mathrm{Ker}L,0)=\deg(I,\Omega\cap\mathrm{Ker}L,0)=1,$$

since $0\in\Omega\cap\mathrm{Ker}L$. The remainder of the proof is the same.

By using the same method as in the proof of Theorem 2.2 and Lemmas 2.1 − 2.4, we can show Lemma 2.5 and Theorem 2.3, when BVP (1.1), (1.2) satisfies the case (ii).

Lemma 2.5. *If $\alpha=1,\sum_{j=1}^{m-2}\beta_j=1$, then $L:domL\subset Y\longrightarrow Z$ is a Fredholm operator of index zero. Furthermore, the linear continuous projector operator $Q:Z\longrightarrow Z$ can be defined by*

$$Qy=\frac{2}{1-\sum\limits_{j=1}^{m-2}\beta_j\eta_j^2}\sum_{j=1}^{m-2}\beta_j\int_{\eta_j}^{1}\int_0^{\tau}y(v)dvd\tau,$$

and the linear operator $K_P:ImL\longrightarrow domL\cap KerP$ can be written by

$$K_Py=-\frac{t^2}{\xi^2}\int_0^{\xi}\int_0^{s}\int_0^{\tau}y(v)dvd\tau ds+\int_0^{t}\int_0^{s}\int_0^{\tau}y(v)dvd\tau ds.$$

Furthermore

$$\|K_P\|\le\Delta_1\|y\|_1,\ \text{for all } y\in ImL,$$

here $\Delta_1=\frac{2}{\xi}+1$.

Notice that the $\mathrm{Ker}L=\{x\in\mathrm{dom}L:x=d,d\in R\}$, $\mathrm{Im}L=\{y\in Z:\sum_{j=1}^{m-2}\beta_j\int_{\eta_j}^{1}\int_0^{\tau}y(v)dvd\tau=0\}$.

Theorem 2.3. *Let $f:[0,1]\times R^3\longrightarrow R$ be a continuous function, assume that*

(1) *The condition* (1) *in Theorem* 2.2 *is satisfied.*

(2) *There exists a constant* $M > 0$*, such that for* $x \in domL$*, if* $|x(t)| > M$*, for all* $t \in [0,1]$*, then*

$$\sum_{j=1}^{m-2} \beta_j \left[\int_0^{\eta_j} (1-\eta_j) f(v, x(v), x'(v), x''(v)) dv + \int_{\eta_j}^1 (1-v) f(v, x(v), x'(v), x''(v)) dv \right] \neq 0,$$

(3) *There exists a constant* $M^* > 0$*, such that either*

$$d \cdot \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^\tau f(v, d, 0, 0) dv d\tau < 0, \text{ for all } |d| > M^*,$$

or else

$$d \cdot \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^\tau f(v, d, 0, 0) dv d\tau > 0, \text{ for all } |d| > M^*.$$

Then BVP (1.1), (1.2) with $\alpha = 1$, $\sum_{j=1}^{m-2} \beta_j = 1$, has at least one solution in $C^2[0,1]$ provided that

$$\|a\|_1 + \|b\|_1 + \|c\|_1 < \frac{1}{\Delta_2},$$

where $\Delta_2 = \Delta_1 + 1$, Δ_1 as in Lemma 2.5.

Proof. Let

$$\Omega_1 = \{x \in \text{dom}L \backslash \text{Ker}L : Lx = \lambda Nx \text{ for some } \lambda \in [0,1]\}.$$

Then for $x \in \Omega_1, Lx = \lambda Nx$, thus $\lambda \neq 0, Nx \in \text{Im}L = \text{Ker}Q$, hence

$$\sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^\tau f(v, x(v), x'(v), x''(v)) dv d\tau = 0,$$

that is

$$\sum_{j=1}^{m-2} \beta_j \left[\int_0^{\eta_j} (1-\eta_j) f(v, x(v), x'(v), x''(v)) dv + \int_{\eta_j}^1 (1-v) f(v, x(v), x'(v), x''(v)) dv \right] = 0.$$

Thus, from condition (2), there exists $t_0 \in [0,1]$, such that $|x(t_0)| < M$, in view of $x(0) = x(t_0) - \int_0^{t_0} x'(t) dt$, we obtain

$$|x(0)| \leq M + \|x'\|_\infty. \tag{2.20}$$

From $x(0) = \alpha x(\xi) = x(\xi)$, there exists $t_1 \in (0, \xi)$, such that $x'(t_1) = 0$, thus from $x'(t) = x'(t_1) + \int_{t_1}^{t} x''(t)dt$, one has

$$\|x'\|_\infty \leq \|x''\|_1. \tag{2.21}$$

Again from $x''(0) = 0$, thus from $x''(t) = x''(0) + \int_{t_2}^{t} x'''(t)dt$, we obtain

$$\|x''\|_\infty \leq \|x'''\|_1. \tag{2.22}$$

We let $Px = x(0)$, hence from (2.20), (2.21) and (2.22), we have

$$\begin{aligned}\|Px\| = |x(0)| &\leq M + \|x'\|_\infty \leq M + \|x''\|_1 \leq M + \|x''\|_\infty \\ &\leq M + \|x'''\|_1 = M + \|Lx\|_1 \leq M + \|Nx\|_1,\end{aligned}$$

thus, by using the same method as in the proof of Lemma 2.2, we can prove that Ω_1 is bounded too. Similar to the other proof of Lemmas 2.3-2.4 and Theorem 2.2, we can verify Theorem 2.3.

3. Example

Example. Consider the following boundary value problem:

$$x''' = t^2 + 4 + \sin(x)^2 + \frac{1}{5}(t+1)x' + \cos(x'')^3, \quad t \in (0,1), \tag{3.1}$$

$$x'(0) = 0, \quad x''(0) = 0, \quad x(1) = \frac{1}{4}x(\frac{1}{4}) + \frac{1}{6}x(\frac{1}{3}) + \frac{7}{12}x(\frac{1}{2}), \tag{3.2}$$

where

$$f(t, x, y, z) = t^2 + 4 + \sin(x)^2 + \frac{1}{5}(t+1)y + \cos z^3, t \in (0,1),$$

$\alpha = 0, \beta_1 = \frac{1}{4}, \beta_2 = \frac{1}{6}, \beta_3 = \frac{7}{12}, \eta_1 = \frac{1}{4}, \eta_2 = \frac{1}{3}, \eta_3 = \frac{1}{2}$, then $\beta_1 + \beta_2 + \beta_3 = 1$, $\beta_1\eta_1 + \beta_2\eta_2 + \beta_3\eta_3 = \frac{59}{144} < 1$, we can choose $a(t) = c(t) = 0$, $b(t) = \frac{2}{5}$, $r(t) = 7$, for $t \in [0, 1]$, thus

$$|f(t, x, y, z)| \leq \frac{2}{5}|y| + 7,$$

$$\|a\|_1 + \|b\|_1 + \|c\|_1 = \frac{2}{5} < \frac{1}{2}.$$

Since f has the same sign as $x'(t)$ when $|x'(t)| > 35$, we may choose $M = M^* = 35$, and then the conditions (1) - (3) of Theorem 2.2 are satisfied. Theorem 2.2 implies that the BVP (3.1)-(3.2) has at least one solution $x \in C^2[0, 1]$.

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